Mathematical Logic
(A Berkeley-Harvard undergraduate course)

Theodore A. Slaman
W. Hugh Woodin

6 May 2021

Contents

Foreword

This text provides an undergraduate level introduction to mathematical logic from the perspective of “model theory”. Mathematical logic is studied by mathematicians, logicians, and philosophers alike. It is the backbone of most logical and mathematical arguments, and provides insight into the extent of our reasoning. Model theory presents a formalization of syntax, proofs, and truth, so hopefully the reader can understand its importance.

More practically, the study of model theory can provide a great comfort to budding mathematicians. No doubt you have at one point come across some obtuse description that and wondered if it is really useful. After all, one can read the properties of some objects like the hyperreals, or an infinite-dimensional vector space, but what good is it if no examples of such objects can be found? The results in this text assure us that these structures in a sense “exist.” Conversely, how can we know the worth of exploring the logical consequences of some sentences, especially if we might run into a contradiction? Model theory again provides an answer, as we will show that, for example, the axioms of a group will not imply a contradiction so long as we can provide an example of a group.

At the heart of this text is the proof of the Gödel Completeness Theorem. Naively, the notion of what statements are “true” and what statements are “provable” seem related, but in some sense different. After all, why would a statement about the real numbers, for example, be falsifiable or not? But as we will see, to an extent, these two notions are equivalent.

But be forewarned since (mathematical) life is always full of surprises. This is *not* to say that mathematical truth is exactly captured by proof. But that is the story of the Gödel Incompleteness Theorem, which is the sequel to the Gödel Completeness Theorem.

Acknowledgements

We would like to thank Java Darleen Villano and Matt F. Park for working through all of the exercises in an earlier version.

Theodore Slaman: Professor of Mathematics, UC Berkeley.

- Sea Ranch CA: 6 May 2021

W. Hugh Woodin: Professor of Mathematics (emeritus), UC Berkeley. Professor of Philosophy and of Mathematics, Harvard University.

- Ironwood Forest National Monument AZ: 6 May 2021

1

Propositional logic

Propositional logic governs the way by which propositions are combined in compound sentences.

Informally, a proposition is a declarative sentence, such as any of the following.

- Life is nothing but a competition to be the criminal rather than the victim. (B. Russell)
- Life is as tedious as a twice-told tale. (W. Shakespeare)
- Life is a dead-end street. (H. L. Menken)
- Life is too short to learn German. (R. Porson)

Propositions may be combined by logical connectives to form more complicated statements, such as "If life is a dead-end street, then life is too short to learn German" or "If it is not the case that life is too short to learn German, then life is as tedious as a twice told tale". The truth or falsity of a compound statement depends solely on that of its parts. Understanding propositional logic is just understanding that dependence.

In the next section, we will introduce symbols A_n for propositions, $\neg$ for negation, and $\to$ for implication. Using $A_1, \ldots, A_4$ to denote the above propositions, our two compound sentences would be denoted by $(A_3 \to A_4)$ and $((\neg A_4) \to A_2)$.

If one masters the elements of propositional logic then the solution to the following typical logic puzzle becomes clear.

- Two physicists, A and B, and a logician C, are wearing hats, which they know are either black or white but not all white. A can see the hats of B and C; B can see the hats of A and C; C is blind. Each is asked in turn if they know the color of their own hat. The answers are: A:"No." B: "No." C: "Yes." What color is C's hat and how does C know?

1.1 The language

Our language for propositional logic consists of (certain) finite sequences of symbols. The allowed symbols form the *alphabet* of the language. The actual ontology of these symbols is at this stage irrelevant.

Definition 1.1 • The *logical symbols* are the following symbols.

$$(\quad) \quad \neg \quad \rightarrow$$

- The *propositional symbols* are A_n, for n in $\mathbb{N}$. ($\mathbb{N}$ is the set of non-negative integers; i.e. the set of *natural numbers*). ☐

We frequently use the notation $\vec{A}$ to denote a finite sequence $\langle A_{i_1}, \dots, A_{i_n} \rangle$ of (not necessarily distinct) propositional symbols.

Definition 1.2 If $\vec{s} = \langle s_1, \dots, s_n \rangle$ and $\vec{t} = \langle t_1, \dots, t_m \rangle$ are finite sequences, we let $\vec{s} + \vec{t}$ denote the finite sequence

$$\vec{s} + \vec{t} = \langle s_1, \dots, s_n, t_1, \dots, t_m \rangle.$$

For a sequence $\vec{s}$, let $|\vec{s}|$ denote the length of s. ☐

Definition 1.3 The *propositional language* $\mathcal{L}_0$ is the smallest set L of finite sequences of the above symbols satisfying the following properties.

(1) For each propositional symbol A_n with $n \in \mathbb{N}$, $A_n \in L$.
(2) For each pair of finite sequences s and t, if s and t belong to L, then
- $(\neg s) \in L$
- $(s \rightarrow t) \in L$. ☐

Remark 1.4 A more precise definition of $\mathcal{L}_0$ is that $\mathcal{L}_0$ is the smallest set L of finite sequences of the above symbols satisfying the following properties.

(1) For each propositional symbol A_n with $n \in \mathbb{N}$, $\langle A_n \rangle \in L$.
(2) For each pair of finite sequences s and t, if s and t belong to L, then
- $\langle (\neg \rangle + s + \langle) \rangle \in L$
- $\langle (\rangle + s + \langle \rightarrow \rangle + t + \langle) \rangle \in L$.

This is of course what we meant by Definition 1.3. ☐

For the duration of Chapter 1, we will use *propositional formula* or just *formula* to refer to an element of $\mathcal{L}_0$. A formula is typically denoted φ or ψ. We will occasionally also refer to, for example, the propositional formula A_i and in doing so we mean of course the finite sequence s of length 1 where $s = \langle A_i \rangle$. Strictly speaking this renders the notion "A_i" as potentially ambiguous, but the context will always make this clear.

The notation $\vec{\varphi} = \langle \varphi_1, \dots, \varphi_n \rangle$ and $\vec{\psi} = \langle \psi_1, \dots, \psi_n \rangle$ can refer to a finite sequence of (not necessarily distinct) formulas.

In Definition 1.3, we defined the propositional language $\mathcal{L}_0$ as the smallest set which is closed under the two conditions, Condition (1) and Condition (2). In the following, we show that $\mathcal{L}_0$ is well defined.

Theorem 1.5 *$\mathcal{L}_0$ is the intersection of all of the sets which satisfy the two conditions of Definition 1.3.*

Proof. Let L_0 be the intersection of all of the sets which satisfy the two conditions of Definition 1.3. There is at least one such set, since the set of all finite sequences of symbols does satisfy the two conditions. We claim that L_0 is a set which satisfies those two conditions.

For each $n \in \mathbb{N}$, A_n is an element of every set which satisfies Condition 1. Consequently, A_n is an element of the intersection of all such sets, and thus it is an element of L_0.

Now, suppose that s and t belong to L_0. Then they belong to every set which satisfies Conditions 1 and 2. But then, for every such set, we can apply Condition 2 to conclude that $(\neg s)$ and $(s \to t)$ also belong to that set. Therefore, $(\neg s)$ and $(s \to t)$ belong to the intersection of all such sets, and thus belong to L_0.

Hence, L_0 satisfies Conditions 1 and 2. Since it is contained in every set which also satisfies those conditions, it must the smallest such set. Consequently, $\mathcal{L}_0$ is equal to L_0. □

1.1.1 Readability and subformulas

Definition 1.6 (1) A sequence $\langle a_1, \ldots, a_k \rangle$ is an *initial segment* of another sequence $\langle b_1, \ldots, b_m \rangle$ if and only if k is less than or equal to m and for all $i \leq k$, $a_i = b_i$. In other words, $\langle b_1, \ldots, b_m \rangle$ is equal to $\langle a_1, \ldots, a_k \rangle + \langle b_{k+1}, \ldots, b_m \rangle$, where $\langle b_{k+1}, \ldots, b_m \rangle$ could be the empty sequence.

(2) When m is greater than k, we say that $\langle a_1, \ldots, a_k \rangle$ is a *proper initial segment* of $\langle a_1, \ldots, a_k, b_{k+1}, \ldots, b_m \rangle$. □

Lemma 1.7 (Readability for Formulas) *Suppose that φ is a formula in $\mathcal{L}_0$. Then exactly one of the following conditions applies.*

(1) *There is an n such that $\varphi = \langle A_n \rangle$.*
(2) *There is a $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$.*
(3) *There are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$.*

Proof. Consider the subset L of $\mathcal{L}_0$ which consists of those formulas which satisfy the above three clauses. By the first clause, if $n \in \mathbb{N}$, then $\langle A_n \rangle \in L$. Consequently, L satisfies Condition 1 of Definition 1.3. Secondly, if ψ is in L, then $\psi \in \mathcal{L}_0$ and so $(\neg\psi) \in \mathcal{L}_0$. But then $(\neg\psi)$ is an element of $\mathcal{L}_0$ which satisfies the second of the above clauses, and hence $(\neg\psi) \in L$. Similarly, if ψ_1 and ψ_2 belong to L, then so does $(\psi_1 \to \psi_2)$. Thus, L satisfies Condition 2 of Definition 1.3. It follows that $\mathcal{L}_0 \subseteq L$, and so $\mathcal{L}_0 = L$.

It remains to show that the three possibilities are mutually exclusive.

Clearly, the first case excludes the other two since both of the formulas in the latter two cases begin with the symbol (. Now, if $\varphi = (\neg\psi)$, then the second symbol in φ is $\neg$. However, if $\varphi = (\psi_1 \to \psi_2)$, then the second symbol in φ is the first symbol in ψ_1, which by the above is either a propositional symbol A_n or a left-parenthesis (. Consequently, these two cases are mutually exclusive. □

We now prove a technical lemma which we will use to prove Unique Readability for Formulas.

Lemma 1.8 *If $\varphi \in \mathcal{L}_0$, then no proper initial segment of φ is an element of $\mathcal{L}_0$.*

Proof. We prove Lemma 1.8 by induction on the length of φ.

If φ has length 1 then the only subsequence to be considered is the empty sequence, which by Lemma 1.7 is not an element of $\mathcal{L}_0$.

Now suppose that $\varphi \in \mathcal{L}_0$ has length n, $n > 1$, and Lemma 1.8 holds for all elements of $\mathcal{L}_0$ with length less than n. By Lemma 1.7, since φ has length greater than 1, φ has one of two forms: $(\neg\psi)$ or $(\psi_1 \to \psi_2)$.

Suppose that φ is $(\neg\psi)$. For a contradiction, suppose that $\theta \in \mathcal{L}_0$ is a proper initial segment of $(\neg\psi)$. Then the first symbol in θ is (, so θ is not of the form $\langle A_i \rangle$, and by Lemma 1.7 the length of θ is greater than one. Thus, the second symbol in θ is $\neg$, which by Lemma 1.7 is not the first symbol of any element of $\mathcal{L}_0$, and so θ cannot be of the form $(\theta_1 \to \theta_2)$. Consequently, there exists $\theta_1 \in \mathcal{L}_0$ such that θ is equal to $(\neg\theta_1)$. But then $(\neg\psi)$ has $(\neg\theta_1)$ as a proper initial segment, and so ψ has θ_1 as a proper initial segment, contradiction to the induction hypothesis.

Finally, suppose that φ is $(\psi_1 \to \psi_2)$ and that $\theta \in \mathcal{L}_0$ is a proper initial segment of φ. We can apply Lemma 1.7 and argue as in the previous paragraph that there are θ_1 and θ_2 in $\mathcal{L}_0$ such that $\theta = (\theta_1 \to \theta_2)$. But then either θ_1 is a proper initial segment of ψ_1 (a contradiction), ψ_1 is a proper initial segment of θ_1 (a contradiction), or $\psi_1 = \theta_1$ and θ_2 is a proper initial segment of ψ_2 (a contradiction).

In either case, φ has no proper initial segment in $\mathcal{L}_0$. □

Theorem 1.9 (Unique Readability for Formulas) *Suppose that φ is a formula in $\mathcal{L}_0$. Then exactly one of the following conditions applies.*

(1) *There is an n such that $\varphi = \langle A_n \rangle$.*
(2) *There is a $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$.*
(3) *There are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$.*

Further, in (2), the formula ψ is unique, and similarly in (3), the both the formulas ψ_1 and ψ_2 are unique.

Proof. By Lemma 1.7, it is enough to check the claim of uniqueness.

First, suppose that $\varphi = (\neg\psi)$ and $\varphi = (\neg\theta)$. Thus, the sequence of symbols φ can be read as $\langle (, \neg \rangle + \psi + \langle) \rangle$ and as $\langle (, \neg \rangle + \theta + \langle) \rangle$. The occurrences of ψ and θ within φ have the same length and the same elements, and therefore are equal.

Finally, suppose that $\varphi = (\psi_1 \to \psi_2)$ and $\varphi = (\theta_1 \to \theta_2)$. Since both ψ_1 and θ_1 belong to $\mathcal{L}_0$, by Lemma 1.8 neither can be a proper initial segment of the other. Since they are initial segments of each other, they must be equal. As in the case of negation, it follows that ψ_2 and θ_2 are also equal. □

By Lemma 1.9, if φ is a formula and the length of φ is not 1 then φ has a unique decomposition into either $(\neg\psi)$ for some formula ψ, or as $(\varphi_1 \to \varphi_2)$ for some formulas φ_1 and φ_2. These formulas into turn have uniques decompositions, and so forth.

A natural question is whether all the formulas which arise in the iterated decomposition of the formula φ can easily be identified by simply examning φ itself.

Definition 1.10 Suppose that $s = \langle s_1, \ldots, s_n \rangle$ is a finite sequence. A finite sequence t is a *block-subsequence* of s if there exist non-negative integers i and j such that

(1) $i + j \leq n$,
(2) $t = \langle s_i, s_{i+1}, \ldots, s_{i+j} \rangle$. □

Example 1.11 (1) $\langle 3 \rangle$ is a block-subsequence of $\langle 1, 2, 3, 4, 5, 6 \rangle$.
(2) $\langle 3, 4, 5 \rangle$ is a block-subsequence of $\langle 1, 2, 3, 4, 5, 6 \rangle$.
(3) $\langle 1, 6 \rangle$ is not a block-subsequence of $\langle 1, 2, 3, 4, 5, 6 \rangle$.
(4) If s is a finite sequence and s has length n, then there are at most

$$\sum_{i=1}^{n}(n - i) + 1 = n^2 + n - (1/2)n(n + 1) = 1/2(n + 1)n$$

block-subsequences of s. □

Definition 1.12 Suppose that φ is a formula. A formula ψ is a *subformula* of φ if ψ is a block-subsequence of φ. □

The subformulas of φ are precisely the formulas which arise in the iterated decomposition of φ. We make this claim precise as Lemma 1.18, the statement of which involves the notion of a "formula-witness".

Definition 1.13 Suppose that

$$\vec{\psi} = \langle \psi_0, \ldots, \psi_n \rangle$$

is a finite sequence of finite sequences. Then $\vec{\psi}$ is a *formula-witness* if for all $i \leq n$, one of the following hold. following hold.

(1) $\psi_i = \langle A_k \rangle$ for some $k \in \mathbb{N}$.
(2) For some $j < i$, $\psi_i = (\neg \psi_j)$.
(3) For some $j_1, j_2 < i$, $\psi_i = (\psi_{j_1} \to \psi_{j_2})$. □

Lemma 1.14 *Suppose that*

$$\vec{\psi} = \langle \psi_0, \ldots, \psi_n \rangle$$

is a formula-witness. Then for all $i \leq n$, ψ_i is a formula.

Proof. By induction on $i \leq n$, this is immediate appealing to the the definition of a formula. $\square$

Suppose φ is a finite sequence and $\vec{\psi} = \langle \psi_1, \ldots, \psi_n \rangle$ is a formula-witness. Then we say $\vec{\psi}$ is a formula-witness for φ if $\varphi = \psi_n$.

Lemma 1.15 *Suppose φ is a finite sequence. Then the following are equivalent.*

(1) *φ is a formula.*

(2) *There is a formula-witness for φ.*

Proof. By Lemma 1.15, (2) implies (1). Thus it suffices to prove that (1) implies (2).

Note that if
$$\langle \alpha_0, \ldots, \alpha_n \rangle$$
is a formula-witness then so is:
$$\langle \alpha_0, \ldots, \alpha_n \rangle + \langle (\neg \alpha_n) \rangle.$$
Similarly, if
$$\langle \alpha_0, \ldots, \alpha_n \rangle$$
and
$$\langle \beta_0, \ldots, \beta_m \rangle$$
are each formula-witnesses, then so is
$$\langle \alpha_0, \ldots, \alpha_n \rangle + \langle \beta_0, \ldots, \beta_m \rangle.$$
In particular,
$$\langle \alpha_0, \ldots, \alpha_n \rangle + \langle \beta_0, \ldots, \beta_m \rangle + \langle (\alpha_n \to \beta_m) \rangle$$
is a formula-witness.

Let $\mathcal{L}_0^*$ be the set of all finite sequences for which there is a formula-witness for φ. Then by the above, $\mathcal{L}_0^*$ satisfies the closure requirements of Definition 1.3 and so by the minimality of the set of formulas, $\mathcal{L}_0 \subseteq \mathcal{L}_0^*$. This proves that (1) implies (2). $\square$

Suppose $\vec{s} = \langle s_0, \ldots, s_m \rangle$ is a finite sequence. Then a finite sequence $\vec{t}$ is a *final segment* of $\vec{s}$ if for some $j \leq m$,
$$\vec{t} = \langle s_j, \ldots, s_m \rangle.$$
If $0 < j$ then $\vec{t}$ is a *proper final segment* of $\vec{s}$.

We complete the analysis of subformulas in Lemma 1.18. This lemma answers the natural questions which arise about subformulas and block-subsequences etc.

The proof of Lemma 1.18 requires the following variation of Lemma 1.8. We leave the proof of this lemma to the exercises.

Lemma 1.16 *Suppose that φ is a formula and that σ is a proper final segment of φ. Then σ is not a formula.* □

We also note the following lemma which is easily proved by induction on the length of formulas. This lemma simply shows that every occurrence of the symbol $\rightarrow$ in a formula ψ, is associated with a subformula of ψ of the form $(\theta_1 \rightarrow \theta_2)$. By Lemma 1.8 and Lemma 1.16, this subformula is unique.

Lemma 1.17 *Suppose that ψ is a formula and that $\varphi = s + \langle \rightarrow \rangle + t$. Then there exist formulas θ_1 and θ_2 and finite sequences α and β such that*

$$s = \alpha + \langle (\rangle + \theta_1$$

and such that

$$t = \theta_2 + \langle) \rangle + \beta$$

Putting everything together we obtain the following two lemmas. The first lemma shows that the subformulas of φ are exactly the formulas which occur in **every** formula-witness for φ. The proof of the second lemma is left to the exercises.

Lemma 1.18 *Suppose φ is a formula and that ψ is a formula. Then the following are equivalent.*

(1) *ψ is a subformula of φ.*
(2) *Suppose $\langle \psi_1, \ldots, \psi_n \rangle$ is a formula-witness for φ. Then $\psi = \psi_k$ for some k such that $1 \leq k \leq n$.*

Proof. (2) implies (1) is immediate from the definitions and so it suffices to prove that (1) implies (2).

Suppose ψ is a subformula of φ and let $\langle \psi_1, \ldots, \psi_n \rangle$ is a formula-witness for φ. Let $k \leq n$ be least such that ψ is a subformula of ψ_k. It suffices to show that $\psi_k = \psi$. If ψ_k has length 1, then necessarily $\psi = \psi_k$ and so we trivially can reduce to the following 2 cases.

Case 1: $\psi_k = (\neg \psi_j)$ for some $j < k$.

Since ψ is not a subformula of ψ_j, either ψ is an initial segment of ψ_k, or ψ is a final segment of ψ_k. In either case, by Lemma 1.8 and Lemma 1.16, $\psi = \psi_k$.

Case 2: $\psi_k = (\psi_{j_1} \rightarrow \psi_{j_2})$ for some $j_1, j_2 < k$.

There are three subcases. Note that if γ is a formula which begins with the symbol (then there is a formula α such that $\langle (\rangle + \alpha$ is an initial segment of γ. Similarly if the last symbol of γ is) then there is a formula α such that $\alpha + \langle) \rangle$ is a final segment of γ. Both claims are easily proved by induction on the length of γ. Of course, for any formula γ, the first symbol of γ is (if and only if the last symbol of γ is).

Case 2.1: ψ is a block-subsequence of $\langle (\rangle + \psi_{j_1}$.

Since $j_1 < k$, ψ must be an initial segment of $\langle (\rangle + \psi_{j_1}$. Further ψ must be a proper initial segment of $\langle (\rangle + \psi_{j_1}$ since otherwise ψ_{j_1} is a proper final segment of ψ which contradicts Lemma 1.16.

The first symbol of ψ is (and so there is a formula α such $\langle (\rangle + \alpha$ is an initial segment of ψ. But then α is a proper initial segment of ψ_{j_1} which contradicts Lemma 1.8.

Case 2.2: ψ is a block-subsequence of $\psi_{j_2} + \langle) \rangle$.

Since $j_2 < k$, ψ must be a proper final initial segment of $\psi_{j_2} + \langle) \rangle$ (otherwise ψ_{j_2} a proper initial segment of ψ which is impossible). Therefore the last symbol of ψ is) and so there is a formula α such that α is a proper final segment of ψ_{j_2} which contradicts Lemma 1.16.

Case 2.3: ψ is not a block-subsequence of $\langle (\rangle + \psi_{j_1}$, and ψ is not a block-subsequence of $\psi_{j_2} + \langle) \rangle$.

Therefore, there are finite sequences s, t such that

- $\psi_k = s + \psi + t$,
- s is an initial segment of $\langle (\rangle + \psi_{j_1}$,
- t is final segment of $\psi_{j_2} + \langle) \rangle$.

Then by Lemma 1.16.5, there is a formula $\theta = (\theta_1 \to \theta_2)$ such that θ is a block-subsequence of ψ and such that:

- Either θ_1 is a final segment of ψ_{j_1} or ψ_{j_1} is a final segment of θ_1,
- Either θ_2 is an initial segment of ψ_{j_2} or ψ_{j_2} is an initial segment of θ_2.

Thus by Lemma 1.8 and Lemma 1.16, $\theta_1 = \psi_{j_1}$ and $\theta_2 = \psi_{j_2}$. But this implies that $\psi = (\psi_{j_1} \to \psi_{j_2}) = \psi_k$. □

Lemma 1.19 *Suppose that φ is a formula. Then there exists a formula-witness $\langle \psi_1, \ldots, \psi_n \rangle$ such that:*

(1) $\varphi = \psi_n$,
(2) *For all $i < n$, ψ_i is a subformula of φ.*

Proof. This lemma is easily proved by induction of the length of φ, the details are left to exercises. □

1.1.2 Exercises

(1) For which natural numbers n are there elements of $\mathcal{L}_0$ of length n?
(2) Prove Lemma 1.16.
(3) Consider the set of symbols $*$ and $\#$. Let $\mathcal{L}^*$ be the smallest set L of sequences of these symbols with the following properties.
   (a) The length one sequences $\langle * \rangle$ and $\langle \# \rangle$ belong to L.
   (b) If σ and τ belong to L, then so do $\langle * \rangle + \sigma + \langle \# \rangle$ and $\langle * \rangle + \sigma + \tau + \langle \# \rangle$.
   State Readability for $\mathcal{L}^*$ and determine if it holds.

   Hint: Consider **## and ***###

(4) Consider the set of sequences defined as in Definition 1.3 except that the first part of the second clause is changed to read, "If $\varphi \in L$ is an element of L then $\neg\varphi$ is in L" in which the parentheses are omitted.
   (a) Is this set readable?
   (b) Is this set uniquely readable?
(5) Consider the set of sequences defined as in Definition 1.3 except that the second part of the second clause is changed to read, "If φ_1 and φ_2 are elements of L, then $\varphi_1 \to \varphi_2$ is an element of L" in which the parentheses are omitted.
   (a) Is this set readable?
   (b) Is this set uniquely readable?

   Hint: For (a), show that a proper initial segment of φ is not a "formula" if $\varphi = (\neg\psi)$, by induction on the length of φ.
(6) (Polish Notation) Let $\mathcal{P}_0$ be the smallest set of sequences P such that the following conditions hold.
   (a) For each n, $\langle A_n \rangle \in P$.
   (b) If ψ_1 and ψ_2 belong to P, then so do both:
      - $\neg\psi_1 = \langle\neg\rangle + \psi_1$.
      - $\to\psi_1\psi_2 = \langle\to\rangle + \psi_1 + \psi_2$.

   State and prove the unique readability theorem for $\mathcal{P}_0$.

   Hint: Show that a proper initial segment of an element of $\mathcal{P}_0$ is not in $\mathcal{P}_0$.

   Note that the Polish system of notation does away with parentheses.
(7) Prove Lemma 1.19.

1.2 Truth assignments

We can now describe the semantics for propositional logic.

Definition 1.20 A *truth assignment for* $\mathcal{L}_0$ is a function ν from the set of propositional symbols $\{A_n : n \in \mathbb{N}\}$ into the set $\{T, F\}$. □

Now, $(\neg\psi)$ should have the opposite truth value from that of ψ and the truth value of $(\psi_1 \to \psi_2)$ should reflect whether, if ψ_1 has truth value T, then ψ_2 has truth value T.

Theorem 1.21 *Suppose that ν is a truth assignment for $\mathcal{L}_0$. Then there is a unique function $\overline{\nu}$ defined on $\mathcal{L}_0$ with the following properties.*
(1) *For all n, $\overline{\nu}(\langle A_n \rangle) = \nu(A_n)$.*
(2) *For all $\psi \in \mathcal{L}_0$,*

$$\overline{\nu}((\neg\psi)) = \begin{cases} T, & \text{if } \overline{\nu}(\psi) = F; \\ F, & \text{otherwise.} \end{cases}$$

(3) *For all* ψ_1 *and* ψ_2 *in* $\mathcal{L}_0$*,*

$$\overline{\nu}((\psi_1 \to \psi_2)) = \begin{cases} F, & \textit{if } \overline{\nu}(\psi_1) = T \textit{ and } \overline{\nu}(\psi_2) = F; \\ T, & \textit{otherwise.} \end{cases}$$

Proof. We can define $\overline{\nu}(\psi)$ by induction of the length of ψ.

Base step. For each $n \in \mathbb{N}$, define $\overline{\nu}(\langle A_n \rangle) = \nu(A_n)$.

Induction step. Suppose that $s \geq 1$, that $\overline{\nu}$ is defined on all sequences from $\mathcal{L}_0$ of length less than or equal to s, and that φ is an element of $\mathcal{L}_0$ of length $s + 1$.
If $\varphi = (\neg\psi)$, we define $\overline{\nu}(\varphi)$ as in (2); if $\varphi = (\psi_1 \to \psi_2)$, we define $\overline{\nu}(\varphi)$ as in (3).

By Unique Readability for Formulas, Theorem 1.9, $\overline{\nu}$ is well defined.

We finish by proving $\overline{\nu}$ is unique. Suppose that $\widehat{\nu} : \mathcal{L}_0 \to \{T, F\}$ and satisfies (1), (2), and (3). For the sake of a contradiction, suppose that $\widehat{\nu}$ is not equal to $\overline{\nu}$. Fix φ so that $\widehat{\nu}(\varphi) \neq \overline{\nu}(\varphi)$ and so that there is no $\psi \in \mathcal{L}_0$ such that ψ is strictly shorter than φ and $\widehat{\nu}(\psi) \neq \overline{\nu}(\psi)$.

Since $\widehat{\nu}$ satisfies (1), for every n, $\widehat{\nu}(\langle A_n \rangle) = \nu(A_n)$. By definition,

$$\overline{\nu}^(\langle A_n \rangle) = \nu(A_n).$$

Hence, for every n, $\widehat{\nu}(\langle A_n \rangle) = \overline{\nu}(\langle A_n \rangle)$.

Consequently, the length of φ must be greater than 1. By Readability for Formulas, Lemma 1.7, φ is either a negation $(\neg\psi)$ or an implication $(\psi_1 \to \psi_2)$.

Case 1 $\varphi = (\neg\psi)$ for some formula ψ.

ψ has shorter length than φ and so $\overline{\nu}(\psi) = \widehat{\nu}(\psi)$. But both $\overline{\nu}$ and $\widehat{\nu}$ satisfy (2) and so $\overline{\nu}(\varphi) = \widehat{\nu}(\varphi)$, this is a contradiction.

Case 2 $\varphi = (\psi_1 \to \psi_2)$, for some formulas ψ_1 and ψ_2.

Both ψ_1 and ψ_2 have shorter length than φ. Therefore $\overline{\nu}(\psi_1) = \widehat{\nu}(\psi_1)$ and $\overline{\nu}(\psi_2) = \widehat{\nu}(\psi_2$. But both $\overline{\nu}$ and $\widehat{\nu}$ satisfy (3) and so $\overline{\nu}(\varphi) = \widehat{\nu}(\varphi)$, this is a again contradiction.

Thus in each case, we have a contradiction and so $\overline{\nu} = \widehat{\nu}$, and this proves the theorem. □

Theorem 1.22 *Suppose that* $\varphi \in \mathcal{L}_0$ *and that* ν *and* μ *are truth assignments which agree on the propositional symbols which occur in* φ*. Then* $\overline{\nu}(\varphi) = \overline{\mu}(\varphi)$*.*

Proof. Proceed just as in the uniqueness part of the proof of Theorem 1.9. Show that there cannot be a shortest subformula of φ where $\overline{\nu}$ and $\overline{\mu}$ disagree. □

1.2.1 Satisfiability

Definition 1.23 (1) A truth assignment ν *satisfies* a formula φ if and only if $\overline{\nu}(\varphi) = T$. Similarly, ν *satisfies* a set of formulas Γ if and only if it satisfies all of the elements of Γ.

(2) φ is a *tautology* if and only if every truth assignment satisfies φ.
(3) $\varphi \in \mathcal{L}_0$ or $\Gamma \subset \mathcal{L}_0$ are *satisfiable* if and only if there is a truth assignment which satisfies φ or Γ, respectively.
(4) φ is a *contradiction* if and only if there is no truth assignment which satisfies φ. □

To give an example, consider the formula $(\neg((\neg A_1) \to A_2))$ and a truth assignment ν such that $\nu(A_1) = \nu(A_2) = F$. By Theorem 1.22, the values of ν on A_1 and A_2 determine the value of $\overline{\nu}$ on $(\neg((\neg A_1) \to A_2))$. In Figure 1.1, we show the values of $\overline{\nu}$ on $(\neg((\neg A_1) \to A_2))$ and its subformulas.

A_1	A_2	$(\neg A_1)$	$((\neg A_1) \to A_2)$	$(\neg((\neg A_1) \to A_2))$
F	F	T	F	T

Fig. 1.1 Extending a truth assignment

We can expand the table to systematically examine all possible truth assignments on $(\neg((\neg A_1) \to A_2))$, as in Figure 1.2.

A_1	A_2	$(\neg A_1)$	$((\neg A_1) \to A_2)$	$(\neg((\neg A_1) \to A_2))$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	F
F	F	T	F	T

Fig. 1.2 The truth table for $(\neg((\neg A_1) \to A_2))$

Truth tables, such as the one in Figure 1.2, provide a systematic method to examine all the possible truth assignments for a given formula. Given a formula φ, we generate a truth table for φ as follows.

(1) The top row of the table consists of a list $\psi_1, \psi_2, \ldots, \psi_n = \varphi$ consisting of the subformulas of φ, ordered from left to right as follows.
   (a) The subformulas of φ of the form $\langle A_m \rangle$ appear in the list without repetition before any of the other subformulas of φ.
   (b) For each $i \leq n$ all of the proper subformulas of ψ_i appear in the list $\psi_1, \psi_2, \ldots, \psi_{i-1}$.
   (c) The last element of the list is φ.

   Thus $\langle \varphi_1, \ldots, \varphi_n \rangle$ is any formula-witness for φ such that for all $1 \leq i \leq j \leq n$, φ_i is a subformula of φ, $\text{length}(\varphi_i) \leq \text{length}(\varphi_j)$, and such that if both $\text{length}(\varphi_j) = 1$ and $i < j$, then $\varphi_i \neq \varphi_j$.
(2) Letting k be the number of subformulas of φ of the form $\langle A_m \rangle$, we consider all of the 2^k possible truth assignments for their propositional symbols. We use a row in the table for each such truth assignment ν, and we fill in the cell below $\langle A_m \rangle$ in that row with the value of ν at A_m.

(3) Finally, we work our way across each row and fill in the values of $\overline{\nu}$ at ψ_i as determined by the values already filled in for its subformulas.

A_1	A_2	$(A_1 \to A_2)$	$(\neg A_1)$	$(\neg A_2)$	$((\neg A_2) \to (\neg A_1))$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

A_1	A_2	$((A_1 \to A_2) \to ((\neg A_2) \to (\neg A_1)))$
T	T	T
T	F	T
F	T	T
F	F	T

Fig. 1.3 The truth table for $((A_1 \to A_2) \to ((\neg A_2) \to (\neg A_1)))$

We give another example in Figure 1.3. This time we have chosen the tautology expressing the principle that if A_1 implies A_2, then the contrapositive implication from $(\neg A_2)$ to $(\neg A_1)$ also holds.

Theorem 1.24 *There are algorithms to determine whether a propositional formula φ is a tautology, satisfiable, or a contradiction.*

Proof. Starting with a formula φ, we can systematically generate its truth table. Then φ is a tautology if and only if every entry in the last column of its truth table is equal to T. It is satisfiable if and only if there is an entry in the last column of its truth table which is equal to T. It is a contradiction if and only if every entry in the last column of its truth table is equal to F. $\square$

Remark 1.25 Roughly speaking, if φ has n many symbols, then the analysis of φ by the method of truth tables involves 2^n many steps. A question which has received a considerable amount of attention is whether there is a more efficient method which when given φ determines whether φ is satisfiable. For more information on this problem, known as the $P = NP$ problem, and even a cash prize, see the following web site.

http://claymath.org/millennium-problems/p-vs-np-problem $\square$

1.2.2 Truth functions

Definition 1.26 An n-place *truth function* is a function whose domain is the set of sequences of T's and F's of length n, written $\{T, F\}^n$ and whose range is contained in $\{T, F\}$. $\square$

If φ is a formula in $\mathcal{L}_0$ and the propositional symbols which occur in φ are contained in the set $\{A_0, \ldots, A_{n-1}\}$, then we can define the truth function f_φ derived from φ. Given $\sigma \in \{T, F\}^n$, we let ν be the truth assignment on $\{A_0, \ldots, A_{n-1}\}$ such that $\nu(A_{i-1})$ is equal to the ith element of σ, and we define $f_\varphi(\sigma)$ to be $\overline{\nu}(\varphi)$.

In the next theorem, we show that $\mathcal{L}_0$ is as expressive as is possible. By this, we mean that every truth function is represented by a formula in $\mathcal{L}_0$.

Theorem 1.27 *Suppose that $f : \{T, F\}^n \to \{T, F\}$ is a truth function. Then there is a formula φ such that $f_\varphi = f$.*

Proof. We build up to the formula φ by a sequence of smaller steps.

For $\sigma \in \{T, F\}^n$, define $\theta_{\sigma,i}$ so that

$$\theta_{\sigma,i} = \begin{cases} A_{i-1}, & \text{if } \sigma(i) = T; \\ (\neg A_{i-1}), & \text{if } \sigma(i) = F. \end{cases}$$

Given two formulas ψ_1 and ψ_2, we define the conjunction of ψ_1 and ψ_2 to be the formula $(\neg(\psi_1 \to (\neg\psi_2)))$. As is seen in Figure 1.4, a truth assignment satisfies the conjunction of ψ_1 and ψ_2 if and only if it satisfies both ψ_1 and ψ_2.

ψ_1	ψ_2	$(\neg\psi_2)$	$(\psi_1 \to (\neg\psi_2))$	$(\neg(\psi_1 \to (\neg\psi_2)))$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

Fig. 1.4 The conjunction of ψ_1 and ψ_2.

Given more than two formulas $\psi_1, \ldots, \psi_n$, we use recursion and define their conjunction to be the conjunction of ψ_1 with the conjunction of $\psi_2, \ldots, \psi_n$. For example, the conjunction of ψ_1, ψ_2, and ψ_3 is the formula

$$(\neg(\psi_1 \to (\neg(\neg(\psi_2 \to (\neg\psi_3)))))).$$

By induction, if ν is a truth assignment, then $\overline{\nu}$ maps the conjunction of $\psi_1, \ldots, \psi_n$ to T if and only if $\overline{\nu}$ maps each of $\psi_1, \ldots, \psi_n$ to T.

For $\sigma \in \{T, F\}^n$, we let ψ_σ be the conjunction of the formulas $\theta_{\sigma,i}$ for i less than or equal to n. The only truth assignments that satisfy ψ_σ are those which assign $\sigma(i)$ to A_{i-1}.

Given two formulas ψ_1 and ψ_2, we define the disjunction of ψ_1 and ψ_2 to be the formula $((\neg\psi_1) \to \psi_2)$. As is seen in Figure 1.2.2, a truth assignment satisfies the conjunction of ψ_1 and ψ_2 if and only if it satisfies at least one of ψ_1 or ψ_2. As above, when n is greater than two, we define the disjunction of $\psi_1, \ldots, \psi_n$ to be the disjunction of ψ_1 with the disjunction of $\psi_2, \ldots, \psi_n$. By another induction, if ν is a truth assignment, then $\overline{\nu}$ maps the disjunction of $\psi_1, \ldots, \psi_n$ to T if and only if it maps at least one of $\psi_1, \ldots, \psi_n$ to T.

ψ_1	ψ_2	$(\neg\psi_1)$	$((\neg\psi_1) \to \psi_2)$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

Fig. 1.5 The disjunction of ψ_1 and ψ_2.

Now we let φ_f be the disjunction of the set of formulas ψ_σ for which $f(\sigma) = T$. By construction, if ν is a truth assignment that satisfies φ_f, then there is a σ such that $f(\sigma) = T$ and for all i less than or equal to n, $\nu(A_{i-1})$ is equal to the ith element of σ. Consequently, f is equal to f_{φ_f}, as required. □

Remark 1.28 It is not unusual to include symbols $\wedge$ for conjunction, $\vee$ for disjunction, and $\leftrightarrow$ for "if and only if". By Theorem 1.27, these and all other logical connectives can be expressed in the language with only $\neg$ and $\to$.

Of course, the fewer symbols there are in the language, the fewer the number of cases there are in proofs by induction, so we decided in favor a small number of logical symbols. Occasionally, we pay a price for that decision: for example, with the lengths of the formulas that appeared in the proof of Theorem 1.27. □

Definition 1.29 *Disjunctive normal form* is a formula that consists solely of disjunctions of conjunctions of atomic formulas and their negations, or more informally, an OR statement of AND statements. By the proof of Theorem 1.27, every truth function can be realized by a propositional formula in disjunctive normal form. □

Remark 1.30 In some applications, it important to find the best possible formula φ for a given truth function f. Best possible could mean having the shortest length or having the fewest logical connectives of a certain type. When n is large, it is computationally prohibitive to verify for various candidate formulas φ that it represents a given truth function f. Thus finding the optimal φ for a specified f remains an interesting problem. □

Remark 1.31 The next section concerns the notion of proof. However, we already have enough definitions and concepts to prove an interesting theorem, the property of Craig interpolation. The following lemma isolates the key point.

There are a number of different approaches to proving Theorem 1.32. A second approach which is based on substitution is outlined in Exercise (6) on page 20. □

Lemma 1.30.5 *Suppose that $(\varphi \to \psi)$ is a tautology and that φ and ψ share at least one propositional symbol. Suppose that ν, μ are truth assignments which agree on the propositional symbols occurring in both φ and ψ, and that $\bar{\nu}(\varphi) = T$. Then $\bar{\mu}(\psi) = T$*

Theorem 1.32 (Craig Interpolation Theorem for Propositional Logic) *Suppose that $(\varphi \to \psi)$ is a tautology and that φ and ψ share at least one propositional symbol. Then there exists a θ, called an interpolant, that contains only propositional symbols occur in both φ and in ψ, and both $(\varphi \to \theta)$ and $(\theta \to \psi)$ are tautologies.*

Proof. We first suppose that φ is a contradiction. Let A_i be a propositional symbol which occurs in both φ and ψ. Let $\theta = (A_i \to (\neg A_i))$. Thus θ is a contradiction as well. Since φ is a contradiction, $(\varphi \to \theta)$ is a tautology, and $(\theta \to \alpha)$ is a tautology for any propositional formula α. In particular, $(\theta \to \psi)$ is a tautology. This proves the theorem in the case that φ is a contradiction.

We now suppose that φ is not a contradiction and let $\mathbb{A} = \{A_{n_0}, A_{n_2}, ..., A_{n_k}\}$ be the set of propositional letters that appear in both formulas. Let $\nu_0, \nu_2, ..., \nu_n$ be truth assignments that satisfy φ such that any other truth assignment that satisfies φ agrees with a ν_i on $\mathbb{A}$.

We construct the interpolant θ as follows. First, for each $i \leq k$, define

$$\alpha_{n_i} = \begin{cases} A_{n_i} & \text{if } \nu_j(A_{n_i}) = T \\ (\neg A_{n_i}) & \text{if } \nu_j(A_{n_i}) = F \end{cases}$$

Next, for each $j \leq n$ define

$$\theta_j = \bigwedge_{i \leq k} \alpha_{n_i}$$

Finally, define

$$\theta = \bigvee_{j \leq n} \theta_j$$

That is, θ is a disjunction of conjunctions of propositional letters in A or their negations. θ is satisfied if and only if at least one of the the θ_j's is satisfied, and a θ_j is satisfied if and only if all its α_{n_i} are satisfied. By definition, for each j, $\overline{\nu}_j(\varphi) = T$, and θ_j was constructed to be satisfied by ν_j. Therefore, for an arbitrary truth assignment ν, if $\overline{\nu}$ satisfies φ, then ν agrees with some ν_j on $\mathbb{A}$ and so $\overline{\nu}$ satisfies θ. In other words, $(\varphi \to \theta)$ is a tautology.

Now suppose towards contradiction that $(\theta \to \psi)$ is not a tautology. Then there is some truth assignment μ such that $\overline{\mu}(\theta) = T$ but $\overline{\mu}(\psi) = F$. By choice of $\nu_1, ..., \nu_n$, since $\overline{\mu}(\theta) = T$, there is a j such that μ and ν_j agree on $\mathbb{A}$ and $\overline{\nu}_j(\varphi) = T$. The truth assignment μ can then be transformed into a truth assignment μ^*, defined as:

$$\mu^* = \begin{cases} \mu(A_{n_i}) & \text{for } A_{n_i} \text{ in } \psi \text{ but not in } \varphi \\ \nu_j(A_{n_i}) & \text{otherwise} \end{cases}$$

Observe that $\overline{\mu^*}(\psi) = \overline{\mu}(\psi) = F$ since they assign the same truth values to the symbols which appear in ψ. Similarly, $\overline{\mu^*}(\varphi) = T$ since ν_j and μ^* agree on the truth assignment for the propositional symbols which appear in φ but not in ψ. However, the existence of the truth assignment, μ^*, contradicts the fact that $(\varphi \to \psi)$ is a tautology, thus there is no truth assignment μ, and the result follows. $\square$

1.2.3 Exercises

(1) Let $\square$ (this is "NOR") be the propositional connective with the truth table in Figure 1.6. Suppose that $f : \{T, F\}^n \to \{T, F\}$ is a truth function. Show

A	B	$A \square B$
T	T	F
T	F	F
F	T	F
F	F	T

Fig. 1.6 The truth table for $\square$

that there is a formula φ whose only propositional connective is $\square$ such that $f_\varphi = f$.

(2) Using the original language of $\neg$ and $\to$, exhibit a formula φ that contains propositional letters A_0, A_1, and A_2 such that $\overline{\nu}(\varphi)$ is the truth value ν assigns to the majority of A_0, A_1, and A_2.

(3) Given two different truth assignments ν_1 and ν_2, construct an infinite set Γ such that Γ is satisfied by *only* ν_1 and ν_2.

(4) Give an example of a propositional formula φ (in the language $\mathcal{L}_0$ with only $\neg$ and $\to$) such that for every truth assignment $\nu, \overline{\nu}(\varphi) = T$ if and only if either $\nu(A_1) = T$ or both $\nu(A_2) = \nu(A_3) = F$

(5) Given an enumeration of different truth assignments $\{\nu_i : i \in \mathbb{N}\}$ is there a set Γ such that Γ is satisfied by *only* $\{\nu_i : i \in \mathbb{N}\}$?

    Hint: Consider the case where for each number i, $\nu_i(A_k) = T$ if and only if $k \geq i$.

(6) Find a formula φ such that the following hold.
   (a) The propositional symbol A_0 occurs in φ, and no other propositional symbol occurs in φ.
   (b) φ is satisfiable.
   (c) If ψ does not contain any propositional symbol A_i with $i > 0$ then either $(\varphi \to \psi)$ is a tautology or $(\varphi \to (\neg\psi))$ is a tautology.

(7) Does there exist a $\varphi \in \mathcal{L}_0$ such that the following conditions hold:
   (a) φ is satisfiable.
   (b) If ψ contains only propositional letters that appear in φ, then if φ does not imply φ, then ψ implies φ.

(8) Show that if $(\varphi \to \psi)$ is a tautology and ψ is not a tautology, then either φ is a contradiction or φ and ψ have at least one propositional symbol in common.

1.2.4 Substitution

Outside of negating a formula or linking two by logical implication, another action on propositional formulas is substitution. For example, suppose we have the formula $\varphi = (A_0 \rightarrow A_1)$, and we want to replace the instance of A_1 with an instance of A_2. We denote this action by $\varphi(A_1; A_2)$. We can generalize substitution to replacing propositional letters with any formula. We state this formally in the following definition.

Definition 1.33 Suppose that φ and $\psi_1, ..., \psi_k$ are formulas, and $A_{n_1}, ..., A_{n_k}$ are distinct propositional letters. Then $\varphi(A_{n_1}, ..., A_{n_k}; \psi_1, ..., \psi_k)$ denotes the formula obtained by simultaneously, for each i, substituting the formula ψ_i for each occurrence of A_{n_i} in φ. For the sake of (human) readability, we will often just write $\varphi(\bar{A}; \vec{\psi})$ and in doing so the convention is that $|\bar{A}| = |\vec{\psi}|$. Here we note that since the propositional symbols A_i are naturally ordered by $A_{i_1} < A_{i_2}$ if $i_1 < i_2$, each finite set $\bar{A}$ of propositional symbols corresponds uniquely to a finite sequence $\vec{A}$ of propositional symbols. □

Note that our notion of substitution does not allow one to distinguish different occurrences of A_i; more precisely in defining $\varphi(A_i; A_j)$, we cannot specify particular occurrences of A_i to replace and also specify occurrences of A_i to preserve. The following lemma is left to the exercises.

Lemma 1.34 *Suppose φ is a formula and every propositional symbol occurring in φ is included in $\{A_0, \ldots, A_n\}$. Let $\langle \psi_0, \ldots, \psi_n \rangle$ be a sequence of formulas and let $\varphi^* = \varphi(A_0, ..., A_n; \psi_0, ..., \psi_n)$. Suppose ν is a truth assignment and let ν^* be the truth assignment where*

$$\nu^*(A_m) = \begin{cases} \overline{\nu}(\psi_i) & \text{if } m = i \text{ and } i \leq n \\ \nu(A_m) & \text{otherwise} \end{cases}$$

Then $\overline{\nu^*}(\varphi) = \overline{\nu}(\varphi^*)$. □

1.2.5 Exercises

(1) Show that substitution of propositional symbols with formulas yields a formula.

(2) Prove Lemma 1.34.

Hint: Use induction on the length of φ.

(3) Show that substitution on the propositional symbols of a tautology produces another tautology. Deduce that tautologies and contradictions are invariant under substitution of propositional symbols.

Hint: Use Lemma 1.34.

(4) Suppose φ is satisfiable. Produce a tautology by substitution on the propositional symbols.

Hint: Use Lemma 1.34.

(5) Show that if $(\varphi \rightarrow \psi)$ is a tautology and ψ is not a tautology, then either φ is a contradiction or φ and ψ have at least one propositional symbol in common.

Note: This is the generalization of the Craig Interpretation Theorem to the case where φ and ψ have no propositional symbols in common.

(6) Prove the Craig interpolation theorem by induction on the number of propositional symbols which occur in φ and which do not occur in ψ.

Hint: The base step is when every propositional symbol which occurs in φ also occurs in ψ.

For the inductive step, fix a propositional symbol A_i which occurs in φ and not in ψ and fix a propositional symbol A_k such that A_k occurs in both φ and ψ. Consider two substitutions on A_i in φ: one defined using the tautology $(A_k \rightarrow A_k)$, and the other defined using the negation $(\neg(A_k \rightarrow A_k))$ of that tautology.

1.3 A proof system for $\mathcal{L}_0$

Suppose that Γ is a subset of $\mathcal{L}_0$ so that Γ is a set of propositional formulas. We shall define a formal notion of proof. Intuitively a proof from Γ will be a finite sequence, $\langle \varphi_1, \ldots, \varphi_n \rangle$, of propositional formulas which satisfies certain conditions. In order to make the definition precise we need to first define the set of *Logical Axioms*.

Definition 1.35 Suppose that φ_1, φ_2 and φ_3 are propositional formulas. Then each of the following propositional formulas is a logical axiom:
(Group I axioms)

(1) $((\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3)) \rightarrow ((\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3)))$
(2) $(\varphi_1 \rightarrow \varphi_1)$
(3) $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_1))$

(Group II axioms)

(1) $(\varphi_1 \rightarrow ((\neg\varphi_1) \rightarrow \varphi_2))$

(Group III axioms)

(1) $(((\neg\varphi_1) \rightarrow \varphi_1) \rightarrow \varphi_1)$

(Group IV axioms)

(1) $((\neg\varphi_1) \rightarrow (\varphi_1 \rightarrow \varphi_2))$
(2) $(\varphi_1 \rightarrow ((\neg\varphi_2) \rightarrow (\neg(\varphi_1 \rightarrow \varphi_2))))$ □

It is easily verified that each logical axiom is a tautology. Notice also that the set of all logical axioms is generated from just the 7 logical axioms (where $\varphi_1 = A_1$ and $\varphi_2 = A_2$) by substitutions. Thus unlike the case of tautologies, it is quite feasible to check if a formula is a logical axiom.

While we could have taken potentially the whole set of tautologies to constitute our axioms, we will see these are robust enough to give our proof system the property of “completeness”.

Definition 1.36 Suppose that $\Gamma \subseteq \mathcal{L}_0$.

(1) Suppose that

$$s = \langle \varphi_1, \ldots, \varphi_n \rangle$$

is a finite sequence of propositional formulas. The finite sequence s is a Γ-proof if for each $i \leq n$ at least one of

(a) $\varphi_i \in \Gamma$; or

(b) φ_i is a logical axiom; or

(c) there exist $j_1 < i$ and $j_2 < i$ such that

$$\varphi_{j_2} = (\varphi_{j_1} \to \varphi_i).$$

(2) $\Gamma \vdash \varphi$ if and only if there exists a finite sequence

$$s = \langle \varphi_1, \ldots, \varphi_n \rangle$$

such that s is a Γ-proof and such that $\varphi_n = \varphi$. □

Notice that if $s = \langle \varphi_1, \ldots, \varphi_n \rangle$ is a Γ-proof and if $t = \langle \psi_1, \ldots, \psi_m \rangle$ is a Γ-proof then so is $s + t = \langle \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \rangle$.

We shall prove a sequence of simple lemmas about the proof system. For each lemma we shall note which logical axioms are actually used.

The first lemma, which concerns inference, requires no logical axioms whatsoever.

Lemma 1.37 (Inference) *Suppose that $\Gamma \subseteq \mathcal{L}_0$, φ is a formula and that ψ is a formula. Suppose that $\Gamma \vdash \psi$ and that $\Gamma \vdash (\psi \to \varphi)$.*

Then $\Gamma \vdash \varphi$.

Proof. Let $\langle \varphi_1, \ldots, \varphi_n \rangle$ be a Γ-proof of ψ, thus $\varphi_n = \psi$. Let $\langle \psi_1, \ldots, \psi_m \rangle$ be a Γ-proof of $(\psi \to \varphi)$. Then $\langle \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m, \varphi \rangle$ is a Γ-proof and $\Gamma \vdash \varphi$. □

The second lemma is the Soundness Lemma, which ensures proofs indeed yield valid logical consequences. This also is independent of the choice of logical axioms, provided that every logical axiom is a tautology.

Lemma 1.38 (Soundness) *Suppose that $\Gamma \subseteq \mathcal{L}_0$, φ is a formula and that $\Gamma \vdash \varphi$. Suppose that*

$$\nu : \{A_1, \ldots, A_n, \ldots\} \to \{T, F\}$$

is a truth assignment such that $\overline{\nu}(\psi) = T$ for all $\psi \in \Gamma$.

Then $\overline{\nu}(\varphi) = T$.

Proof. Let $\langle \varphi_1, \ldots, \varphi_n \rangle$ be a Γ-proof of ψ. One proves by induction on $i \leq n$ that $\overline{\nu}(\varphi_i) = T$. We leave the details as an exercise. $\square$

The next lemma is the Deduction Lemma. This lemma requires the logical axioms from Group I.

Lemma 1.39 (Deduction) *Suppose that* $\Gamma \subseteq \mathcal{L}_0$*,* φ *is a formula,* ψ *is a formula and*

$$\Gamma \cup \{\varphi\} \vdash \psi.$$

Then $\Gamma \vdash (\varphi \to \psi)$.

Proof. Let

$$\langle \psi_1, \ldots, \psi_n \rangle$$

be a $(\Gamma \cup \{\varphi\})$-proof of ψ. We prove by induction on $i \leq n$ that

$$\Gamma \vdash (\varphi \to \psi_i).$$

First we consider the case $i = 1$. Either $\psi_1 \in \Gamma \cup \{\varphi\}$ or ψ_1 is a logical axiom (possibly both). So there are three subcases of this case.

Subcase 1.1: $\psi_1 \in \Gamma$. So we must show that $\Gamma \vdash (\varphi \to \psi_1)$. However

$$\Gamma \vdash (\psi_1 \to (\varphi \to \psi_1))$$

since $(\psi_1 \to (\varphi \to \psi_1))$ is a logical axiom. Further

$$\Gamma \vdash \psi_1$$

since $\psi_1 \in \Gamma$. Therefore by the Inference Lemma 1.37, $\Gamma \vdash (\varphi \to \psi_1)$.

Subcase 1.2: $\psi_1 = \varphi$. Note that $(\varphi \to \varphi)$ is a logical axiom and so

$$\Gamma \vdash (\varphi \to \varphi).$$

Subcase 1.3: ψ_1 is a logical axiom. This is just like subcase 1.1; $(\psi_1 \to (\varphi \to \psi_1))$ is a logical axiom and so

$$\Gamma \vdash (\psi_1 \to (\varphi \to \psi_1)).$$

Since ψ_1 is a logical axiom, $\Gamma \vdash \psi_1$. Therefore by the inference Lemma, $\Gamma \vdash (\varphi \to \psi_1)$.

We now suppose that $k \leq n$ and assume as an induction hypothesis that for all $i < k$,

$$\Gamma \vdash (\varphi \to \psi_i).$$

There are two subcases.

Subcase 2.1: $\psi_k \in \Gamma \cup \{\varphi\}$ or ψ_k is a logical axiom. But then exactly as in the case of ψ_1, $\Gamma \vdash (\varphi \to \varphi_k)$.

Subcase 2.2: There exist $j_1 < k$ and $j_2 < k$ such that $\psi_{j_2} = (\psi_{j_1} \to \psi_k)$.

By the induction hypothesis; $\Gamma \vdash (\varphi \to \psi_{j_1})$ and $\Gamma \vdash (\varphi \to \psi_{j_2})$. Now we use the logical axiom

$$((\varphi \to (\psi_{j_1} \to \psi_k)) \to ((\varphi \to \psi_{j_1}) \to (\varphi \to \psi_k))).$$

By the induction hypothesis,

$$\Gamma \vdash (\varphi \to (\psi_{j_1} \to \psi_k)),$$

and so by the Inference Lemma,

$$\Gamma \vdash ((\varphi \to \psi_{j_1}) \to (\varphi \to \psi_k)).$$

Again by the induction hypothesis,

$$\Gamma \vdash (\varphi \to \psi_{j_1}),$$

and so by the Inference Lemma one last time,

$$\Gamma \vdash (\varphi \to \psi_k).$$

This completes the induction and so $\Gamma \vdash (\varphi \to \psi)$. Finally we note that only Group I logical axioms were used. $\square$

Definition 1.40 Suppose that $\Gamma \subseteq \mathcal{L}_0$.

(1) Γ is *inconsistent* if for some formula φ, $\Gamma \vdash \varphi$ and $\Gamma \vdash (\neg\varphi)$.

(2) Γ is *consistent* if Γ is not inconsistent. $\square$

If Γ is an inconsistent set of formulas then $\Gamma \vdash \psi$ for *every* formula ψ. This is the content of the next lemma the proof of which appeals to the Deduction Lemma and logical axioms in Group II. Therefore only logical axioms from Groups I and II are needed.

Lemma 1.41 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is inconsistent. Suppose that ψ is a formula. Then $\Gamma \vdash \psi$.*

Proof. Since Γ is inconsistent there exists a formula φ such that

$$\Gamma \vdash \varphi$$

and $\Gamma \vdash (\neg\varphi)$.

But

$$\Gamma \vdash (\varphi \to ((\neg\varphi) \to \psi))$$

since $(\varphi \to ((\neg\varphi) \to \psi))$ is a logical axiom. Therefore by the Inference Lemma,

$$\Gamma \vdash ((\neg\varphi) \to \psi)$$

and by the Inference Lemma again,

$$\Gamma \vdash \psi.$$

This completes the proof. □

Definition 1.42 Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent. Then Γ is *maximally consistent* if and only if for each formula ψ if $\Gamma \cup \{\psi\}$ is consistent then $\psi \in \Gamma$. □

Lemma 1.43 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent. Suppose that φ is a formula.*

Then at least one of $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{(\neg\varphi)\}$ is consistent, possibly both.

Proof. Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. Therefore, for each formula ψ,

$$\Gamma \cup \{(\neg\varphi)\} \vdash \psi$$

and in particular, $\Gamma \cup \{(\neg\varphi)\} \vdash \varphi$.

Thus by the Deduction Lemma, $\Gamma \vdash ((\neg\varphi) \to \varphi)$. But $(((\neg\varphi) \to \varphi) \to \varphi)$ is a logical axiom (Group III), and so by the Inference Lemma, $\Gamma \vdash \varphi$.

Now assume toward a contradiction that $\Gamma \cup \{\varphi\}$ is inconsistent. By Lemma 1.41, for each formula ψ,

$$\Gamma \cup \{\varphi\} \vdash \psi.$$

By the Deduction Lemma, for each formula ψ,

$$\Gamma \vdash (\varphi \to \psi).$$

But $\Gamma \vdash \varphi$ and so by the Inference Lemma, for each formula ψ, $\Gamma \vdash \psi$. Thus Γ is inconsistent, which is a contradiction. Therefore $\Gamma \cup \{\varphi\}$ is consistent.

So we have proved, assuming the consistency of Γ, that if $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent then $\Gamma \cup \{\varphi\}$ is consistent. □

Corollary 1.44 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is maximally consistent. Suppose that φ is a formula.*

Then:

(1) *Either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$;*
(2) *If $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$.*

Proof. We first prove (1). By Lemma 1.43, either $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{(\neg\varphi)\}$ is consistent. Therefore since Γ is maximally consistent (1) must hold.

We finish by proving (2). We are given that $\Gamma \vdash \varphi$. By (1), if $\varphi \notin \Gamma$ then $(\neg\varphi) \in \Gamma$ which implies that $\Gamma \vdash (\neg\varphi)$. But $\Gamma \vdash \varphi$ and so this contradicts the consistency of Γ. □

We now use the logical axioms in Group IV.

Lemma 1.45 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is maximally consistent. Suppose that φ_1 and φ_2 are formulas.*

Then $(\varphi_1 \to \varphi_2) \in \Gamma$ if and only if at least one of $\varphi_1 \notin \Gamma$ or $\varphi_2 \in \Gamma$.

Proof. We first suppose that $\varphi_1 \notin \Gamma$. We must show that $(\varphi_1 \to \varphi_2) \in \Gamma$.

Since $\varphi_1 \notin \Gamma$, by Corollary 1.44, $(\neg\varphi_1) \in \Gamma$.

Thus $\Gamma \vdash (\neg\varphi_1)$. But

$$\Gamma \vdash ((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$$

since $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$ is a logical axiom, and so by the Inference Lemma,

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Therefore by Corollary 1.44, $(\varphi_1 \to \varphi_2) \in \Gamma$.

Next we suppose that $\varphi_2 \in \Gamma$. Now

$$(\varphi_2 \to (\varphi_1 \to \varphi_2))$$

is a logical axiom and so $\Gamma \vdash (\varphi_2 \to (\varphi_1 \to \varphi_2))$. By the Inference Lemma, since $\varphi_2 \in \Gamma$,

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Therefore, again by Corollary 1.44, $(\varphi_1 \to \varphi_2) \in \Gamma$.

To finish, we suppose that $\varphi_1 \in \Gamma$ and $\varphi_2 \notin \Gamma$. Now we must show that $(\varphi_1 \to \varphi_2) \notin \Gamma$.

Since $\varphi_2 \notin \Gamma$, by Corollary 1.44, $(\neg\varphi_2) \in \Gamma$.

Thus $\Gamma \vdash \varphi_1$ and $\Gamma \vdash (\neg\varphi_2)$. But

$$\Gamma \vdash (\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2)))),$$

since $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$ is a logical axiom. Therefore by the Inference Lemma,

$$\Gamma \vdash ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))),$$

and by the Inference Lemma once again,

$$\Gamma \vdash (\neg(\varphi_1 \to \varphi_2)).$$

Finally by Corollary 1.44, $(\neg(\varphi_1 \to \varphi_2)) \in \Gamma$ and so $(\varphi_1 \to \varphi_2) \notin \Gamma$ as required.

This completes the proof of the lemma. □

Our goal is to show that if Γ is consistent then Γ is satisfiable. We first consider the special case that Γ is maximally consistent. This case will turn out to be an easy case for Γ uniquely specifies the truth assignment which witnesses that Γ is satisfiable.

Lemma 1.46 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is maximally consistent. Then Γ is satisfiable.*

Proof. Define a truth assignment ν as follows. For each $i \in \mathbb{N}$, let

$$\nu(A_i) = \begin{cases} T, & \text{if } \langle A_i \rangle \in \Gamma; \\ F, & \text{if } \langle A_i \rangle \notin \Gamma. \end{cases}$$

We claim that for each formula φ, $\overline{\nu}(\varphi) = T$ if $\varphi \in \Gamma$ and $\overline{\nu}(\varphi) = F$ if $\varphi \notin \Gamma$. We organize our proof of this claim by induction on the length of φ.

The case that φ has length 1 is immediate.

Suppose that φ has length $n > 1$ and that as induction hypothesis, for all formulas ψ if ψ has length less than n then $\overline{\nu}(\psi) = T$ if $\psi \in \Gamma$ and $\overline{\nu}(\psi) = F$ if $\psi \notin \Gamma$.

There are two cases.

Case 1: $\varphi = (\neg\psi)$. Since Γ is maximally consistent, $\varphi \in \Gamma$ if and only if $\psi \notin \Gamma$. But $\overline{\nu}(\varphi) = T$ if and only if $\overline{\nu}(\psi) = F$. By the induction hypothesis $\overline{\nu}(\psi) = T$ if and only if $\psi \in \Gamma$.

Thus if $\varphi \in \Gamma$ then $\overline{\nu}(\varphi) = T$ and $\overline{\nu}(\varphi) = F$ if $\varphi \notin \Gamma$.

Case 2: $\varphi = (\psi_1 \to \psi_2)$. Since Γ is maximally consistent, $\varphi \in \Gamma$ if and only if at least one of $\psi_1 \notin \Gamma$ or $\psi_2 \in \Gamma$. This is by Lemma 1.45.

By the definition of $\overline{\nu}$, $\overline{\nu}(\varphi) = T$ if and only if either $\overline{\nu}(\psi_1) = F$ or $\overline{\nu}(\psi_2) = T$. Therefore by the induction hypothesis, $\overline{\nu}(\varphi) = T$ if and only if either $\psi_1 \notin \Gamma$ of $\psi_2 \in \Gamma$.

Thus, $\overline{\nu}(\varphi) = T$ if and only if $\varphi \in \Gamma$.

This completes the induction. □

Theorem 1.47 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent. Then there exists a set $\Gamma^* \subset \mathcal{L}_0$ such that:*

(1) $\Gamma \subseteq \Gamma^*$,

(2) Γ^* *is maximally consistent.*

Proof. Let $(\varphi_i : i \in \mathbb{N})$ be an enumeration of all of the formulas of $\mathcal{L}_0$. For example, we could enumerate the finitely many length 1 formulas which use only the propositional symbol A_0; then, we could enumerate the finitely many formulas of length less than or equal to 2 which use no propositional symbols other than A_0 and A_1; and in subsequent steps, enumerate the finitely many formulas of length less than or equal to n which use no propositional symbols other than $A_0, \ldots, A_n$.

We construct a sequence of sets of formulas $(\Gamma_n : m \in \mathbb{N})$ by induction on n. To begin, let Γ_0 equal Γ. Given Γ_n, let Γ_{n+1} be defined as follows.

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

We check by induction that each Γ_n is consistent. Clearly, Γ_0 is consistent, since we are given that Γ is consistent. Assuming that Γ_n is consistent, we can apply Lemma 1.43 to conclude that at least one of $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is also consistent. But then Γ_{n+1} is also consistent.

Now, define Γ^* so that

$$\Gamma^* = \cup_{n\in\mathbb{N}}\, \Gamma_n.$$

Assume toward a contradiction that Γ^* is not consistent and fix $\varphi \in \Gamma^*$. Thus by Lemma 1.41,

$$\Gamma^* \vdash (\neg\varphi).$$

Let $\langle \psi_1, \ldots, \psi_m \rangle$ be a Γ^*-proof of $(\neg\varphi)$ and let

$$\Sigma = \{\psi_i \mid i \leq m \text{ and } \psi_i \in \Gamma^*\} \cup \{\varphi\}.$$

Thus Σ is a finite subset of Γ^*, $\Sigma \vdash \varphi$, and $\Sigma \vdash (\neg\varphi)$.

Therefore Σ is also inconsistent. But $\Gamma_n \subseteq \Gamma_{n+1}$ for all $n \in \mathbb{N}$ and so since Σ is a finite subset of Γ^*, necessarily

$$\Sigma \subseteq \Gamma_n$$

for all sufficiently large n. But thus implies that Γ_n is inconsistent for all sufficiently large $n \in \mathbb{N}$, which is a contradiction. Thus Γ^* is consistent.

Finally we prove that Γ^* is maximally consistent. For every formula φ, there is an n such that φ is equal to φ_n. By the definition of Γ_{n+1}, either $\varphi_n \in \Gamma_{n+1}$ or $(\neg\varphi_n) \in \Gamma_{n+1}$. Since $\varphi = \varphi_n$ and $\Gamma_{n+1} \subseteq \Gamma^*$, either $\varphi \in \Gamma^*$ or $(\neg\varphi) \in \Gamma^*$, as required for maximality. □

Theorem 1.48 (Completeness for $\mathcal{L}_0$; Version I) *Suppose that* $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is consistent.*

Then Γ *is satisfiable.*

Proof. By Theorem 1.47 extend Γ to a maximally consistent set, and then apply Lemma 1.46. □

1.3.1 Exercises

(1) Determine for each of the following formulas if that formula is a tautology.
   (a) $(((A_1 \to A_1) \to A_2) \to A_2)$
   (b) $((((A_1 \to A_2) \to A_2) \to A_2) \to A_2)$

(2) Let φ be a propositional formula and let $\Gamma = \{(\neg(\neg\varphi))\}$. Find a Γ-proof of φ.

Hint: Use a Group III axiom and a Group IV axiom.

(3) Let φ be a propositional formula. Let Γ be the empty set of propositional formulas. Find a Γ-proof of the tautology $((\neg(\neg\varphi)) \to \varphi)$.

Hint: Use the proof of the Deduction Lemma together with the previous exercise.

1.4 Logical implication and compactness

Definition 1.49 Let Γ be a subset of $\mathcal{L}_0$ and let φ be an element of $\mathcal{L}_0$. Then Γ *logically implies* φ if and only if $\Gamma \cup \{(\neg\varphi)\}$ is not satisfiable. □

For example, $\{\varphi\}$ logically implies φ, and $\{\varphi_1, (\varphi_1 \to \varphi_2)\}$ logically implies φ_2.

If Γ is a set a formulas, we write $\Gamma \models \varphi$ to indicate that Γ logically implies φ. We note the following.

- If Γ is empty then $\Gamma \models \varphi$ if and only if φ is a tautology.
- If Γ is not empty and if Γ is not satisfiable, then $\Gamma \models \varphi$ for all propositional formulas φ.

We now prove a second version of the Completeness Theorem for $\mathcal{L}_0$.

Theorem 1.50 (Completeness for $\mathcal{L}_0$; Version II) *Suppose that* $\Gamma \subseteq \mathcal{L}_0$, φ *is a formula and that* $\Gamma \models \varphi$.

Then $\Gamma \vdash \varphi$.

Proof. Since $\Gamma \models \varphi$, $\Gamma \cup \{(\neg\varphi)\}$ is not satisfiable. Therefore by the Completeness Theorem, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent and so by Lemma 1.41,

$$\Gamma \cup \{(\neg\varphi)\} \vdash \varphi.$$

By Lemma 1.39, the Deduction Lemma,

$$\Gamma \vdash ((\neg\varphi) \to \varphi).$$

But $(((\neg\varphi) \to \varphi) \to \varphi)$ is a logical axiom and so by Lemma 1.37, the Inference Lemma, $\Gamma \vdash \varphi$. □

Thus we obtain the following theorem as a special case of Theorem 1.50. This verifies that indeed all tautologies can be generated from the rather simple collection of the logical axioms, by a (possibly rather long) series of easy steps. In particular, for every tautology, there is a "tautology witness" which is easily verified to be a witness.

Theorem 1.51 *Suppose that* φ *is a propositional formula.*

Then φ *is a tautology if and only if* $\emptyset \vdash \varphi$. □

We now consider two versions of *compactness* for the $\mathcal{L}_0$.

Theorem 1.52 (Compactness for $\mathcal{L}_0$; Version I) *Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$, and Γ logically implies φ. Then there is a finite set Γ_0 such that $\Gamma_0 \subseteq \Gamma$ and Γ_0 logically implies φ.*

Proof. Since $\Gamma \vDash \varphi$, $\Gamma \cup \{(\neg\varphi)\}$ is *not satisfiable*. Therefore by the Completeness Theorem, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. But this implies that there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \cup \{(\neg\varphi)\}$ is inconsistent. Therefore by Lemma 1.38, the Soundness Lemma, $\Gamma_0 \cup \{(\neg\varphi)\}$ is not satisfiable and so $\Gamma_0 \vDash \varphi$. □

Definition 1.53 A nonempty subset Γ of $\mathcal{L}_0$ is *finitely satisfiable* if and only if for every finite subset Γ_0 of Γ, there is a truth assignment ν such that for all $\psi \in \Gamma_0$, $\overline{\nu}(\psi) = T$. □

We end this chapter with a second version of the Compactness Theorem. Note that this theorem doen not involve any formal notion of proof.

Theorem 1.54 (Compactness for $\mathcal{L}_0$; Version II) *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and Γ is not empty. Then Γ is satisfiable if and only if Γ is finitely satisfiable.*

Proof. By the Soundness Lemma, Lemma 1.38, if Γ is finitely satisfiable then Γ is consistent. Therefore by the Completeness Theorem, if Γ is finitely satisfiable then Γ is satisfiable. Trivially, if Γ is satisfiable then Γ is finitely satisfiable. □

1.4.1 Exercises

(1) Suppose that $\langle \nu_i : i \in \mathbb{N} \rangle$ is a sequence of truth assignments. Show that there is a truth assignment ν such that for all $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that
$$\nu \restriction \{A_0, \ldots, A_k\} = \nu_m \restriction \{A_0, \ldots, A_k\}$$
Hint: Construct $\nu \restriction \{A_0, \ldots, A_k\}$ by induction on k such that there are infinitely many m such that $\nu \restriction \{A_0, \ldots, A_k\} = \nu_m \restriction \{A_0, \ldots, A_k\}$.

(2) Prove the compactness theorem (Version II) without referring to the formal notion of proof we have defined for $\mathcal{L}_0$.
Hint: Use the previous exercise.

(3) For $\Gamma \subseteq \mathcal{L}_0$ and φ and ψ in $\mathcal{L}_0$, show that $\Gamma \cup \{\varphi\}$ logically implies ψ if and only if Γ logically implies $(\varphi \to \psi)$.
The next three exercises refer to the following definitions.

Definition 1.55 Suppose Γ_1 and Γ_2 are subsets of $\mathcal{L}_0$. Then Γ_1 is *logically equivalent* to Γ_2 if and only if, for all $\varphi \in \mathcal{L}_0$, Γ_1 logically implies φ if and only if Γ_2 logically implies φ. □

Definition 1.56 Suppose $\Gamma \subseteq \mathcal{L}_0$. Then Γ is *independent* if it is not logically equivalent to any of its proper subsets. □

(4) Suppose Γ is a finite set of formulas. Show that there is a Γ_0 such that $\Gamma_0 \subseteq \Gamma$, Γ and Γ_0 are logically equivalent, and such thay Γ_0 is independent.

(5) Show that there is an infinite set Γ of formulas such that Γ has no independent and logically equivalent subset.

(6) Show that for every set $\Gamma \subseteq \mathcal{L}_0$, there is a $\Delta \subseteq \mathcal{L}_0$ such that Δ is independent and logically equivalent to Γ.

(7) Show that the set of logical consequences of

$$\{A_i : i \neq 1 \text{ and } i \in \mathbb{N}\}$$

is consistent but not maximally consistent. Show that the set of logical consequences of

$$\{A_i : i \in \mathbb{N}\}$$

is maximally consistent.

(8) Let Δ be a set of propositional formulas in $\mathcal{L}_0$ such that the symbol $\neg$ does not appear in any element of Δ
   (a) Give an example of a formula φ such that $\{\varphi\}$ is not logically equivalent $\{\psi\}$, for any formula ψ in Δ.
   (b) Show that $\{A_i : i \in \mathbb{N}\}$ and Δ are logically equivalent.

(9) Let Δ be the set of propositional formulas of the form $(\theta_0 \wedge \theta_1 \wedge ... \wedge \theta_k)$ where $\theta_i = A_i$ or $\theta_i = (\neg A_i)$. Show that an arbitrary subset of $\mathcal{L}_0$ is logically equivalent to a set containing only negations of formulas in Δ.

2

First order logic—syntax

While propositional logic can conceivably cover a wide range of applications, first order logic is most commonly used as the language to describe internal workings and properties of groups, vector spaces, ordered sets, and other mathematical structures.

Our language consists of (certain) finite sequences of symbols, as described below.

- The *logical symbols* are the following.

$$(\quad) \quad \neg \quad \rightarrow \quad \forall$$

- $\hat{=}$ is the *equality symbol*.
- The *variable symbols* are x_i, for $i \in \mathbb{N}$.
- The *constant symbols* are c_i, for $i \in \mathbb{N}$.
- The *function symbols* are F_i, for $i \in \mathbb{N}$.
- The *predicate symbols* are P_i, for $i \in \mathbb{N}$.

We fix a function π mapping the set of function and predicate symbols to $\mathbb{N}$ so that for each $k \geq 1$, each of the sets

$$\{i \in \mathbb{N} \mid \pi(F_i) = k\}$$

and

$$\{i \in \mathbb{N} \mid \pi(P_i) = k\}$$

is infinite, and such that no symbol is mapped to 0.

For example, we could define $\pi(F_i) = n+1$, where the nth prime is the least prime which divides $i+2$. (Here the 0-th prime is 2, and so $\pi(j) = 1$ if j is even etc.) The purpose of the function π is to specify the number of arguments or *arity* of each function and predicate symbol.

We will frequently use the following notation.

- $\vec{x}$ denotes a finite sequence of (not necessarily distinct) variables. Similarly, $\vec{c}$ denotes a finite sequence of (not necessarily distinct) constants.
- $\bar{x}$ denotes a finite set of variables and $\bar{c}$ denotes a finite set of constants.

We will need to consider the cases where we restrict our language.

Definition 2.1 An *alphabet*, typically denoted as $\mathcal{A}$, is a (perhaps infinite, perhaps empty) subset of the set of all constant, function, and predicate symbols. These are the *nonlogical* symbols of our language. □

Remark 2.2 Given an alphabet $\mathcal{A}$ we may want to add more symbols, whether they be for constants, function, or variables. □

2.1 Terms

Recall our notation; if $\vec{s} = \langle s_1, \ldots, s_n \rangle$ and $\vec{t} = \langle t_1, \ldots, t_m \rangle$ are finite sequences of symbols, then $\vec{s} + \vec{t}$ denotes the finite sequence $\langle s_1, \ldots, s_n, t_1, \ldots, t_m \rangle$.

Definition 2.3 The set of *terms*, T, is defined as the smallest set of finite sequences T satisfying the following properties.

(1) For each $i \in \mathbb{N}$, the sequences of length one,

$$\langle x_i \rangle$$

and

$$\langle c_i \rangle$$

belong to T.

(2) If F_i is a function symbol, $n = \pi(F_i)$, and $\tau_1, \ldots, \tau_n$ belong to T, then

$$\langle F_i \rangle + \langle (\rangle + \tau_1 + \cdots + \tau_n + \langle) \rangle$$

belongs to T. More briefly, the concatenation $F_i(\tau_1 \ldots \tau_n)$ belongs to T. □

We will assume familiarity with the methods of the previous chapter and omit the proof that T is well defined.

Remark 2.4 We shall adopt several notational conventions.

(1) Often we shall say that x_i is a term. Of course we are referring to the sequence of length 1, $\langle x_i \rangle$.

(2) For the sake of readability, we will often denote a finite sequence of (not necessarily distinct) terms as $\vec{\tau} = \langle \tau_1, \ldots, \tau_n \rangle$. Further, let $\bar{\tau} = \{\tau_{n_1}, \ldots\}$ denote sets of terms.

(3) More generally we shall indicate terms informally and use

$$F_i(\tau_1, \ldots, \tau_n)$$

or

$$F_i(\vec{\tau})$$

to indicate the term

$$\langle F_i \rangle + \langle (\rangle + \tau_1 + \cdots + \tau_n + \langle) \rangle$$

□

The elements of T are uniquely readable, as is pointed out in the next sequence of lemmas.

Lemma 2.5 (Readability for Terms) *Suppose τ in* T*. Then one and only one of the following conditions holds.*

(1) *There is an $i \in \mathbb{N}$ greater than or equal to* 1 *such that τ is x_i or τ is c_i.*

(2) *There is exist $i, n \in \mathbb{N}$ with $n \geq 1$ and there is a finite sequence $\langle \tau_1, \ldots, \tau_n \rangle$ in* T *such that $\pi(F_i) = n$ and such that*

$$\tau = F_i(\tau_1, \ldots, \tau_n).$$

Proof. As in the proof of Lemma 1.7, we let T be the subset of T whose elements satisfy one of the above clauses. We observe that T satisfies the closure properties of Definition 2.3. Consequently, $\mathrm{T} \subseteq T$, as required.

The two conditions are mutually exclusive, as the first symbol in τ determines which condition holds. □

Lemma 2.6 *If $\tau \in \mathrm{T}$, then no proper initial segment of τ is an element of* T*.*

Proof. We proceed by induction on the length of $\tau \in \mathrm{T}$.

If τ is a term of length 1, then the only proper initial segment is the null sequence, which by Lemma 2.5 is not an element of T.

Suppose that τ has length greater than 1 and assume the lemma for all terms of length less than that of τ. By Lemma 2.5, τ is of the form $F_i(\tau_1, \ldots, \tau_n)$. Suppose that σ is a proper initial segment of τ such that $\sigma \in \mathrm{T}$. As above, σ is not the null sequence, so the first symbol in σ is F_i. By Lemma 2.5, σ must have the form $F_i(\sigma_1, \ldots, \sigma_n)$, where each σ_i belongs to T. But then σ_1 and τ_1 must be identical, since neither can be a proper initial segment of the other. It follows by an induction up to n, that for each i, σ_i is equal to τ_i. But then $\sigma = \tau$, contradicting the choice of σ. Thus, τ has no proper initial segment in T, as required. □

Theorem 2.7 (Unique Readability for Terms) *Suppose $\tau \in T$. Then one and only one of the following conditions holds.*

(1) *There is an $i \in \mathbb{N}$ greater than or equal to* 1 *such that τ is x_i or τ is c_i.*

(2) *There exist $i, n \in \mathbb{N}$ with $n \geq 1$ and a finite sequence of terms $\langle \tau_1, \ldots, \tau_n \rangle$ such that $\pi(F_i) = n$ and such that*

$$\tau = F_i(\tau_1, \ldots, \tau_n).$$

Further, in (2)*, the function symbol F_i and the finite sequence $\langle \tau_1, \ldots, \tau_n \rangle$ are unique.*

Proof. By Lemma 2.5, it will be sufficient to verify the uniqueness of the finite sequence $\langle \tau_1, \ldots, \tau_n \rangle$. This follows as in the proof of Lemma 2.6. Suppose that τ could be written as $F_i(\tau_1, \ldots, \tau_n)$ and as $F_j(\sigma_1, \ldots, \sigma_m)$. Then F_i and F_j both occur as the first symbol in τ, and hence are equal. Consequently, $n = m = \pi(F_i)$. Then τ_1 and σ_1 must also be equal, as neither can be a proper initial segment of the other. By induction on i less than or equal to n, for each i, τ_i is equal to σ_i, as required. □

Example 2.8 If we allow ourselves to talk about the integers with the usual addition and multiplication, and allow ourselves a variable, a term is a polynomial. Thus terms generalize the familiar notion of polynomials. □

2.2 Formulas

Definition 2.9 The set of formulas, $\mathcal{L}$, is the smallest set L of finite sequences of symbols as above satisfying the following properties.

(1) **Atomic Formulas:**
   (a) If P_i is a predicate symbol, $n = \pi(P_i)$ is the arity of P_i and $\tau_1, \ldots, \tau_n$ are terms, then

   $$P_i(\tau_1, \ldots, \tau_n)$$

   is an element of L.
   (b) If τ_1 and τ_2 are terms, then

   $$(\tau_1 \hat{=} \tau_2)$$

   is an element of L.
(2) **Inductive Formulas:**
   (a) If $\varphi \in L$, then

   $$(\neg\varphi)$$

   is an element of L
   (b) If φ_1 and φ_2 are elements of L, then

   $$(\varphi_1 \rightarrow \varphi_2)$$

   is an element of L
   (c) If $\varphi \in L$ and x_i is a variable symbol, then

   $$(\forall x_i \varphi)$$

   is an element of L. □

As in the case of T, we will not repeat the argument to show that $\mathcal{L}$ is well defined.

2.3 Readability and subformulas

We will give an abbreviated proof that every formula in $\mathcal{L}$ is uniquely readable, as stated in Theorem 2.12. As above, we proceed by proving a readability lemma, a proper initial segment lemma, and then a uniqueness lemma.

Lemma 2.10 (Readability for Formulas) *Suppose that φ is a formula. Then one and only one of the following conditions holds.*

(1) *There exist $i, n \in \mathbb{N}$ with $n \geq 1$ and a sequence of terms $\langle \tau_1, \ldots, \tau_n \rangle$ such that $n = \pi(P_i)$ and such that*

$$\varphi = P_i(\tau_1, \ldots, \tau_n).$$

(2) *There are terms τ_1 and τ_2 such that $\varphi = (\tau_1 \hat{=} \tau_2)$.*
(3) *There is a formula ψ such that $\varphi = (\neg\psi)$.*
(4) *There are formulas ψ_1 and ψ_2 such that $\varphi = (\psi_1 \rightarrow \psi_2)$.*
(5) *There is a formula ψ and a variable x_i such that $\varphi = (\forall x_i \psi)$.*

The proof Lemma 2.10 is analogous to that of Lemma 1.7.

Lemma 2.11 *If $\varphi \in \mathcal{L}$, then no proper initial segment of φ is an element of $\mathcal{L}$.*

Proof. We consider the cases of Lemma 2.10.

Suppose that φ is of the form $P_i(\tau_1 \ldots \tau_n)$ and ψ is a proper initial segment of φ which also belongs to $\mathcal{L}$. Then the first symbol in ψ is P_i and so ψ must also be of the form $P_i(\sigma_1 \ldots \sigma_n)$. But then τ_1 must equal σ_1, or they would be a pair of distinct terms for which one is a proper initial segment of the other, contradicting Lemma 2.6. It follows by induction on i less than or equal to n that each σ_i is equal to τ_i, and hence that φ is equal to ψ.

The case when φ is an equality between terms can be analyzed similarly, using Lemma 2.6.

The cases when φ is $(\neg\psi)$ or $(\psi_1 \rightarrow \psi_2)$ are analogous to the same cases in the propositional case. See Lemma 1.8.

Finally, consider the case when φ is $(\forall x_i \varphi_1)$. If ψ is an initial segment of φ, then ψ must be of the form $(\forall x_i \psi_1)$, as φ and ψ must have the same first three symbols. But then induction applies to φ_1, and ψ_1 must equal φ_1. It follows that φ is equal to ψ. □

Theorem 2.12 (Unique Readability for Formulas) *Suppose that φ is a formula. Then one and only one of the following conditions holds.*

(1) *There exist $i, n \in \mathbb{N}$ with $n \geq 1$ and a sequence of terms $\langle \tau_1, \ldots, \tau_n \rangle$ such that $n = \pi(P_i)$ and such that*

$$\varphi = P_i(\tau_1, \ldots, \tau_n).$$

(2) *There are terms τ_1 and τ_2 such that $\varphi = (\tau_1 \hat{=} \tau_2)$.*

(3) *There is a formula* ψ *such that* $\varphi = (\neg\psi)$.
(4) *There are formulas* ψ_1 *and* ψ_2 *such that* $\varphi = (\psi_1 \to \psi_2)$.
(5) *There is a formula* ψ *and a variable* x_i *such that* $\varphi = (\forall x_i \psi)$.

Further, in each of the above cases; the terms, the finite sequence of terms, or the subformulas which are mentioned in that case, are unique. □

We leave the proof of Theorem 2.12 to the Exercises.

We define the relation ψ *is a subformula of* φ for formulas in $\mathcal{L}$. We then verify (just as we did for propositional formulas) that this definition captures exactly the formulas which appear in the iterated decomposition of φ, as given by Unique Readability for Formulas. Equivalently, this definition captures exactly the formulas used to construct φ.

Definition 2.13 Suppose that φ is a formula. A formula ψ is a *subformula* of φ if ψ is a block-subsequence of φ. (See Definition 1.10.) □

Definition 2.14 Suppose that

$$\vec{\psi} = \langle \psi_0, \ldots, \psi_n \rangle$$

is a finite sequence of finite sequences. Then $\vec{\psi}$ is a *formula-witness* if for all $i \leq n$, one of the following hold.

(1) ψ_i is an atomic formula.
(2) For some $j < i$, $\psi_i = (\neg\psi_j)$.
(3) For some $j_1, j_2 < i$, $\psi_i = (\psi_{j_1} \to \psi_{j_2})$.
(4) For some $j < i$ and for some $k \in \mathbb{N}$, $\psi_i = (\forall x_k \psi_j)$. □

Just as for the case for propositional formulas:

Lemma 2.15 *Suppose that*

$$\vec{\psi} = \langle \psi_0, \ldots, \psi_n \rangle$$

is a formula-witness. Then for all $i \leq n$, ψ_i *is a formula.*

Proof. By induction on $i \leq n$, this is immediate from the Definition 2.9. □

Suppose φ is a finite sequence and $\vec{\psi} = \langle \psi_1, \ldots, \psi_n \rangle$ is a formula-witness. Then we say $\vec{\psi}$ is a formula-witness for φ if $\varphi = \psi_n$. Thus we have the following lemma.

Lemma 2.16 *Suppose* φ *is a finite sequence. Then the following are equivalent.*

(1) φ *is a formula.*
(2) *There is a formula-witness for* φ.

Proof. Let $\mathcal{L}^*$ be the set of all finite sequences for which there is a formula-witness for φ. Then $\mathcal{L}^*$ satisfies the closure requirements of Definition 2.9 and so by the minimality of the set of formulas, $\mathcal{L} \subseteq \mathcal{L}^*$.

Now suppose $\varphi \in \mathcal{L}^*$. Let $\langle \psi_1, \ldots, \psi_n \rangle$ be a formula-witness for φ. It follows easily by induction on $i \leq n$ that ψ_i is a formula for all $1 \leq i \leq n$. Thus $\mathcal{L}^* \subseteq \mathcal{L}$. This proves the lemma. □

Suppose $\vec{s} = \langle s_0, \ldots, s_m \rangle$ is a finite sequence. Then a finite sequence $\vec{t}$ is a *final segment* of $\vec{s}$ if for some $j \leq m$,

$$\vec{t} = \langle s_j, \ldots, s_m \rangle.$$

If $0 < j$ then $\vec{t}$ is a *proper final segment* of $\vec{s}$.

We complete the analysis of subformulas in Lemma 2.19. We do not actually need this lemma, but it does answer the natural questions which arise about subformulas and block-subsequences etc.

The proof of Lemma 2.19 requires the following variations of Lemma 2.6 and Lemma 2.11. We leave the proofs of these two lemmas to the exercises.

Note that in the case of final segments of formulas, the conclusion refers to both terms and formulas. The difference (between the case of initial segments versus final segments) of course is that it is an immediate consequence of Readability for Formulas that an initial segment of a formula is not a term.

Lemma 2.17 *Suppose that τ is a term and that σ is a proper final segment of τ. Then σ is not a term.* □

Lemma 2.18 *Suppose that φ is a formula and that σ is a proper final segment of φ. Then σ is not a formula and σ is not a term.* □

Putting everything together we obtain the following.

Lemma 2.19 *Suppose φ is a formula and that ψ is a formula. Then the following are equivalent.*

(1) *ψ is a subformula of φ.*
(2) *Suppose $\langle \psi_1, \ldots, \psi_n \rangle$ is a formula-witness for φ. Then $\psi = \psi_k$ for some k where $1 \leq k \leq n$.*

Proof. (2) implies (1) is immediate from the definitions and so it suffices to prove that (1) implies (2).

Suppose ψ is a subformula of φ and let $\langle \psi_1, \ldots, \psi_n \rangle$ is a formula-witness for φ. Let $k \leq n$ be least such that ψ is a subformula of ψ_k. It suffices to show that $\psi_k = \psi$.

Case 1: ψ_k is an atomic formula.

There are two subcases. First suppose that $\psi_k = P_m(\tau_1, \ldots, \tau_M)$. Then since ψ is a formula, by Readability For Formulas, Lemma 2.10, ψ must be an initial segment of ψ_k and so by Lemma 2.11, $\psi = \psi_k$.

Now suppose that $\psi_k = (\tau_1 \hat{=} \tau_2)$. Then by Readability for Formulas again, there must exist terms σ_1, σ_2 such that $\psi = (\sigma_1 \hat{=} \sigma_2)$. Thus either σ_1 is a final segment of τ_1 or τ_1 is a final segment of σ_1. Therefore $\sigma_1 = \tau_1$ by Lemma 2.18.

Similarly, either σ_2 is an initial segment of τ_2 or τ_2 is an initial segment of σ_2. Therefore by Lemma 2.11, $\sigma_2 = \tau_2$. Thus $\psi_k = \psi$.

For the remaining cases we have reduced to the case that ψ_k is an inductive formula.

Case 2: $\psi_k = (\forall x_m \psi_j)$ for some $j < k$.

Since ψ is not a subformula of ψ_j, either ψ is an initial segment of ψ_k, or ψ is a final segment of ψ_k. In either case, by Lemma 2.11 and Lemma 2.18, $\psi = \psi_k$.

Case 3: $\psi_k = (\neg\psi_j)$ for some $j < k$.

Then again, either ψ is an initial segment of ψ_k, or ψ is a final segment of ψ_k. In either case, by Lemma 2.11 and Lemma 2.18, $\psi = \psi_k$.

Case 4: $\psi_k = (\psi_{j_1} \to \psi_{j_2})$ for some $j_1, j_2 < k$.

There are three subcases.

Subcase 4.1: ψ is an initial segment of $\langle (\rangle + \psi_{j_1}$.

Note that ψ must be a proper initial segment of $\langle (\rangle + \psi_{j_1}$, since otherwise ψ_{j_1} is a proper final segment of ψ, which is impossible by Lemma 2.18. Therefore the first symbol of ψ is (and so ψ is an inductive formula. The first symbol of ψ_{j_1} cannot be $\forall$ and so either $\psi = (\neg\alpha)$ for some formula α, or $\psi = (\alpha \to \beta)$ for some formulas α and β. But in either case α is a proper initial segment of ψ_{j_1}, which is a contradiction.

Case 4.2: ψ is a final segment of $\psi_{j_2} + \langle) \rangle$.

Note that ψ must be a proper final initial segment of $\psi_{j_2} + \langle) \rangle$ since otherwise ψ_{j_2} a proper initial segment of ψ which is impossible by Lemma 2.11. Therefore the last symbol of ψ is). If ψ is an inductive formula then there must be a formula α such that α is a proper final segment of ψ_{j_2} which contradicts Lemma 2.18. Therefore ψ is an atomic formula. But then there is a term τ such that τ is a proper final segment of ψ_{j_2} which again contradicts Lemma 2.18.

Case 4.3: ψ is not a block-subsequence of $\langle (\rangle + \psi_{j_1}$, and ψ is not a block-subsequence of $\psi_{j_2} + \langle) \rangle$.

Therefore, there are finite sequences s, t such that

- $\psi_k = s + \psi + t$,
- s is an initial segment of $\langle (\rangle + \psi_{j_1}$,
- t is final segment of $\psi_{j_2} + \langle) \rangle$.

But then by arguing as in the proof of Lemma 1.16.5, there is a formula $\theta = (\theta_1 \to \theta_2)$ such that θ is a subformula of ψ and such that:

- Either θ_1 is a final segment of ψ_{j_1} or ψ_{j_1} is a final segment of θ_1,
- Either θ_2 is an initial segment of ψ_{j_2} or ψ_{j_2} is an initial segment of θ_2.

Thus by Lemma 2.11 and Lemma 2.18, $\theta_1 = \psi_{j_1}$ and $\theta_2 = \psi_{j_2}$. But this implies that $\psi = (\psi_{j_1} \to \psi_{j_2}) = \psi_k$. $\square$

2.3.1 Exercises

(1) Prove Theorem 2.12.
(2) Prove Lemma 2.17.
(3) Prove Lemma 2.18.
(4) Consider the set of sequences defined as in Definition 2.9 except that the last clause is changed to read, "If $\varphi \in L$ and x_i is a variable symbol, then $\forall x_i \varphi$ is an element of L" in which the parentheses are omitted.
  (a) Is this set readable?
  (b) Is this set uniquely readable?
(5) Consider the set of sequences defined as in Definition 2.9 except that the fourth clause is changed to read, "If φ_1 and φ_2 are elements of L, then $\varphi_1 \rightarrow \varphi_2$ is an element of L" in which the parentheses are omitted.
  (a) Is this set readable?
  (b) Is this set uniquely readable?

  Hint: For (a), show that a proper initial segment of φ is not a "formula" if either $\varphi = (\forall x_k \psi)$ or $\varphi = (\neg\psi)$, by induction on the length of φ.

2.4 Free variables, bound variables

Suppose that φ is a formula and that x_i is a variable. Then each occurrence of $\forall x_i$ in φ defines a unique subformula of φ. This is the content of the next lemma.

Lemma 2.20 *Suppose that φ is a formula, x_i is a variable, s and t are finite sequences, and that*

$$\varphi = s + \langle (, \forall, x_i \rangle + t.$$

Then there is a unique formula ψ such that $s + \psi$ is an initial segment of φ.

Proof. Note that the uniqueness of ψ follows by observing that if there were two such formulas, then one would be a proper initial segment of the other and contradict Lemma 2.11.

We prove the existence claims of Lemma 2.20 by induction on the length of φ. There are no formulas of length 1, and so the lemma is true of all length 1 formulas on trivial grounds. Now assume the lemma is true of every formula which is shorter than φ. By Readability for Formulas, Lemma 2.10, we can analyze φ by considering the various cases of that lemma.

If φ is atomic, then φ does not contain an occurrence of $\langle (, \forall, x_i \rangle$, and again the claim is true on trivial grounds.

If φ is $(\neg\theta)$, then any occurrence of $\langle (, \forall, x_i \rangle$ in φ is also one in θ and it follows from the induction hypothesis that there ψ exists as required.

Similarly, if φ is $(\psi_1 \rightarrow \psi_2)$ and there is an occurrence of $\langle (, \forall, x_i \rangle$ in φ, then it must be contained completely in ψ_1 or in ψ_2 (there is no $\rightarrow$ in $\langle (, \forall, x_i \rangle$) and so the induction hypothesis yields ψ.

Finally, if φ is $(\forall x_j \varphi_1)$, then either the occurrence of $\langle (, \forall, x_i \rangle$ is as an initial segment of φ, s and t are each the empty sequence, and the formula φ is the desired ψ, or the occurrence of $\langle (, \forall, x_i \rangle$ is entirely contained in φ_1 and the induction hypothesis again yields ψ. □

Suppose φ is a formula then by Unique Readability for Formulas, Theorem 2.12, every occurrence of $\langle \forall, x_i \rangle$ as a block-subsequence of φ must be immediately preceded by an occurrence of the symbol (.

This suggests the following definition.

Definition 2.21 Suppose that $\varphi = \langle a_1, \dots, a_n \rangle$ is a formula and x_i is a variable.

(1) An occurrence of $\forall x_i$ in φ is an occurrence of $\langle \forall, x_i \rangle$ in φ (as a block-subsequence).
(2) The *scope* of a particular occurrence of $\forall x_i$ in φ is the unique interval $[j_1, j_2]$ with the following properties.
    (a) $[j_1 + 1, j_1 + 2]$ is the given occurrence of $\forall x_i$.
    (b) $\langle a_{j_1}, \dots, a_{j_2} \rangle$ is a formula. □

Example 2.22 (1) The scope of the first occurrence of $\forall x_1$ in the formula

$$((\forall x_1 (x_1 \hat{=} x_2)) \to (\forall x_1 \psi))$$

is the block-subsequence which contains that occurrence and which gives the subformula $(\forall x_1 (x_1 \hat{=} x_2))$.

(2) The scope of the second occurrence of $\forall x_1$ in the formula

$$((\forall x_1 (x_1 \hat{=} x_2)) \to (\forall x_1 \psi))$$

is the block-subsequence which contains that occurrence and gives the subformula $(\forall x_1 \psi)$. □

Definition 2.23 Suppose that φ is a formula and that x_i is a variable which occurs in φ.

(1) An occurrence of x_i in φ is *free* if and only if it is not within the scope of any occurrence of $\forall x_i$ in φ. Otherwise, the occurrence is *bound*.
(2) x_i is a *free variable* if and only if there is a free occurrence of x_i in φ.
(3) x_i is a *bound variable* of φ if and only if x_i occurs in φ and is not a free variable of φ. □

Example 2.24 Consider the formula

$$\varphi = ((\forall x_1 (x_1 \hat{=} x_2)) \to (\forall x_2 (x_2 \hat{=} x_1)))$$

(1) The first two occurrences of x_1 are bound and the third occurrence of x_1 is free.

(2) The first occurrence x_2 is free and the other two occurrences of x_2 are bound. □

Definition 2.25 (1) Suppose that τ is a term and

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}$$

is a finite set of variables where $n_0 < n_1 < \cdots < x_{n_k}$. We write $\tau(\bar{x})$ to indicate that all the variables of τ are included in the set $\{x_{n_0}, \ldots, x_{n_k}\}$.

(2) Suppose that φ is a formula and

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}$$

is a finite set of variables with $n_0 < n_1 < \cdots < x_{n_k}$. We write $\varphi(\bar{x})$ to indicate that all the **free** variables of φ are included in the set $\{x_{n_0}, \ldots, x_{n_k}\}$.

Another notation we will occasionally use (and this is a standard notation) is $\varphi[\bar{x}]$, where $\bar{x}$ is a finite set of variables, to indicate that every free variable of φ is included in the set $\bar{x}$. □

Formulas with no free variables are particularly interesting especially when defining as we shall the notion of *satisfiability*.

Definition 2.26 A formula φ is a *sentence* if and only if the sentence φ has no free variables. □

Definition 2.27 A set of sentences is a *theory.* □

2.5 Substitution

Definition 2.28 (1) Suppose that τ is a term,

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}$$

is a finite set of variables with $n_0 < n_1 < \cdots < x_{n_k}$, and that

$$\vec{\tau} = \langle \tau_0, \ldots, \tau_k \rangle,$$

sequence of terms $\vec{\tau}$ (with $|\vec{x}| = |\vec{\tau}|$). We write $\tau(\bar{x}; \vec{\tau})$ to indicate the term obtained by simultaneously substituting, for each $i \leq k$, the term τ_i for each occurrence of x_{n_i} in τ.

(2) Suppose that φ is a formula,

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}$$

is a finite set of variables with $n_0 < n_1 < \cdots < x_{n_k}$, and that

$$\vec{\tau} = \langle \tau_0, \ldots, \tau_k \rangle,$$

sequence of terms $\vec{\tau}$ (with $|\bar{x}| = |\vec{\tau}|$). We write $\varphi(\bar{x}; \vec{\tau})$ to indicate the formula obtained by simultaneously substituting, for each $i \leq k$, the term τ_i for each **free** occurrence of x_{n_i} in τ.

We could have chosen to write $\tau(\vec{x}, \vec{\tau})$ where we require that the finite sequence $\vec{x}$ have the special form indicated,

$$\vec{x} = \langle x_{n_0}, \ldots, x_{n_k} \rangle$$

where the ordering of the variables in the sequence *agrees* with the natural ordering of the variables as given by their indicies. Of course such finite sequences $\vec{x}$ are uniquely specified by simply the finite set $\bar{x}$ of variables which occur in $\vec{x}$. Thus using the notation for example, $\varphi(\bar{x}; \vec{\tau})$, instead of $\varphi(\vec{x}; \vec{\tau})$ is just an arbitrary choice.

We actually have more freedom than our definition indicates. For example, we may have a formula $\varphi(\bar{x})$ where

$$\bar{x} = \{x_0, x_1\}$$

and wish to only substitute instances of x_1 with a term τ.

We indicate this action by writing $\varphi(x_1; \tau)$ instead of writing

$$\varphi(\{x_0, x_1\}; \langle x_0, \tau \rangle)$$

which of course expresses exactly this.

Lemma 2.29 (1) *For any term $\tau(\bar{x})$ and sequence of terms $\vec{\tau}$, where $|\bar{x}| = |\vec{\tau}|$, $\tau(\bar{x}; \vec{\tau})$ is a term.*

(2) *For any formula $\varphi(\bar{x})$ and for any sequence of terms $\vec{\tau}$, where $|\bar{x}| = |\vec{\tau}|$, $\varphi(\bar{x}; \vec{\tau})$ is a formula.*

(3) *For any formula $\varphi(\bar{x})$ and for any sequence of constants $\vec{c}$, where $|\bar{x}| = |\vec{c}|$, $\varphi(\bar{x}; \vec{c})$ is a sentence.*

Proof. The proof of (1) is by induction of the length of τ. Having proved (1) for all terms τ and for all sequences of terms $\vec{\tau}$, one proves (2) by indiction on the length of the formula φ.

Finally having proved (2) for all formulas φ and for all sequences of terms $\vec{\tau}$, (3) follows immediately.

The details are left to the reader since at this stage we have proved many similar claims by analogous arguments. $\square$

3

First order logic—semantics

3.1 Formulas and structures

Suppose that $\mathcal{A}$ is an alphabet. Thus $\mathcal{A}$ is simply a subset (possibly empty) of the set

$$\{c_i, F_i, P_i \mid i \in \mathbb{N}\}$$

of all constant, function, and predicate symbols.

Definition 3.1 A finite sequence φ is an $\mathcal{L}_{\mathcal{A}}$-formula if and only if the following hold.

(1) φ is a formula.
(2) The constant, predicate, and function symbols occurring in φ are all in the alphabet, $\mathcal{A}$. □

Definition 3.2 An $\mathcal{L}_{\mathcal{A}}$-*structure* (or simply a *model*) is a pair (M, I) as follows.

(1) $M \neq \emptyset$; this set is referred to as the structure's *universe*
(2) I, called the *interpretation function*, is a function with domain $\mathcal{A}$ such that for each $i \in \mathbb{N}$ the following conditions hold:
   (a) If $c_i \in \mathcal{A}$ then $I(c_i) \in M$;
   (b) if $F_i \in \mathcal{A}$ then $I(F_i)$ is a function

   $$I(F_i) : M^n \to M$$

   where $n = \pi(F_i)$;
   (c) if $P_i \in \mathcal{A}$ then

   $$I(P_i) \subseteq M^n$$

   where $n = \pi(P_i)$. □

Often, when $\mathcal{A}$ is small, we will opt to represent (M, I) as the interpretation of $\mathcal{A}$ under I. For instance, suppose $M = \mathbb{R}$ and $\mathcal{A} = \{c_1, c_2, F_1, F_2, P_1\}$. We can say $(M, I) = (\mathbb{R}, I(c_1), (c_2), (F_1), (F_2), (P_1)) = (\mathbb{R}, 0, 1, +, \times, <)$, so I interprets the real numbers as an ordered field.

Remark 3.3 So far, all we have discussed only symbols that can be written on paper, or perhaps a chalkboard. The interpretation function assigns meaning to these symbols; the reader may realize that this is precisely what they have been doing throughout most of their mathematical career. It is how we recognize the numerals *"5"* and "V" as both referring to five, or both "$\cdot$" or "$\times$" as references to multiplication. □

Let $\mathcal{B}$ be a subset of $\mathcal{A}$ and let (M, I) be an $\mathcal{L}_{\mathcal{A}}$-structure. We then define the $\mathcal{L}_{\mathcal{B}}$-reduct of (M, I):

Definition 3.4 The $\mathcal{L}_{\mathcal{B}}$*-reduct* of an $\mathcal{L}_{\mathcal{A}}$-structure is a pair $(M, I\restriction_{\mathcal{L}_{\mathcal{B}}})$ where M is the same nonempty universe and $I\restriction_{\mathcal{L}_{\mathcal{B}}}$ is the restriction of the original interpretation function I to the language $\mathcal{L}_{\mathcal{B}}$. □

Example 3.5 Consider the following structure, $(\mathbb{N}, 0, 1, +, \cdot, <)$, the set of natural numbers with addition, multiplication, and the less-than order defined on it. Here, our alphabet consists of two constant symbols, interpreted by elements 0 and 1, two function symbols, interpreted by operations $+$ and $\cdot$, and one binary predicate symbol, interpreted by relation $<$.

Now consider the subset of the alphabet consisting of just the one predicate symbol. Then the corresponding reduct of our original structure is the set of natural numbers with just the less-than order defined. □

3.2 The satisfaction relation

3.2.1 Interpreting terms

Definition 3.6 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure. A function ν is an $\mathcal{M}$*-assignment* if ν is a map from $\{x_i : i \in \mathbb{N}\}$ to M. □

Thus, an $\mathcal{M}$-assignment is simply a function which associates to each variable symbol an element of the universe of the structure $\mathcal{M}$.

Readers may already be familiar with this action. For example, if one has the polynomial $x^2 + 1$, and a ν assignment such that $\nu(x) = 2$, then $\overline{\nu}(x^2 + 1) = 5$

Definition 3.7 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and that ν is an $\mathcal{M}$-assignment. We would like all terms to refer to elements of M, so we define a function

$$\overline{\nu} : \{\tau : \tau \text{ is an } \mathcal{L}_{\mathcal{A}}\text{-term}\} \to M$$

by induction on the length of $\mathcal{L}_{\mathcal{A}}$-terms as follows.

Base step. If τ has length 1, then we define $\overline{\nu}(\tau)$ by whichever equation applies.

$$\overline{\nu}(\langle x_i \rangle) = \nu(x_i)$$
$$\overline{\nu}(\langle c_i \rangle) = I(c_i)$$

Induction step. For a finite sequence of terms $\vec{\tau}$, then

$$\overline{\nu}(\vec{\tau}) = \overline{\nu}(\langle \tau_1, \dots, \tau_n \rangle) = \langle \overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_n) \rangle$$

If $\tau = F_i(\vec{\tau})$, where $|\vec{\tau}| = \pi(F_i)$, then

$$\overline{\nu}(\tau) = I(F_i)(\overline{\nu}(\vec{\tau})).$$

By unique readability for terms, $\overline{\nu}$ is well defined.

Definition 3.8 Suppose that $\mathcal{M}$ is an $\mathcal{L}_{\mathcal{A}}$-structure, τ is an $\mathcal{L}_{\mathcal{A}}$-term, ν and μ are $\mathcal{M}$-assignments, and $\bar{x} = \{x_{n_0}, x_{n_2} \dots x_{n_k}\}$ is a finite set of variables.

- Then ν and μ *agree on* $\bar{x}$ if and only if for all $i \le k$, $\nu(x_{n_i}) = \mu(x_{n_i})$. Denote this as $\nu|_{\bar{x}} = \mu|_{\bar{x}}$.
- Furthermore, ν and μ *agree on the variables of* $\tau(\vec{x})$ if and only if $\overline{\nu}(\vec{x}) = \overline{\mu}(\vec{x})$.
- Similarly, ν and μ *agree on the free variables of* $\varphi(\vec{x})$ if and only if $\overline{\nu}(\vec{x}) = \overline{\mu}(\vec{x})$.

Lemma 3.9 *Consider an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M} = (M, I)$ and an $\mathcal{L}_{\mathcal{A}}$-term with free variables $\tau(\vec{x})$. Suppose that ν and μ are $\mathcal{M}$-assignments such that $\nu(\vec{x}) = \mu(\vec{x})$. Then $\overline{\mu}(\tau) = \overline{\nu}(\tau)$*

Proof. We prove Lemma 3.9 by induction on the length of τ. If τ has length 1, then by Theorem 2.7, unique readability for terms, there is an i such that either τ is $\langle x_i \rangle$ and $\overline{\mu}(\tau) = \mu(x_i) = \nu(x_i) = \overline{\nu}(\tau)$ or τ is $\langle c_i \rangle$ and $\overline{\mu}(\tau) = I(c_i) = \overline{\nu}(\tau)$. In either case, the lemma is verified. Now, suppose that τ has length greater than 1 and assume the lemma for all terms which are shorter than τ. Again by Theorem 2.7, τ has the form $F_i(\vec{\tau})$, and we use the inductive hypothesis as follows.

$$\begin{aligned}
\overline{\mu}(\tau) &= \overline{\mu}(F_i(\vec{\tau})) \\
&= I(F_i)(\overline{\mu}(\vec{\tau})) \\
&= I(F_i)(\overline{\nu}(\vec{\tau})) && \text{(by induction)} \\
&= \overline{\nu}(F_i(\vec{\tau})) \\
&= \overline{\nu}(\tau)
\end{aligned}$$

Thus, $\overline{\mu}(\tau) = \overline{\nu}(\tau)$ as required. □

Definition 3.10 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and ν is an $\mathcal{M}$-assignment. We define the *satisfaction relation*,

$$(\mathcal{M}, \nu) \models \varphi$$

by induction on the length of φ as follows.

Atomic cases. Suppose that φ is an atomic formula.

(1) Suppose that $\vec{\tau}$ is a sequence of terms, and $\varphi = P_i(\vec{\tau})$ where $|\vec{\tau}| = \pi(P_i)$. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \overline{\nu}(\vec{\tau}) \in I(P_i).$$

That is, predicates are satisfied by the elements it contains.

(2) Suppose that $\varphi = (\sigma \mathrel{\hat{=}} \tau)$ where σ and τ are terms. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \overline{\nu}(\sigma) = \overline{\nu}(\tau).$$

That is, a statement of equality is satisfied if and only if two terms are interpreted as the same element of M.

Inductive cases. Suppose that φ is not an atomic formula.

(3) Suppose that $\varphi = (\neg\psi)$. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } (\mathcal{M}, \nu) \not\models \psi.$$

Here, we use $(\mathcal{M}, \nu) \not\models \psi$ to indicate that it is not the case that $(\mathcal{M}, \nu) \models \psi$.

(4) Suppose that $\varphi = (\psi_1 \to \psi_2)$. Then

$$\begin{aligned}&(\mathcal{M}, \nu) \models \varphi \text{ if and only if}\\ &\qquad\qquad \text{either} (\mathcal{M}, \nu) \not\models \psi_1 \text{ or } (\mathcal{M}, \nu) \models \psi_2.\end{aligned}$$

(5) Suppose that $\varphi = (\forall x_i \psi)$. Then $(\mathcal{M}, \nu) \models \varphi$ if and only if for all $\mathcal{M}$-assignments μ, if ν and μ agree on the free variables of φ, then $(\mathcal{M}, \mu) \models \psi$. Since x_i is bound in φ, the values of these μ's on x_i range over all of M.

By Theorem 2.12, unique readability for formulas, $(\mathcal{M}, \nu) \models \varphi$ is well defined for all $\mathcal{M}$-assignments ν and $\mathcal{L}_{\mathcal{A}}$-formulas φ. We will sometimes say that $(\mathcal{M}, \nu)$ *satisfies* φ to indicate $(\mathcal{M}, \nu) \models \varphi$.

Example 3.11 Say we are talking about the integers with the usual ordering, and have constants for all the integers. Refer to this structure as $(\mathbb{Z}, 0, 1, -1, 2, \ldots, <)$. While " $<$ " is nothing more than a symbol, $I(<)$ is a subset of $\mathbb{Z} \times \mathbb{Z}$, a set of ordered pairs (a, b) where $(a, b) \in I(<)$ if and only if a is less than b. Predicates can be interpreted as subsets of the underlying set, or of the finite Cartesian product of a set, and satisfaction of a predicate is defined by membership. □

Recall that if $\bar{x}$ is a finite set of variables then $\varphi(\bar{x})$ indicates that φ is a formula such that for all $i \in \mathbb{N}$, if x_i is a free variable of φ then x_i is in $\bar{x}$.

Theorem 3.12 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, $\varphi(\bar{x})$ is an $\mathcal{L}_{\mathcal{A}}$-formula, and ν and μ are $\mathcal{M}$-assignments such that $\overline{\nu}|_{\bar{x}} = \overline{\mu}|_{\bar{x}}$. Then*

$$(\mathcal{M}, \nu) \models \varphi \leftrightarrow (\mathcal{M}, \mu) \models \varphi.$$

Proof. We proceed by induction on the length of φ. We now suppose that the length of φ is n and that the theorem holds for all $\mathcal{L}_{\mathcal{A}}$-formulas of length less than n. To be precise, we can assume the following.

Induction hypothesis. *For all $\psi(\vec{x})$, ν_1, and μ_1, if $\psi(\vec{x})$ is an $\mathcal{L}_{\mathcal{A}}$-formula of length less than the length of φ, ν_1 and μ_1 are $\mathcal{M}$-assignments such that $\nu_1|_{\bar{x}} = \mu_1|_{\bar{x}}$, then $(\mathcal{M}, \nu_1) \vDash \psi$ if and only if $(\mathcal{M}, \mu_1) \vDash \psi$.*

We begin with the case that φ is an atomic formula (in which case the induction hypothesis is vacuously true).

Atomic cases. Suppose φ is an atomic formula.

There are two further subcases.

Predicate Case: $\varphi = P_i(\vec{\tau})$.

Then every variable in φ is free. By hypothesis, $\overline{\nu}(\vec{\tau}) = \overline{\mu}(\vec{\tau})$. Thus

$$(\mathcal{M}, \nu) \vDash P_i(\vec{\tau}) \;\leftrightarrow\; \overline{\nu}(\vec{\tau}) \in I(P_i) \;\leftrightarrow\; \overline{\mu}(\vec{\tau}) \in I(P_i) \;\leftrightarrow\; (\mathcal{M}, \mu) \vDash P_i(\vec{\tau});$$

by definition, Lemma 3.9, and definition, respectively. It follows that

$$(\mathcal{M}, \nu) \vDash \varphi \;\leftrightarrow\; (\mathcal{M}, \mu) \vDash \varphi.$$

Equality Case: $\varphi = (\tau \mathrel{\hat{=}} \sigma)$, where σ and τ are terms.

Again, every variable in φ is free. By Lemma 3.9, $\overline{\nu}(\tau) = \overline{\mu}(\tau)$ and $\overline{\nu}(\sigma) = \overline{\mu}(\sigma)$. It follows from the definition of satisfaction as above that $(\mathcal{M}, \nu) \vDash (\tau \mathrel{\hat{=}} \sigma)$ if and only if $(\mathcal{M}, \mu) \vDash (\tau \mathrel{\hat{=}} \sigma)$.

Inductive cases. There are three inductive cases, negations, implications, and quantifications.

Negation: $\varphi = (\neg\psi)$.

Then φ and ψ have the same free variables, and so we may apply the induction hypotheses as follows.

$$\begin{aligned}
(\mathcal{M}, \nu) \vDash \varphi \;&\leftrightarrow\; (\mathcal{M}, \nu) \nvDash \psi && \text{(by definition)}\\
&\leftrightarrow\; (\mathcal{M}, \mu) \nvDash \psi && \text{(by induction)}\\
&\leftrightarrow\; (\mathcal{M}, \mu) \vDash \varphi. && \text{(by definition)}
\end{aligned}$$

Implication: $\varphi = (\psi_1 \to \psi_2)$.

In this case, the free variables of φ are the free variables of ψ_1 or ψ_2, and by the induction hypothesis, $(\mathcal{M}, \nu) \vDash \psi_i$ if and only if $(\mathcal{M}, \mu) \vDash \psi_i$.

By definition, $(\mathcal{M}, \nu) \vDash \varphi$ if and only if, either $(\mathcal{M}, \nu) \nvDash \psi_1$ or $(\mathcal{M}, \nu) \vDash \psi_2$, and similarly for μ. By the induction hypothesis, $(\mathcal{M}, \nu) \vDash \varphi$ if and only if $(\mathcal{M}, \mu) \vDash \varphi$.

Quantification: $\varphi = (\forall x_i \psi)$.

By assumption, $\mu|_{\bar{x}} = \nu|_{\bar{x}}$. By definition, $(\mathcal{M}, \nu) \vDash \varphi$ if and only if for all $\mathcal{M}$-assignments ρ, $\rho|_{\bar{x}} = \nu|_{\bar{x}}$ implies $(\mathcal{M}, \rho) \vDash \psi$, and similarly for μ. Therefore (trivially) for all $\mathcal{M}$-assignments ρ, $\rho|_{\bar{x}} = \mu|_{\bar{x}} \;\leftrightarrow\; \rho|_{\bar{x}} = \nu|_{\bar{x}}$. But then, $(\mathcal{M}, \nu) \vDash \varphi \;\leftrightarrow\; (\mathcal{M}, \mu) \vDash \varphi$

This completes the proof of the theorem. □

Lemma 3.13 *If φ is a sentence, the satisfaction relation is independent of $\mathcal{M}$-assignments.*

Proof. This follows from from Lemma 3.9 and Theorem 3.12. If φ is a sentence, then satisfiability is determined solely by I, and not any ν, as φ has no free variables. □

Lemma 3.13 allows for two notational conventions which we shall frequently use from this point on. Suppose that

$$\mathcal{M} = (M, I)$$

is an $\mathcal{L}_\mathcal{A}$-structure and that φ is an $\mathcal{L}_\mathcal{A}$-formula.

First, if φ is a sentence then we write either

- $\mathcal{M} \vDash \varphi$.
- $(M, I) \vDash \varphi$;

to indicate $(\mathcal{M}, \nu) \vDash \varphi$ for some (equivalently, any) $\mathcal{M}$-assignment ν.

Second, we write

$$\mathcal{M} \vDash \varphi[a_0, \ldots, a_n]$$

to indicate the following.

- The free variables of φ are included in the set $\{x_i \mid i \leq n\}$.
- $\{a_i \mid i \leq n\} \subseteq M$.
- Suppose ν is an $\mathcal{M}$-assignment such that $\nu(x_i) = a_i$ for all $i \leq n$. Then

$$(\mathcal{M}, \nu) \vDash \varphi.$$

That latter differs slightly from our convention that $\varphi(x_{n_0}, \ldots, x_{n_k})$ indicates that φ is a formula with all its free variables included in the set $\{x_{n_i} \mid i \leq k\}$.

The reason of course is that this is unambiguous whereas if, for example, the only free variable of φ is say x_5 and $a \in M$, then the notation

$$\mathcal{M} \vDash \varphi[a]$$

is potentially ambiguous; it could indicate

$$(\mathcal{M}, \nu) \vDash \varphi$$

for any (or some) $\mathcal{M}$-assignment ν such that $\nu(x_5) = a$, or it could indicate

$$(\mathcal{M}, \nu) \vDash \varphi$$

for any (or some) $\mathcal{M}$-assignment ν such that $\nu(x_0) = a$.

But in general these two assertions are very different assertions.

3.3 Substitution and the satisfaction relation

This section formalizes the relationship between the syntactic action of substitution with the semantic notion of satisfaction. We shall use the following notation. Suppose $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and ν is an $\mathcal{M}$-assignment.

Definition 3.14 (1) Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, $\tau = \tau(\bar{x})$ is an $\mathcal{L}_{\mathcal{A}}$-term, and that

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}.$$

Suppose $\vec{a} = \langle a_0, \ldots, a_k \rangle$ is a sequence of (not necessarily distinct) elements of M. Then

$$\tau[\vec{a}]$$

indicates $\overline{\nu}(\tau)$, where ν is any $\mathcal{M}$-assignment such that $\nu(x_{n_i}) = a_i$, for $i = 0, \ldots, k$.

(2) Suppose $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, $\varphi = \varphi(\bar{x})$ is an $\mathcal{L}_{\mathcal{A}}$-formula, and that

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}.$$

Suppose $\vec{a} = \langle a_0, \ldots, a_k \rangle$ is a sequence of (not necessarily distinct) elements of M. Then

$$\mathcal{M} \vDash \varphi[\vec{a}]$$

indicates that

$$(\mathcal{M}, \nu) \vDash \varphi,$$

where ν is any $\mathcal{M}$-assignment such that $\nu(x_{n_i}) = a_i$, for $i = 0, \ldots, k$.

The definitions of $\tau[\vec{a}]$ and of the relation $\mathcal{M} \vDash \varphi[\vec{a}]$ given above are well defined, as by Lemma 3.5 and Theorem 3.7, they depend only on $\nu|_{\bar{x}}$.

Remark 3.15 Strictly speaking, the relation

$$\mathcal{M} \vDash \varphi[\vec{a}]$$

depends on the pair $(\varphi, \bar{x})$. This differes from writing

$$\mathcal{M} \vDash \varphi[a_0, \ldots, a_n]$$

which by our convention (see page 48) is really

$$\mathcal{M} \vDash \varphi[\vec{a}]$$

where $\vec{a} = \langle a_0, \ldots, a_n \rangle$ but only in the context of the pair $(\varphi, \bar{x})$ where

$$\bar{x} = \{x_0, \ldots, x_n\}.$$

After sufficient experience with these notions, these kind of notational distinctions are usually ignored. But at this introductory stage, that seems unwise. □

Example 3.16 We can think of $\varphi(x_1;\tau)$ as saying that φ holds of τ. However, blind substitution can have unintended results. Consider the following formula φ.

$$\varphi = \varphi(x_1) = (\forall x_2(x_1 \hat{=} x_2))$$

Now, we substitute x_2 for x_1.

$$\varphi(x_1;x_2) = (\forall x_2(x_2 \hat{=} x_2))$$

There is a substantial difference between the two formulas. Every structure satisfies $\varphi(x_1;x_2)$, but for every structure $\mathcal{M}$ and every $a \in M$, $\mathcal{M} \models \varphi[a]$ if and only if $M = \{a\}$. □

Definition 3.17 Suppose φ is a formula, x_i is a free variable of φ, and τ is a term. The term τ is *free for x_i in φ* if for each variable x_j occurring in τ, no free occurrence of x_i in φ is within the scope of an occurrence of $\forall x_j$. □

The penultimate theorem of this chapter establishes the commutative nature of evaluating along with substituting.

Theorem 3.18 (Substitution) *Let $\mathcal{M} = (M, I)$ be an $\mathcal{L}_{\mathcal{A}}$-structure and ν be an $\mathcal{M}$-assignment.*

(1) *Suppose that $\tau = \tau(\bar{x})$ is an $\mathcal{L}_{\mathcal{A}}$-term and that*

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}.$$

Suppose $\vec{\sigma} = \langle \sigma_0, \ldots, \sigma_k \rangle$ is a sequence of (not necessarily distinct) $\mathcal{L}_{\mathcal{A}}$-terms. Then

$$\overline{\nu}(\tau(\bar{x};\vec{\sigma})) = \tau[\vec{b}],$$

where $\vec{b} = \langle \overline{\nu}(\sigma_0) \ldots \overline{\nu}(\sigma_k) \rangle$.

(2) *Suppose that $\varphi = \varphi(\bar{x})$ is an $\mathcal{L}_{\mathcal{A}}$-formula, and that*

$$\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}.$$

Suppose $\vec{\sigma} = \langle \sigma_0, \ldots, \sigma_k \rangle$ is a sequence of $\mathcal{L}_{\mathcal{A}}$-terms, possibly not all distinct, and that that σ_i is free for x_{n_i} for all $i \leq k$. Then

$$(\mathcal{M}, \nu) \models \varphi(\bar{x};\vec{\sigma}) \;\leftrightarrow\; \mathcal{M} \models \varphi[\vec{b}],$$

where $\vec{b} = \langle \overline{\nu}(\sigma_0) \ldots \overline{\nu}(\sigma_k) \rangle$.

Proof. The two parts are proven by induction on the lengths of τ and φ, respectively. We leave the proof of the first to the reader and present the proof of the second.

So, we assume that (1) holds and prove (2) by induction on the length of the formula φ.

Atomic Case: For every atomic formula φ, (2) follows directly from the definitions and (1).

Inductive Case: There are three cases to consider: negation, implication, and quantification.

Negation: $\varphi = (\neg\psi)$.

The free variables of φ are exactly the same as those of ψ, and for each $i \leq k$, σ_i is free for x_{n_i} in ψ. Thus,

$$\varphi(\bar{x};\vec{\sigma}) = (\neg\psi(\bar{x};\vec{\tau})).$$

By the induction hypothesis,

$$(\mathcal{M},\nu) \vDash \psi(\bar{x};\vec{\sigma}) \;\leftrightarrow\; \mathcal{M} \vDash \psi[\vec{b}].$$

But then by definition of the satisfaction of a negation,

$$(\mathcal{M},\nu) \vDash \varphi(\bar{x};\vec{\sigma}) \;\leftrightarrow\; \mathcal{M} \vDash \varphi[\vec{b}].$$

Implication: $\varphi = (\psi_1 \to \psi_2)$.

In this case, the free variables of φ are the free variables of ψ_1 or ψ_2, Further, $\varphi(\bar{x};\vec{\sigma}) = (\psi_1(\bar{x};\vec{\sigma}) \to \psi_2(\bar{x};\vec{\sigma}))$. We may apply the induction hypothesis to obtain the following equivalences.

$$(\mathcal{M},\nu) \vDash \psi_1(\bar{x};\vec{\sigma}) \;\leftrightarrow\; \mathcal{M} \vDash \psi_1[\vec{b}]$$
$$(\mathcal{M},\nu) \vDash \psi_2(\bar{x};\vec{\sigma}) \;\leftrightarrow\; \mathcal{M} \vDash \psi_2[\vec{b}]$$

By definition, $(\mathcal{M},\nu) \vDash \varphi(\bar{x};\vec{\sigma}) \;\leftrightarrow\;$ either $(\mathcal{M},\nu) \nvDash \psi_1(\bar{x};\vec{\sigma})$ or $(\mathcal{M},\nu) \vDash \psi_2(\bar{x};\vec{\sigma})$. Similarly, $\mathcal{M} \vDash \varphi[\vec{b}] \;\leftrightarrow\;$ either $\mathcal{M} \nvDash \psi_1[\vec{b}]$ or $\mathcal{M} \vDash \psi_2[\vec{b}]$.

Thus, we have the required equivalence:

$$(\mathcal{M},\nu) \vDash \varphi(\bar{x};\vec{\sigma}) \;\leftrightarrow\; \mathcal{M} \vDash \varphi[\vec{b}]$$

The final case inductive case is quantification.

Quantification: $\varphi = (\forall x_m \psi)$.

We can reduce to the case that x_m occurs in $\bar{x} = \{x_{n_0}, \ldots, x_{n_k}\}$ by just adding if necessary, x_m to the set $\bar{x}$. The reason of couse is that assumption in (2) is only that the free variables of φ are included in the set $\bar{x}$, not that every variable in the set $\bar{x}$ is a free variable of the formula φ.

Further we can further reduce to the case that $\sigma_i = x_m$ where $n_i = m$ in which case

$$\varphi(\bar{x};\vec{\sigma}) = (\forall x_m \psi(\bar{x};\vec{\sigma})).$$

We can do this because x_m is not a free variable of φ and so this change does not change either $\varphi(\bar{x};\vec{\sigma})$ or whether $\mathcal{M} \vDash \varphi[\vec{b}]$.

Note that for all $i \leq k$, if $n_i \neq m$ then no free occurrence of x_{n_i} in ψ is within the scope of any occurrence of $\forall x_j$ in ψ for any variable x_j which occurs in σ_i. This is because the same is true for φ by assumption. If $n_i = m$ then again (and now it is trivial since we have set $\sigma_i = x_m$) no free occurrence of x_{n_i} in ψ within the scope of any occurrence of $\forall x_j$ in ψ for any variable x_j which occurs in σ_i. Therefore, we can apply the induction hypothesis to $\psi(\bar{x}; \vec{\sigma})$ and we also have that:

$$\varphi(\bar{x}, \vec{\sigma}) = (\forall x_m \psi(\bar{x}, \vec{\sigma}).$$

By the basic theorem on page 46, Theorem 3.12, $(\mathcal{M}, \nu)$ satisfies $\varphi(\bar{x}; \vec{\sigma})$ if and only if Condition A holds.

Condition A. For all $\mathcal{M}$-assignments μ, if μ and ν agree on the free variables of $\varphi(\bar{x}; \vec{\sigma})$, then

$$(\mathcal{M}, \mu) \vDash \psi(\bar{x}; \vec{\sigma}).$$

Now, we can apply the inductive hypothesis in the conclusion of Condition A and see that it is equivalent to Condition B.

Condition B. For all $\mathcal{M}$-assignments μ, if μ and ν agree on the free variables of $\varphi(\bar{x}; \vec{\sigma})$, then

$$\mathcal{M} \vDash \psi[\vec{b}]$$

since necessarily $\vec{b} = \langle \overline{\mu}(\sigma_0), \ldots, \overline{\mu}(\sigma_k) \rangle$.

We next show **B** is equivalent to the following one, Condition C.

Condition C. For all $\mathcal{M}$-assignments ρ, if for each $i \leq k$ such that x_{n_i} appears freely in φ, $\rho(x_{n_i}) = \overline{\nu}(\sigma_i)$, then $(\mathcal{M}, \rho) \vDash \psi$

B $\rightarrow$ **C** : Suppose that ρ is an $\mathcal{M}$-assignment such that for each $i \leq k$, if x_{n_i} occurs freely in φ then $\rho(x_{n_i}) = \overline{\nu}(\sigma_i)$. Let μ_ρ be the $\mathcal{M}$-assignment defined as follows.

$$\mu_\rho(x_j) = \begin{cases} \nu(x_j), & \text{if } x_j \text{ occurs freely in } \varphi(\bar{x}; \vec{\sigma}); \\ \rho(x_j), & \text{otherwise.} \end{cases}$$

Thus μ_ρ and ν agree on the free variables of $\varphi(\bar{x}; \vec{\sigma})$. Therefore by **B**,

$$\mathcal{M} \vDash \psi[\vec{b}]$$

since necessarily $\vec{b} = \langle \overline{\mu_\rho}(\sigma_0), \ldots, \overline{\mu_\rho}(\sigma_k) \rangle$. For each $i \leq k$, σ_i is free for x_{n_i} in the formula φ. Thus for each $i \leq k$, if x_{n_i} is a free variable of φ then every variable occurring in σ_i is a free variable of $\varphi(\bar{x};\vec{\sigma})$. Therefore by (1), for each $i \leq k$, if x_{n_i} is a free variable of φ then

$$\overline{\mu_\rho}(\sigma_i) = \overline{\nu}(\sigma_i) = \rho(x_{n_i}).$$

Therefore since $\mathcal{M} \vDash \psi[\vec{b}]$,

$$(\mathcal{M}, \rho) \vDash \psi$$

as required.

C $\rightarrow$ **B** : Suppose that μ is an $\mathcal{M}$-assignment which agrees with ν on the free variables of $\varphi(\bar{x};\vec{\sigma})$. Define ρ_μ as follows.

$$\rho_\mu(x_j) = \begin{cases} \overline{\mu}(\sigma_i), & \text{if } j = n_i \text{ and } x_{n_i} \text{ occurs freely in } \varphi; \\ \mu(x_j), & \text{otherwise.} \end{cases}$$

By definition of ρ_μ, for each $i \leq k$, if x_{n_i} is a free variable of φ then

$$\rho_\mu(x_{n_i}) = \overline{\mu}(\sigma_{n_i}).$$

For each $i \leq k$, if x_{n_i} is a free variable of φ then, since σ_i is free for x_{n_i} in φ, every variable of σ_i is a free variable of $\psi(\bar{x};\vec{\sigma})$. Therefore for all $i \leq k$, if x_{n_i} is a free variable of φ then

$$\overline{\mu}(\sigma_i) = \overline{\nu}(\sigma_i),$$

since μ is an $\mathcal{M}$-assignment which agrees with ν on the free variables of $\varphi(\bar{x};\vec{\sigma})$, and hence on all the variables occurring in σ_i.

Therefore by **C**, $(\mathcal{M}, \rho_\mu) \vDash \psi$. Finally, for all $i \leq k$, if x_{n_i} is a free variable of φ or if $n_i = m$, then

$$\overline{\mu}(\sigma_i) = \overline{\nu}(\sigma_i);$$

Thus for all $i \leq k$, if x_{n_i} is a free variable of ψ then

$$\overline{\mu}(\sigma_i) = \overline{\nu}(\sigma_i);$$

Therefore by Theorem 3.12, necessarily $\mathcal{M} \vDash \psi[\vec{b}]$, as required.

This finishes the proof that **B** and **C** are equivalent.

By Theorem 3.12, and since

$$\vec{b} = \langle \overline{\nu}(\sigma_0) \ldots \overline{\nu}(\sigma_k) \rangle,$$

C is equivalent to $\mathcal{M} \vDash \varphi[\vec{b}]$. Therefore, we have the desired equivalence:

$$(\mathcal{M}, \nu) \vDash \varphi(\bar{x};\vec{\sigma}) \;\leftrightarrow\; \mathcal{M} \vDash \varphi[\vec{b}]$$

This completes the final case (and hence the proof). □

Finally we end this chapter with the theorem that connects the satisfaction relation with the interpretation of the constants. This theorem shows that by substituting variables for constants, truth in a structure under an assignment to variables can be reduced to ignoring the interpretation function of the structure on the constants. The precise formulation obscures the simple idea.

Theorem 3.19 is an easy consequence of Theorem 3.18.

Theorem 3.19 *Let $\mathcal{M} = (M, I)$ be an $\mathcal{L}_\mathcal{A}$-structure and ν be an $\mathcal{M}$-assignment. Suppose φ is an $\mathcal{L}_\mathcal{A}$ formula and $\langle c_{n_1}, \ldots, c_{n_k} \rangle$ is the increasing enumeration (by index) of the constants occurring in φ. Let m be large enough so that all the variables occurring in φ are contained in the set $\{x_i \mid i < m\}$.*

Let $\hat{\varphi}$ be the $\mathcal{L}_\mathcal{A}$-formula obtained by substituting x_{m+i} for all occurrences of c_{n_i} in φ, for all $i = 1, \ldots, k$.

Suppose $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and ν is an $\mathcal{M}$-assignment. Let $\hat{\nu}$ be the $\mathcal{M}$-assignment where

$$\hat{\nu}(x_j) = \begin{cases} \nu(x_j), & \text{if } x_j \neq x_{m+i} \text{ for any } i = 1, \ldots, k; \\ I(c_{n_i})), & \text{if } x_j = x_{m+i} \text{ for some } i = 1, \ldots, k. \end{cases}$$

Then $(\mathcal{M}, \nu) \vDash \varphi$ if and only if $(\mathcal{M}, \hat{\nu}) \vDash \hat{\varphi}$. □

Definition 3.20 Suppose that φ and ψ are $\mathcal{L}_\mathcal{A}$-formulas. Then *φ and ψ are logically equivalent* if and only if f for all $\mathcal{L}_\mathcal{A}$-structures $\mathcal{M}$ and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \vDash \varphi \ \leftrightarrow \ (\mathcal{M}, \nu) \vDash \psi$$ □

We are only concerned with formulas up to logical equivalence.

Lemma 3.21 *Suppose that φ is an $\mathcal{L}_\mathcal{A}$-formula. Then there is an $\mathcal{L}_\mathcal{A}$-formula ψ such that the following hold.*

(1) *φ and ψ are logically equivalent.*
(2) *φ and ψ have the same free variables.*
(3) *For each free variable x_i of ψ, ψ has no bound occurrences of x_i.*

Proof. Choose $n \in \mathbb{N}$ large enough such that all the variables which occur in φ are included in the set $\{x_0, \ldots, x_n\}$. Let ψ be the formula where for each $i \leq n$, every *bound* occurrence of x_i in φ is replaced by x_{n+1+i}. Then ψ has the required properties and the details are left to the exercises. □

Definition 3.22 Suppose that φ is an $\mathcal{L}_\mathcal{A}$-formula. Then *φ is logically valid* if and only if

$$(\mathcal{M}, \nu) \vDash \varphi$$

for all $\mathcal{L}_\mathcal{A}$-structures $\mathcal{M}$ and all $\mathcal{M}$-assignments ν. □

Suppose $\mathcal{A}$ is an alphabet which contains at least one function symbol, or which contains a predicate symbol P_i such that $\pi(P_i) > 1$. Then there is an $\mathcal{L}_\mathcal{A}$-formula φ such that φ is *not* logically valid but such that for all $\mathcal{L}_\mathcal{A}$-structures $\mathcal{M} = (M, I)$, if M is finite then

$$(\mathcal{M}, \nu) \vDash \varphi$$

for all $\mathcal{M}$-assignments ν. The case where $\mathcal{A}$ contains a function symbol is in the exercises below (problem 4).

This seems to suggest that the problem of determining whether or not an $\mathcal{L}_{\mathcal{A}}$-formula φ is logically valid is intrinsically a very hard problem.

3.3.1 Exercises

(1) Let $\mathcal{A}$ be an alphabet with one binary relation symbol. More precisely, suppose $\mathcal{A} = \{P_i\}$ for some $i \in \mathbb{N}$ such that $\pi(P_i) = 2$.
Give an example of an $\mathcal{L}_{\mathcal{A}}$-sentence φ and $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \vDash \varphi$ and $\mathcal{M}_2 \nvDash \varphi$, and such that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the **same** universe.

(2) Do there exist an alphabet $\mathcal{A}$, an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$, an $\mathcal{M}$-assignment ν, and an $\mathcal{L}_{\mathcal{A}}$-formula φ such that $(\mathcal{M}, \nu) \vDash \varphi$ and $(\mathcal{M}, \nu) \vDash (\neg\varphi)$? Do there exist such $\mathcal{M}$ and ν such that $(\mathcal{M}, \nu) \nvDash \varphi$ and $(\mathcal{M}, \nu) \nvDash (\neg\varphi)$?

(3) Suppose that $A_1, \ldots, A_n$ are propositional symbols, that θ is a propositional tautology, and that $\varphi_1, \ldots, \varphi_n$ are $\mathcal{L}_{\mathcal{A}}$-formulas. Let ψ be the result of substituting for each i, the formula φ_i for each occurrence of the propositional symbol A_i in θ. Prove that for every $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ and every $\mathcal{M}$-assignment ν, $(\mathcal{M}, \nu) \vDash \psi$.

(4) Suppose $\mathcal{A} = \{F_i, c_0\}$ and $\pi(F_i) = 1$.
Give an example of an $\mathcal{L}_{\mathcal{A}}$-formula φ such that all of the following conditions hold:
(a) φ is a sentence.
(b) There is at least one $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$.
(c) For all $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M}$, if $\mathcal{M} \vDash \varphi$, then the universe of $\mathcal{M}$ is infinite.

Hint: Suppose $F : M \to M$ is a function and M is finite. Then F is a surjection iff F is 1-to-1.

(5) Suppose that Δ is a set of $\mathcal{L}_{\mathcal{A}}$-formulas and c_0 is a constant symbol which belongs to $\mathcal{A}$ and that does not occur in any formula of $\Delta \cup \{\varphi\}$. Suppose that $\Delta \cup \{(\neg(\forall x_k(\neg\varphi)))\}$ is satisfiable. Show that $\Delta \cup \{\varphi(x_k; c_0)\}$ is satisfiable.

(6) Prove Lemma 3.21.

4

The logic of first order structures

4.1 Isomorphisms between structures

In mathematics, there is the notion of two structures, such as groups, topological spaces, etc., being isomorphic. This chapter explores that notion in full generality by identifying several notions, including isomorphism, which the perspective of mathematical logic naturally isolates.

Remarkably this more general perspective leads to new and fundamental insights to the mathematical properties of even the most classical of structures, that of the real numbers themselves.

We begin with the fundamental notion of isomorphism for $\mathcal{L}_\mathcal{A}$-structures. Notice that if $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure, then (trivially) $\mathcal{L}_\mathcal{A}$ is uniquely specified by the domain of I.

Our focus will be generally in the abstract context of an alphabet $\mathcal{A}$ as opposed to the case where $\mathcal{A}$ contains all the constant, function, and predicate symbols. The reason of course is that the familiar mathematical structures are almost always naturally an $\mathcal{L}_\mathcal{A}$-structure, for a finite alphabet $\mathcal{A}$.

Definition 4.1 Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_\mathcal{A}$-structures. A bijection $e : M \to N$ defines an *isomorphism* between $\mathcal{M}$ and $\mathcal{N}$ if and only if the following conditions hold.

(1) For each constant symbol c_i in the domain of I,

$$e(I(c_i)) = J(c_i).$$

(2) For each function symbol F_i in the domain of I, if $n = \pi(F_i)$ then for each $\langle a_1, \ldots, a_{n+1}\rangle \in M^{n+1}$,

$$\langle I(F_i)(a_1, \ldots, a_n) = a_{n+1} \;\leftrightarrow\; \langle J(F_i)(e(a_1), \ldots, e(a_n)) = e(a_{n+1}).$$

(3) For each predicate symbol P_i in the domain of I, for each $\vec{a} \in M^n$,

$$\langle a_1, \ldots, a_n\rangle \in I(P_i) \;\leftrightarrow\; \langle e(a_1), \ldots, e(a_n)\rangle \in J(P_i).$$

For simplicity, we can denote $e(\vec{a}) = e\langle a_1, \ldots, a_n\rangle = \langle e(a_1), \ldots, e(a_n)\rangle$

We write $\mathcal{M} \cong \mathcal{N}$ to indicate that M and N are isomorphic. When $\mathcal{M}$ is equal to $\mathcal{N}$, we say that e is an *automorphism*.

Remark 4.2 For any structure $\mathcal{M}$, the identity function $e : x \mapsto x$ is an example, though a trivial one, of an automorphism of $\mathcal{M}$. It follows directly from Definition 4.1 that the inverse of an isomorphism is also an isomorphism and that the composition of two isomorphisms is also an isomorphism. $\square$

Example 4.3 Consider this structure: the field of complex numbers,

$$\mathbf{C} = (\mathbb{C}, 0, 1, i, +, \times).$$

Recall multiplication and addition for 2 by 2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times_m \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \cdot e + b \cdot g & a \cdot f + b \cdot h \\ c \cdot e + d \cdot g & c \cdot f + d \cdot h \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} +_m \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a + e & b + f \\ c + g & d + h \end{pmatrix}$$

Here a, b, c, d, e, f, g are real numbers.

A fundamental result in complex analysis is that

$$\mathbf{C} \cong \left(\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, +_m, \times_m \right)$$

where $a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $+_m$ and $\cdot_m$ is matrix addition and multiplication.[1] $\square$

Lemma 4.4 *Suppose that $e : M \to N$ is an isomorphism of $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$. Suppose that ν is an $\mathcal{M}$-assignment. Then the composition of e and ν, $e \circ \nu$, is an $\mathcal{N}$-assignment, and for each $\mathcal{L}_{\mathcal{A}}$-term τ,*

$$\bar{\nu}(\tau) = \overline{e \circ \nu}(\tau)$$

Proof. As $e \circ \nu$ maps each variable symbol x_i to an element of M via ν and then maps that element of M to an element of N via $e, e \circ \nu$ is a $\mathcal{N}-$assignment.

The proof of the lemma is by induction on the length of the term τ. For the terms of length 1:

Constants - From the definition of an isomorphism,

$$\begin{aligned} e(\overline{\nu}(\langle c_i \rangle)) &= e(I(c_i)) \\ &= J(c_i) \qquad \text{(since } e \text{ is an isomorphism)} \\ &= \overline{e \circ \nu}(\langle c_i \rangle). \end{aligned}$$

Variables - Straight from the definitions,

[1] Ash, Novinger; *Complex Variables* Ch.1

$$\begin{aligned} e(\overline{\nu}(\langle x_i \rangle)) &= e(\nu(x_i)) \\ &= e \circ \nu(x_i) \\ &= \overline{e \circ \nu}(\langle x_i \rangle). \end{aligned}$$

We now assume τ has length longer that 1 and that the lemma holds for all terms σ of length less than the length of τ.

Since $|\tau| > 1$, $\tau = F_i(\vec{\sigma})$ for some function symbol F_i.

$$\begin{aligned} e(\overline{\nu}(F_i(\vec{\sigma}))) &= e(I(F_i)(\overline{\nu}(\vec{\sigma}))) \\ &= J(F_i)(e(\overline{\nu}(\vec{\sigma})) \qquad \text{(since } e \text{ is an isomorphism)} \\ &= J(F_i)(\overline{e \circ \nu}(\vec{\sigma})) \qquad \text{(by induction)} \\ &= \overline{e \circ \nu}(F_i(\vec{\sigma})) \end{aligned}$$

□

Theorem 4.5 *Suppose that $e : M \to N$ is an isomorphism of $\mathcal{L}_\mathcal{A}$-structures $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$. Then for each $\mathcal{L}_\mathcal{A}$-formula φ,*

$$(\mathcal{M}, \nu) \vDash \varphi \;\leftrightarrow\; (\mathcal{N}, e \circ \nu) \vDash \varphi.$$

Proof. The proof is by induction on the length of formulas for all $\mathcal{M}$-assignments ν.

We first suppose φ is an Atomic Formula.

Predicate Case: Suppose $\varphi = P_i(\vec{\tau})$. Then for all $\mathcal{M}$-assignments ν:

$$\begin{aligned} &(\mathcal{M}, \nu) \vDash P_i(\vec{\tau}) \;\leftrightarrow \\ &\quad\leftrightarrow\; \overline{\nu}(\vec{\tau}) \in I(P_i) \qquad \text{(by definition)} \\ &\quad\leftrightarrow\; e(\overline{\nu}(\vec{\tau})) \in J(P_i) \qquad \text{(since } e \text{ is an isomorphism)} \\ &\quad\leftrightarrow\; \overline{e \circ \nu}(\vec{\tau}) \in J(P_i) \qquad \text{(by the observation on terms)} \\ &\quad\leftrightarrow\; (\mathcal{N}, e \circ \nu) \vDash P_i(\vec{\tau}) \end{aligned}$$

Equality Case: Suppose $\varphi = (\tau_1 \hat{=} \tau_2)$. We use the fact that an isomorphism is injective.

$$\begin{aligned} (\mathcal{M}, \nu) \vDash (\tau_1 \hat{=} \tau_2) \;&\leftrightarrow\; \overline{\nu}(\tau_1) = \overline{\nu}(\tau_2) && \text{(by definition)} \\ &\leftrightarrow\; e(\overline{\nu}(\tau_1)) = e(\overline{\nu}(\tau_2)) && \text{(as } e \text{ is injective)} \\ &\leftrightarrow\; \overline{e \circ \nu}(\tau_1) = \overline{e \circ \nu}(\tau_2) && \text{(as above)} \\ &\leftrightarrow\; (\mathcal{N}, e \circ \nu) \vDash (\tau_1 \hat{=} \tau_2) && \text{(by definition)} \end{aligned}$$

Now, we suppose that φ is not an Atomic Formula, and that the theorem holds for all formulas ψ such that $|\psi| < |phi|$, for all $\mathcal{M}$-assignments μ. Suppose μ is an $\mathcal{M}$-assignments .

Negation Case: Straight from the definition,

$$\begin{aligned}
(\mathcal{M},\nu) \models (\neg\psi) &\leftrightarrow (\mathcal{M},\nu) \not\models \psi && \text{(by definition)}\\
&\leftrightarrow (\mathcal{N}, e\circ\nu) \not\models \psi && \text{(by induction)}\\
&\leftrightarrow (\mathcal{N}, e\circ\nu) \models (\neg\psi) && \text{(by definition)}
\end{aligned}$$

The analysis of implication is similar.

Implication Case: Straight from the definition,

$$\begin{aligned}
(\mathcal{M},\nu) \models (\psi_1 \to \psi_2) &\leftrightarrow (\mathcal{M},\nu) \not\models \psi_1 \text{ or } (\mathcal{M},\nu) \models \psi_2 && \text{(by definition)}\\
&\leftrightarrow (\mathcal{N}, e\circ\nu) \not\models \psi_1 \text{ or } (\mathcal{N}, e\circ\nu) \models \psi_2 && \text{(by induction)}\\
&\leftrightarrow (\mathcal{N}, e\circ\nu) \models (\psi_1 \to \psi_2) && \text{(by definition)}
\end{aligned}$$

Quantification Case: Suppose $\varphi(\vec{x}) = (\forall x_i \psi)$. By definition,

$$(\mathcal{M},\nu) \models (\forall x_i \psi)$$

if and only if for every $\mathcal{M}$-assignment μ, if $\nu|_{\bar{x}} = \mu|_{\bar{x}}$ then

$$(\mathcal{M},\mu) \models \psi.$$

Since e is surjective, for every $\mathcal{N}$-assignment μ^* where $\mu^*|_{\bar{x}} = e\circ\mu|_{\bar{x}}$, there exists an $\mathcal{M}$-assignment μ such that $\mu|_{\bar{x}} = \nu|_{\bar{x}}$ and $e\circ\mu = \mu^*$.

By induction, for each such μ and μ^*,

$$(\mathcal{M},\mu) \models \psi \leftrightarrow (\mathcal{N},\mu^*) \models \psi.$$

Thus, if $(\mathcal{M},\nu) \models (\forall x_i \psi)$, then $(\mathcal{N}, e\circ\nu) \models (\forall x_i\psi)$. The same argument shows

$$(\mathcal{M},\nu) \not\models (\forall x_i \psi) \leftrightarrow (\mathcal{M}, e\circ\nu) \not\models \psi$$

and this completes the proof. □

Theorem 4.5 suggests the following definition.

Definition 4.6 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\mathcal{A}}$-structures. Then $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if and only if for each $\mathcal{L}_{\mathcal{A}}$-sentence φ

$$\mathcal{M} \models \varphi \leftrightarrow \mathcal{N} \models \varphi.$$

We write $\mathcal{M} \equiv \mathcal{N}$ to indicate that the structures $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent. □

By Theorem 4.5, if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\mathcal{A}}$-structures then $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent.

4.1.1 Exercises

(1) Suppose $\mathcal{A} = \{F_1\}$ and $\pi(F_i) = 1$. Give an example of an infinite $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M} = (M, I)$ for each of the following properties.
   (a) $\mathcal{M}$ has no nontrivial automorphisms.
   (b) $\mathcal{M}$ has a countably infinite set of automorphisms.
   (c) For each element a of M there are only finitely many b's in M such that there is an automorphism f of $\mathcal{M}$ with $f(a) = b$. However, there are uncountably many automorphisms of $\mathcal{M}$.

(2) Characterize the collection of automorphisms of the integers $\mathbb{Z}$ with the binary relation $<$.

(3) Suppose that $\mathcal{A}$ is finite and that $\mathcal{M}$ is a finite $\mathcal{L}_{\mathcal{A}}$-structure. Prove that there is an $\mathcal{L}_{\mathcal{A}}$-sentence φ such that for every $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{N}$, if $\mathcal{N} \models \varphi$ then $\mathcal{N} \cong \mathcal{M}$.

(4) Suppose that $\mathcal{M}$ and $\mathcal{N}$ are **finite** $\mathcal{L}_{\mathcal{A}}$-structures. Prove that the following are equivalent.
   (a) $\mathcal{M} \cong \mathcal{N}$
   (b) $\mathcal{M} \equiv \mathcal{N}$

   Hint: First assume $\mathcal{A}$ is also finite.

4.2 Substructures and elementary substructures

Definition 4.7 Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures. $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if and only if $M \subseteq N$ and the following conditions hold.

(1) If c_i is a constant symbol of $\mathcal{L}_{\mathcal{A}}$ then $I(c_i) = J(c_i)$.

(2) If F_i is a function symbol of $\mathcal{L}_{\mathcal{A}}$ with $n = \pi(F_i)$, then $I(F_i)$ is the restriction of $J(F_i)$ to M^n.

(3) If P_i is a predicate symbol of $\mathcal{L}_{\mathcal{A}}$ with $n = \pi(P_i)$, then $I(P_i)$ is equal to $J(P_i) \cap M^n$.

We will write $\mathcal{M} \subseteq \mathcal{N}$ to indicate that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Note that we do not allow ourselves to discard constants, predicates, and functions. The only subset we take is of the universe. $\square$

Theorem 4.8 *Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures with $M \subseteq N$. Then the following are equivalent.*

(1) *$\mathcal{M}$ is a substructure of $\mathcal{N}$.*

(2) *For all atomic $\mathcal{L}_{\mathcal{A}}$-formulas, φ and for all $\mathcal{M}$-assignments ν,*

$$(\mathcal{M}, \nu) \models \varphi \;\leftrightarrow\; (\mathcal{N}, \nu) \models \varphi.$$

Proof. The theorem follows directly from the definitions. Notice the specification of *atomic formulas* in statement (2). $\square$

The equivalence given in Theorem 4.8 suggests the following definition.

Definition 4.9 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_\mathcal{A}$-structures with $\mathcal{M} \subseteq \mathcal{N}$. $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, if and only if for all $\mathcal{L}_\mathcal{A}$-formulas φ and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi \;\leftrightarrow\; (\mathcal{N}, \nu) \models \varphi.$$

We write $\mathcal{M} \preceq \mathcal{N}$ to indicate that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$. Notice the specification of *all* $\mathcal{L}_\mathcal{A}$*-formulas* in the definition. □

Example 4.10 Consider these $\mathcal{L}_\mathcal{A}$ structures:

- $\mathcal{N} = (\mathbb{Q}, 0, 1, +, \times, \leq)$,
- $\mathcal{M} = (\mathbb{R}, 0, 1, +, \times, \leq)$.

While $\mathbb{Q} \subseteq \mathbb{R}$ as subsets and substructures, $\mathbb{Q} \not\preceq \mathbb{R}$. Take this sentence:

$$\varphi = (\forall x_0 (x_0 \times x_0 \neq 1 + 1))$$

Note that $\mathcal{N} \models \varphi$ but $\mathcal{M} \not\models \varphi$. □

4.2.1 Exercises

(1) Let $\mathcal{A} = \emptyset$ and let $\mathcal{N}$ be the $\mathcal{L}_\mathcal{A}$ structure whose universe is $\mathbb{N}$, the natural numbers. Show that for every infinite subset S of $\mathbb{N}$, the $\mathcal{L}_\mathcal{A}$-structure with universe S is an elementary substructure of $\mathcal{N}$.

(2) Show that the integers $\mathbb{Z}$ is a substructure, but not an elementary substructure, of $\mathbb{Q}$ as described in Example 4.10.

4.3 Definable sets and Tarski's Criterion

Suppose that $\mathcal{N}$ is an $\mathcal{L}_\mathcal{A}$-structure. The problem of constructing elementary substructures of $\mathcal{N}$ looks difficult because the criterion for success involves truth in the substructure to be constructed and, in particular, anticipating quantification over the whole substructure while still in the process of its construction.

Tarski's Theorem below gives an elegant characterization using definable sets of when a substructure of $\mathcal{N}$ is an elementary substructure.

Definition 4.11 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure.

(1) Suppose that $X \subseteq M$. A set $Y \subseteq M$ is *definable in* $\mathcal{M}$ *with parameters from* X if and only if there are elements $b_1, \ldots, b_m$ of X and an $\mathcal{L}_\mathcal{A}$-formula $\varphi(x_0, \ldots, x_m)$ such that Y is the set of all $b \in M$ such that $\mathcal{M} \models \varphi[b, b_1, \ldots, b_m]$.

(2) A set $Y \subseteq M$ is *definable* in $\mathcal{M}$ without parameters if and only if it is definable with parameters from $\emptyset$. □

One can naturally generalize and define when sets $Y \subset M^{n+1}$ are definable with or without parameters in $\mathcal{M}$. But we shall be mostly interested in the case where n is 0, as is the case above (identifying M^1 with M as usual).

Definition 4.12 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and $n < \omega$.

(1) Suppose that $X \subseteq M$. A set $Y \subseteq M^{n+1}$ is *definable in* $\mathcal{M}$ *with parameters from* X if and only if there are elements $b_1, \ldots, b_m$ of X and an $\mathcal{L}_\mathcal{A}$-formula $\varphi(x_0, \ldots, x_n, \ldots, x_{m+1})$ such that Y is the set of all $\vec{a} \in M^{n+1}$ such that $\mathcal{M} \models \varphi[a_0, \ldots, a_n, b_1, \ldots, b_m]$.

(2) A set $Y \subseteq M^{n+1}$ is *definable* in $\mathcal{M}$ without parameters if and only if it is definable with parameters from $\emptyset$. □

Example 4.13 Consider the structure of the natural numbers with divisibility. That is $(\mathbb{N}, 1, |)$, where $a|b$ if and only if a divides b. The set of primes, P, is definable without parameters, as

$$P = \{n \in \mathbb{N} : (\forall x(x|n \leftrightarrow ((x = n \vee x = 1) \wedge n \neq 1)))\}$$

Likewise, taking P a set of parameters. For $q \in P$, the set $\{n \in \mathbb{N} : q|n\}$, which is the set of all natural numbers divisible by q, is definable with parameters from P. □

The following lemma gives the more traditional version of the definition when $Y \subset M$ is definable in $\mathcal{M}$ with parameters from X. There is the analogous variation (also more traditional) of defining when $Y \subset M^{n+1}$ is definable in $\mathcal{M}$, with parameters from X.

Lemma 4.14 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and that $X \subseteq M$. Suppose that $Y \subseteq M$ and that for some $\mathcal{L}_\mathcal{A}$-formula $\varphi(x_{k_0}, \ldots, x_{k_n})$ and for some $a_1, \ldots, a_n$ in X, Y is the set of all $a \in M$ such that $(\mathcal{M}, \mu) \models \varphi$ for some M-assignment μ such that*

(1) $\mu(x_{k_0}) = a$

(2) $\mu(x_{k_i}) = a_i$ *for all* $i = 1, \ldots, n$.

Then Y is definable in $\mathcal{M}$ with parameters from X.

Proof. We will leave the proof of Lemma 4.14 to the Exercises. □

Theorem 4.15 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and that $X \subseteq M$. Suppose that $n \in \mathbb{N}$, $Y \subset M^{n+1}$ is definable in $\mathcal{M}$ with parameters from X and that $e : M \to M$ is an automorphism of $\mathcal{M}$.*

If for each $b \in X$, $e(b) = b$, then

$$Y = \{\langle e(a_0), \ldots, e(a_n)\rangle : \langle a_0, \ldots, a_n\rangle \in Y\}.$$

Proof. Let $b_0, \ldots, b_m$ be elements of X and let $\varphi = \varphi(x_0, \ldots, x_{n+m+2})$ be an $\mathcal{L}_\mathcal{A}$-formula such that for all $a_0, \ldots, a_n$ in M,

$$\langle a_0, \ldots, a_n\rangle \in Y \leftrightarrow \mathcal{M} \models \varphi[a_0, \ldots, a_n, b_0, \ldots, b_m].$$

Recall our notation: $\vec{a} = \langle a_0, \ldots, a_n\rangle, \vec{a} + \vec{b} = \langle a_0, \ldots, a_n, b_0, \ldots, b_m\rangle$.

Suppose that $\vec{a} \in Y$, then we can apply Theorem 4.5 to conclude that $\mathcal{M} \models \varphi[e(\vec{a} + \vec{b})]$. Since X is a set of fixed points by e, $\mathcal{M} \models \varphi[e(\vec{a}) + \vec{b}]$ and so $e(\vec{a}) \in Y$. Conversely, suppose that $\vec{c} \in Y$, that is $\mathcal{M} \models \varphi[\vec{c} + \vec{b}]$. Since e is an isomorphism, e is surjective. Let $\vec{a}$ be a sequence of elements in M such that $|\vec{a}| = |\vec{c}|$ and $e(a_i) = c_i$. Consequently, $\mathcal{M} \models \varphi[e(\vec{a}) + \vec{b}]$. Applying Theorem 4.5 in the other direction, $\mathcal{M} \models \varphi[\vec{a} + \vec{b}]$ and so $\vec{a} \in Y$. Thus, there is a sequence $\vec{a} \in Y$ such that $\vec{c} = e(\vec{a})$, as required.

Remark 4.16 Definability within a structure is one of the central concepts in Mathematical Logic. In the next section, we shall consider the problem of classifying the definable sets of various specific structures. In many cases, the analysis requires that careful attention be paid to parameters. □

Example 4.17 Suppose that $\mathcal{A} = \emptyset$, so that $\mathcal{L}_{\mathcal{A}}$ is the trivial language. Suppose that M is a nonempty set. Then $\mathcal{M} = (M, \emptyset)$ is an $\mathcal{L}_{\mathcal{A}}$-structure. Further *any* bijection $e : M \to M$ defines an isomorphism of $\mathcal{M}$ to $\mathcal{M}$. We can use Theorem 4.5 to prove the following.

(1) Suppose that $A \subset M$. Then A is definable in $\mathcal{M}$ without parameters if and only if $A = \emptyset$ or $A = M$.
(2) Suppose that $A \subset M$. Then A is definable in $\mathcal{M}$ from parameters if and only if A is finite or $M \setminus A$ is finite.

To verify the first claim, suppose that φ is an $\mathcal{L}_{\mathcal{A}}$-formula and x_1 is the only free variable in φ. If there is no m in $\mathcal{M}$ such that $\mathcal{M} \models \varphi[m]$, then φ defines $\emptyset$ in $\mathcal{M}$. Otherwise, suppose that $m \in M$ and $\mathcal{M} \models \varphi[m]$. If n is another element of M, then the function e from M to M obtained by transposing m and n is a bijection from M to M, and therefore an isomorphism from $\mathcal{M}$ to $\mathcal{M}$. By Theorem 4.5, since $\mathcal{M} \models \varphi[m]$ we also have $\mathcal{M} \models \varphi[e(m)]$, that is $\mathcal{M} \models \varphi[n]$. Consequently, if φ defines a nonempty set, then that set is all of M.

We leave the proof of the second claim to the Exercises. □

Theorem 4.18 (Tarski) *Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures, and $\mathcal{M}$ is a substructure of $\mathcal{N}$. The following are equivalent.*

(1) $\mathcal{M} \preceq \mathcal{N}$
(2) *$\mathcal{M} \subseteq \mathcal{N}$ and for each nonempty set $A \subseteq N$, if A is definable in $\mathcal{N}$ with parameters from M, then $A \cap M \neq \emptyset$.*

Proof. **(1) → (2)**

Fix a nonempty set $A \subseteq N$ such that A is definable in $\mathcal{N}$ with parameters from M.

By Lemma 4.14, there exists an $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(x_0, \ldots, x_{n+1})$ and a finite sequence

$$\vec{b} = \langle b_0, \ldots, b_n \rangle$$

of elements of M such that for all $a \in N$,

$$a \in A \leftrightarrow \mathcal{N} \models \varphi[\langle a\rangle + \vec{b}].$$

Since A is not empty, $\mathcal{N} \not\models (\forall x_0(\neg\varphi))[\vec{b}]$. Since $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$,

$$\mathcal{M} \not\models (\forall x_0(\neg\varphi))[\vec{b}].$$

Fix b in M so that $\mathcal{M} \models \varphi[\langle b\rangle + \vec{b}]$. Since $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$,

$$\mathcal{N} \models \varphi[\langle b\rangle + \vec{b}]$$

and so b is an element of A. Thus $A \cap M \neq \emptyset$, as required.

(2) → (1)

We prove by induction on the length of formulas φ that for all $\mathcal{M}$-assignments ν, $(\mathcal{M}, \nu) \models \varphi$ if and only if $(\mathcal{N}, \nu) \models \varphi$.

The atomic cases follow from Theorem 4.8, and the propositional cases follow directly from the inductive hypothesis.

Therefore we can reduce to inductive step and further reduce to the case when $\varphi(\vec{x}) = (\forall x_i \psi)$.

Note that since $M \subseteq N$, every $\mathcal{M}$-assignment is also an $\mathcal{N}$-assignment.

Suppose that ν is an $\mathcal{M}$-assignment and let $\bar{x}$ be the set of free variables of $(\forall x_i \psi)$.

Case 1: $(\mathcal{N}, \nu) \models (\forall x_i \psi)$.

Thus for every $\mathcal{N}$-assignment (and, in particular, every $\mathcal{M}$-assignment) μ where $\nu|_{\bar{x}} = \mu|_{\bar{x}}$, $(\mathcal{N}, \mu) \models \psi$.

By induction, for these $\mathcal{M}$-assignments,

$$(\mathcal{N}, \mu) \models \psi \leftrightarrow (\mathcal{M}, \mu) \models \psi.$$

Consequently, for every $\mathcal{M}$-assignment μ where $\nu|_{\bar{x}} = \mu|_{\bar{x}}$,

$$(\mathcal{M}, \mu) \models \psi.$$

Thus $(\mathcal{M}, \nu) \models (\forall x_i \psi)$ as required.

Case 2: $(\mathcal{N}, \nu) \not\models (\forall x_i \psi)$.

Thus there is an $\mathcal{N}$-assignment μ such that

(1.1) $\nu|_{\bar{x}} = \mu|_{\bar{x}}$,
(1.2) $(\mathcal{N}, \mu) \not\models \psi$.

Let $\bar{x} = \{x_{k_0}, \ldots, x_{k_m}\}$ be the set of all of the free variables of $(\forall x_i \psi)$, and of course $\bar{x}$ could be the emptyset. Note that $x_i \notin \bar{x}$.

Let Y be the set of $a \in N$ such that there exists an $\mathcal{N}$-assignment ρ such that

(2.1) $\rho(x_i) = a$,
(2.2) for each $i \leq m$, $\rho(x_{k_i}) = \mu(x_{k_i}) = \nu(x_{k_i})$,

(2.3) $(\mathcal{N}, \rho) \models (\neg\psi)$.

Since $(\mathcal{N}, \mu) \not\models \psi$, necessarily

$$(\mathcal{N}, \mu) \models (\neg\psi).$$

Thus $\mu(x_i)$ is an element of Y and so Y is not empty.

By Lemma 4.14, Y is definable in $\mathcal{N}$ with parameters from M, and so by (2), $Y \cap M$ is not empty. Let $b \in Y \cap M$. Therefore if ρ is an $\mathcal{M}$-assignment such that $\rho|_{\bar{x}} = \mu|_{\bar{x}}$ and $\rho(x_i) = b$, then $(\mathcal{N}, \rho) \models \neg\psi$.

By the induction hypothesis, $(\mathcal{M}, \rho) \models \neg\psi$ and so

$$(\mathcal{M}, \nu) \not\models (\forall x_1 \psi).$$

In summary, we have shown that:

If $(\mathcal{N}, \nu) \not\models (\forall x_1 \psi)$ then $(\mathcal{M}, \nu) \not\models (\forall x_1 \psi)$.

Equivalently, we have shown that:

If $(\mathcal{M}, \nu) \models (\forall x_1 \psi)$ then $(\mathcal{N}, \nu) \models (\forall x_1 \psi)$.

By Case 1:

If $(\mathcal{N}, \nu) \models (\forall x_1 \psi)$ then $(\mathcal{M}, \nu) \models (\forall x_1 \psi)$.

Therefore by Case 1 and Case 2:

$(\mathcal{N}, \nu) \models (\forall x_1 \psi)$ if and only $(\mathcal{M}, \nu) \models (\forall x_1 \psi)$.

This proves (2) implies (1). □

4.3.1 Exercises

(1) Let $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times)$. Show that if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.

(2) Suppose $\mathcal{A} = \{P_i\}$ and that $\pi(P_i) = 1$,
Let $\mathcal{M}$ be the finite $\mathcal{L}_{\mathcal{A}}$-structure (M, I) such that $M = \{a, b, c, d, e\}$ and $I(P) = \{a, b\}$. In other words, $\mathcal{M}$ interprets P as holding of a and b and as not holding of c, d, or e.
  (a) Which subsets of M which are definable in $\mathcal{M}$ without parameters?
  (b) Which subsets of M are are definable in $\mathcal{M}$ with parameters?

(3) Suppose $\mathcal{M} \subseteq \mathcal{N}$ are infinite $\mathcal{L}_{\emptyset}$ structures. Show that $\mathcal{M} \preceq \mathcal{N}$

(4) Prove Lemma 4.14.

(5) Prove the second claim of Example 4.17.

(6) Suppose $\mathcal{A} = \{F_i\}$ and that $\pi(F_i) = 1$.
Give an example of infinite $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M} = (M, I)$ such that the only nonempty subset of M which is definable in $\mathcal{M}$ without parameters is M.

(7) Suppose $\mathcal{A} = \{F_i\}$ and $\pi(F_i) = 1$.
Give an example of infinite $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M} = (M, I)$ such that every finite subset of M is definable in $\mathcal{M}$ without parameters.

4.4 The Lowenheim-Skolem Theorem

The (Downward) Lowenheim-Skolem Theorem is an important application of Tarski's Theorem. However, we need to introduce some new definitions and concepts to fully understand the statement.

4.4.1 Countable Sets

Definition 4.19 A set A is *countable* if either it is empty or there is a surjective map from $\mathbb{N}$ to A. $\square$

In particular, every finite set is countable, and every subset of $\mathbb{N}$ is countable. Intuitively, the countable sets are those sets whose size is less than or equal to the size of $\mathbb{N}$.

Theorem 4.20 *Suppose that $\langle A_i : i \in \mathbb{N}\rangle$ is a countable sequence of countable sets. Then $A = \cup\{A_i \mid i \in \mathbb{N}\}$ is a countable set.*

Proof. Clearly we can reduce to the case that each set A_i is nonempty (by replacing A_i with $\mathbb{N}$ for each i such that A_i is empty).

Fix a sequence of functions, so that for each i, f_i is a surjection from $\mathbb{N}$ to A_i.

A side remark. Here we must appeal to the Axiom of Choice (AC). AC is the assertion that if X is a set of nonempty sets, then there is a function F with domain X such that for each element a in X, $F(a) \in a$. In other words, F chooses an element from each element of X.

The relevant set X here, is the set a such that for some $i \in \mathbb{N}$, a is the set of all functions from $\mathbb{N}$ onto A_i.

Appealing to the Axiom of Choice, for this set X, we easily obtain the desired sequence $\langle f_i : i \in \mathbb{N}\rangle$ of functions.

Returning to our proof, let a be an element of A. Define $f : \mathbb{N} \to A$ as follows.

$$f(n) = \begin{cases} f_i(j), & \text{if } n = 2^i 3^j; \\ a, & \text{otherwise.} \end{cases}$$

f is well defined since every element of $\mathbb{N}$ is uniquely factored as a product of prime numbers. If b is an element of A, then there is an i such that $b \in A_i$ and hence there is a j such that $f_i(j) = b$. But then, $f(2^i 3^j) = b$. Consequently, f is a surjection. $\square$

Even though $\mathbb{N}$ is infinite, there are sets whose size is not less than or equal to the size of $\mathbb{N}$.

Theorem 4.21 (Cantor) *The set of real numbers is not countable.*

Proof. We show first that the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ is not countable.

Suppose that

$$f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$$

Define

$$A = \{k \in \mathbb{N} \mid k \notin f(k)\}.$$

We claim that A is not in the range of f. Suppose toward a contradiction that $f(i) = A$. Then $i \in A$ if and only if $i \notin A$ which is a contradiction. This proves that f is not a surjection.

Thus $\mathcal{P}(\mathbb{N})$ is uncountable. Finally we show that set of real numbers is uncountable by producing a function

$$g : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$$

which is one to one. Define g as follows:

$$g(A) = \begin{cases} 0, & \text{if } A = \emptyset; \\ \sum_{i \in A} 3^{-i}, & \text{otherwise.} \end{cases}$$

It follows that for A, B in $\mathcal{P}(\mathbb{N})$, if $A \neq B$ then $g(A) \neq g(B)$ and so g is one to one as required. Thus the range of g is uncountable and so the set of real numbers is uncountable. $\square$

Remark 4.22 In fact one can show that that there is a bijection

$$\pi : \mathcal{P}(\mathbb{N}) \to \mathbb{R}.$$

We will not need this. One can argue for the existence of π by first constructing 1-to-1 functions

$$F : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$$

and

$$G : \mathbb{R} \to \mathcal{P}(\mathbb{N}),$$

and F we already constructed above.

Given the functions F and G, one can then appeal to the Schröder-Bernstein Theorem to get π.

The Schröder-Bernstein Theorem from Set Theory shows that for any nonempty sets X and Y, if there exist 1-to-1 functions

$$F : X \to Y$$

and

$$G : Y \to X,$$

then there is a bijection $\pi : X \to Y$. $\square$

Definition 4.23 If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure, then we say $\mathcal{M}$ *is countable* to indicate that M is a countable set. □

The following lemma is left to the exercises. This lemma does not require the Axiom of Choice.

Lemma 4.24 *Suppose that $\mathcal{N} = (N, J)$ is an $\mathcal{L}_\mathcal{A}$-structure and that $X \subseteq N$ is countable. Let $\mathbb{A}$ be the set of all $A \subseteq N$ such that $A \neq \emptyset$ and such that A is definable in $\mathcal{N}$ with parameters from X. Then $\mathbb{A}$ is countable.* □

The following theorem requires[2] the Axiom of Choice.

Theorem 4.25 (Lowenheim-Skolem) *Suppose that $\mathcal{N} = (N, J)$ is an $\mathcal{L}_\mathcal{A}$-structure and that $X \subseteq N$ is countable. Then there exists an elementary substructure $(M, I) \preceq \mathcal{N}$ such that M is countable and such that $X \subseteq M$.*

Proof. By Tarski's criterion, it suffices to find a countable set $M \subseteq N$ such that $X \subseteq M$ and such that for each nonempty set $A \subseteq N$, if A is definable in the structure $\mathcal{N}$ from parameters in M, then $A \cap M \neq \emptyset$.

We build a set M by recursion, specifying at most countably many of the elements of M during each stage. Define a countable sequence $\langle M_k : k \in \mathbb{N} \rangle$ of countable subsets of N as follows:

(1.1) $M_0 = X$

(1.2) To define M_{k+1}:

a) Let $\langle A_i : i \in \mathbb{N} \rangle$ enumerate the nonempty subsets of N which are definable in $\mathcal{N}$ using parameters from M_k. This collection of sets is countable by Lemma 4.24, since M_k is countable.

b) Let $\langle a_i : i \in \mathbb{N} \rangle$ be such that $a_i \in A_i$ for all $i \in \mathbb{N}$. Note that here we use the Axiom of Choice.

c) Let M_{k+1} be $M_k \cup \{a_i : i \in \mathbb{N}\}$.

(1.3) Let $M = \cup\{M_k : k \in \mathbb{N}\}$

We claim:

(2.1) $M_0 = \emptyset$

(2.2) For each $k \in \mathbb{N}$, $M_k \subseteq M_{k+1}$

(2.3) For each $k \in \mathbb{N}$, if $A \subseteq N$ is definable in $\mathcal{N}$ with parameters from M_k and $A \neq \emptyset$ then $A \cap M_{k+1} \neq \emptyset$

(2.4) For each $k \in \mathbb{N}$, M_k and M are countable.

Claims (2.1)–(2.3) follow by the construction and claim (2.4) follows from Theorem 4.20.

Suppose that $A \subseteq N$ and A is definable in the structure $\mathcal{N}$ from parameters in M. Then since $M = \cup\{M_k : k \in \mathbb{N}\}$, it follows by (3) that for sufficiently large $k \in \mathbb{N}$, A is definable with parameters from M_k. Therefore, if $A \neq \emptyset$, then

[2] In the context of ZF, this is Set Theory without the Axiom of Choice, the theorem is equivalent to the Axiom of Dependent Choice (DC).

$M \cap A \neq \emptyset$. Finally, for each constant symbol c_i and function symbol F_i of $\mathcal{L}_\mathcal{A}$, $J(c_i) \in M$ and for each $\vec{a} \in M^n$ where $n = \pi(F_i)$, $J(F_i)(\vec{a}) \in M$.

Thus, there exists I such that the structure $\mathcal{M} = (M, I)$ is a substructure of $\mathcal{N}$. By Tarski's Theorem, $\mathcal{M} \preceq \mathcal{N}$ and so $\mathcal{M}$ is a countable elementary substructure of $\mathcal{N}$. $\square$

4.4.2 Exercises

(1) Show that $\mathbb{Q}$, the set of rational numbers, is countable.
(2) Show that there is a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$.
Hint: See Remark 4.22
(3) Show that if $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and X is a countable subset of M, then the collection of all sets $A \subseteq M$ such that A is definable in $\mathcal{M}$ with parameters from X is countable. (Hint: Show that there are only countably many formulas and countably many finite sequences from X.)
(4) Prove Lemma 4.24.

4.5 Dense orders

We more deeply explore a classic example of the Lowenheim-Skolem theorem, identifying a countable elementary substructure of $(\mathbb{R}, <)$. This exploration involves showing by constructing automorphisms how one can in this case easily determine which sets are definable. This in turn, using the Tarski Criterion, will show for example that

$$(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$$

where $\mathbb{Q}$ is the set of rational numbers.

Suppose $\mathcal{L}_\mathcal{A}$ has only one 2-place predicate symbol. Thus $\mathcal{L}_\mathcal{A}$-structures are naturally of the form (M, P), where $M \neq \emptyset$ and $P \subseteq M \times M$.

We consider the $\mathcal{L}_\mathcal{A}$-structure $(\mathbb{R}, <)$, given by the set of real numbers with the usual order. Suppose that $X \subset \mathbb{R}$ is finite and non-empty. Define for reals a and b, $a \sim_X b$ if and only if there exists a bijection $e : \mathbb{R} \to \mathbb{R}$ such that e is an automorphism of the $\mathcal{L}_\mathcal{A}$-structure $(\mathbb{R}, <)$, such that $e(a) = b$ and such that for all $t \in X$, $e(t) = t$.

The relation $\sim_X$ is an equivalence relation on $\mathbb{R}$. That is to say that for all t_1, t_2, t_3 in $\mathbb{R}$ the following conditions hold.

(1) $t_1 \sim_X t_1$; since the identity map $x \mapsto x$ is an automorphism.
(2) If $t_1 \sim_X t_2$ then $t_2 \sim_X t_1$; since the inverse of an automorphism is an automorphism.
(3) if $t_1 \sim_X t_2$ and $t_2 \sim_X t_3$ then $t_1 \sim_X t_3$; since the composition of automorphisms is an automorphism.

For each $t \in \mathbb{R}$ let

$$[t]_X = \{w \in \mathbb{R} : w \sim_X t\}$$

be the equivalence class of t.

Definition 4.26 (1) A set $I \subseteq \mathbb{R}$ is an *interval* if for all a, b, c in $\mathbb{R}$, if $a \le b \le c$ and $\{a, c\} \subseteq I$ then $b \in I$.

(2) Suppose $I \subseteq \mathbb{R}$ is an interval, $I \neq \emptyset$, and $I \neq \mathbb{R}$. Then a real number a is a *lower-endpoint* of I if and only if the following hold.
  (a) $a \le r$ for all $r \in I$,
  (b) For all $s > a$, $I \cap [a, s] \neq \emptyset$.

(3) Suppose $I \subseteq \mathbb{R}$ is an interval, $I \neq \emptyset$, and $I \neq \mathbb{R}$. Then a real number a is an *upper-endpoint* of I if and only if the following hold.
  (a) $r \le a$ for all $r \in I$,
  (b) For all $s < a$, $I \cap [s, a]] \neq \emptyset$.

(4) Suppose $I \subseteq \mathbb{R}$ is an interval, $I \neq \emptyset$, and $I \neq \mathbb{R}$. Then a real number a is an *endpoint* of I if a is either an upper-endpoint of I, or a lower-endpoint of I.

Lemma 4.27 *Suppose that $X \subset \mathbb{R}$ is finite and non-empty. Then for each $a \in \mathbb{R}$, $[a]_X$ is an interval. Further*

(1) *if $a \in X$ then $[a]_X = \{a\}$,*

(2) *if $a \notin X$ then $[a]_X$ is the maximum interval $I \subseteq \mathbb{R}$ such that $a \in I$ and $I \cap X = \emptyset$.*

Proof. Suppose $a \in X$, then any automorphism of $(\mathbb{R}, \le)$ which fixes all of the elements of X must fix a. But then for all b, if $a \sim_X b$ then $a = b$. In other words, $[a]_X = \{a\}$.

Otherwise, let I be the maximum interval such that $a \in I$ and $I \cap X = \emptyset$. To show that I is equal to $[a]_X$, let b be an element of I. Let (c, d) be a subinterval of I such that $a, b \in (c, d)$ $[c, d] \subseteq I$. Without loss of generality, assume $a < b$. First, we define an order preserving bijection e_0 from $[c, d]$ to itself so that e_0 maps a to b.

$$e_0(x) = \begin{cases} c + \frac{b-c}{a-c}(x - c), & \text{if } x \in [c, a]; \\ b + \frac{d-b}{d-a}(x - a), & \text{if } x \in [a, d]. \end{cases}$$

The function e_0 consists of stretching the interval $[c, a]$ to match $[c, b]$ and compressing $[a, d]$ to match $[b, d]$. Then we extend e_0 to an order preserving bijection of $\mathbb{R}$ by mapping every real number not in $[c, d]$ to itself. The resulting function, e is an automorphism of $\mathbb{R}$ which shows that $a \sim_X b$. $\square$

Note that if $I = \emptyset$, or if $I = \mathbb{R}$, then I is an interval and I has no endpoints.

Theorem 4.28 *Suppose that $X \subseteq \mathbb{R}$ and that X is non-empty. Suppose that $A \subseteq \mathbb{R}$, $A \neq \emptyset$, and that $A \neq \mathbb{R}$. Then the following are equivalent.*

(1) *A is definable in $(\mathbb{R}, <)$ with parameters from X.*

(2) *A is a finite union of intervals I such that the endpoints of I belong to X.*

Proof. We will take the implication from (2) to (1) as being self-evident, and we will prove the implication from (1) to (2).

Suppose that A is definable in $(\mathbb{R}, <)$ with parameters from X. Let φ be a formula in the first order language with $\leq$, let $a_1, \ldots, a_n$ be elements of X, and suppose that for all real numbers b,

$$b \in A \;\leftrightarrow\; (\mathbb{R}, <) \models \varphi[b, a_1, \ldots, a_n].$$

We first show that for each b, if $b \in A$ then $[b]_X \subseteq A$. So, suppose that b and c are real numbers, $b \in A$, and $b \sim_X c$. By Lemma 4.27, there is an automorphism e of $(\mathbb{R}, \leq)$ such that e maps b to c and e fixes the elements of $a_1, \ldots, a_n$. By Theorem 4.15, $b \in A$ if and only if $e(b) \in A$. Consequently, $b \in A$ implies $c \in A$, as required.

But then, A is a union of $\sim_{\{a_1,\ldots,a_n\}}$ equivalence classes. Each of these classes is an interval, and since $\{a_1, \ldots, a_n\}$ is finite, there are only finitely many of them. Theorem 4.28 follows immediately. $\square$

By applying Tarski's Theorem 4.18, we can characterize the elementary substructures of $(\mathbb{R}, <)$.

Corollary 4.29 *Let $\mathcal{R} = (\mathbb{R}, <)$. Suppose that $M \subseteq \mathbb{R}$ and that $\mathcal{M} = (M, <_M)$ is the induced substructure of $\mathcal{R}$. Then the following are equivalent.*

(1) *$\mathcal{M} \preceq \mathcal{R}$.*

(2) *The following hold.*
   - (a) *For all $a, b \in M$, if $a < b$ then $M \cap (a, b) \neq \emptyset$.*
   - (b) *For all $c \in M$, there exist $a, b \in M$ such that $a < c < b$.*

Proof. The implication from (1) to (2) is immediate since $\mathcal{M} \equiv \mathcal{R}$.

We now prove the implication from (2) to (1). We will apply Tarski's Criterion to show that $\mathcal{M} \preceq \mathcal{R}$. Let $\{m_1, \ldots, m_n\}$ be a finite subset of M, and let A be a nonempty subset of $\mathbb{R}$ which is definable in $\mathcal{R}$ using parameters from $\{m_1, \ldots, m_n\}$. It is sufficient to show that $A \cap M$ is not empty.

By Lemma 4.28, A is a finite union of intervals I in $\mathcal{R}$ whose endpoints belong to $\{m_1, \ldots, m_n\}$. Let I be a nonempty such interval. If I is a singleton $\{m_i\}$, then $m_i \in (A \cap M)$. Secondly, there could be $m_i < m_j$ such every real number between m_i and m_j belongs to I. By (2a), if m_i and m_j are elements of M, there is an $m \in M$ such that $m_i < m < m_j$. Then $m \in (A \cap M)$ as required. Finally, I could be an unbounded interval. By (2a), there must be an element of m in I in this case as well. $\square$

Another corollary is the following version of Theorem 4.28 but for the structure $(\mathbb{Q}, <)$. Here we refer to *intervals* of $(\mathbb{Q}, <)$, and *endpoints* of such intervals, in the obvious generalization of Definition 4.26 to the case of $(\mathbb{Q}, <)$. Of course there are non-empty intervals of $(\mathbb{Q}, <)$ which are both bounded below and bounded above, and which do not have endpoints (in $\mathbb{Q}$).

Rephrasing Theorem 4.27 exactly as Theorem 4.30 is formulated, yields a theorem which is easily verified to be equivalent to Theorem 4.27.

Theorem 4.30 *Suppose that $X \subseteq \mathbb{Q}$ and that X is non-empty. Suppose that $A \subseteq \mathbb{Q}$, $A \neq \emptyset$, and that $A \subset (r, s)$ for some $r < s$. Then the following are equivalent.*

(1) *A is definable in $(\mathbb{Q}, <)$ with parameters from X.*
(2) *A is a finite union of intervals I such that the lower-endpoint of I and the upper-endpoint of I both exist and belong to X.*

Proof. The implication (2) implies (1) is immediate. We assume (1) and prove (2). Let $\varphi(x_0, x_1, \ldots, x_n)$ be a formula and let $q_1, \ldots, q_n$ be elements of X, such that A is the set of all $q \in \mathbb{Q}$ such that

$$(\mathbb{Q}, <) \vDash \varphi[q, q_1, \ldots, q_n]$$

Let B be the set of all $z \in \mathbb{R}$ such that

$$(\mathbb{R}, <) \vDash \varphi[z, q_1, \ldots, q_n]$$

Since $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$, necessarily $A = B \cap \mathbb{R}$. Finally B is a finite union of intervals I such that the endpoints of I belong to $\{q_1, \ldots, q_n\}$. But $A \subset (r, s)$ and so again since $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$, necessarily $B \subset (r, s)$, where of course here (r, s) denotes the interval I in $(\mathbb{R}, <)$ of all real numbers z such that $r < z < s$.

Therefore A is a finite union of (bounded) intervals I of $(\mathbb{Q}, <)$ with endpoints belong to $\{q_1, \ldots, q_n\}$.

4.6 Arbitrary dense total orders

Suppose that $\mathcal{M} = (M, <)$ is a dense total order without endpoints. More precisely where $<$ is a binary relation such that the following hold.

(1) For all $a, b \in M$; either $a < b$, $b < a$, or $a = b$.
(2) For all $a, b, c \in M$; if $a < b$ and if $b < c$ then $a < c$.
(3) For all $a, b \in M$; if $a < b$ then $b \not< b$.
(4) For all $a, b \in$; if $a < b$ then there exists $c \in M$ such that $a < c < b$.
(5) For all $a \in M$, there exists $b \in M$ such that $a < b$ and there exists $c \in M$ such that $c < a$.

Must $\mathcal{M}$ be elementarily equivalent to $(\mathbb{Q}, <)$, and can one characterize the subsets of M which are definable in $\mathcal{M}$?

Our analysis of the definable sets of the structure $\mathcal{R}$ made essential use of the existence of automorphisms. One can construct examples of structures $\mathcal{M} = (M, <)$ which are dense orders without endpoints and with the additional property that if

$$e : M \to M$$

is a bijection which defines an automorphism of the structure $\mathcal{M}$ then e is the identity. In fact one can construct $\mathcal{M}$ as a substructure of $\mathcal{R}$. Thus one cannot hope to use the method of automorphisms to directly analyze the definable sets of an arbitrary dense order without endpoints, *even* for substructures of $\mathcal{R}$.

We begin with a characterization due to Cantor of the countable dense total orders without endpoints.

Theorem 4.31 (Cantor) *Suppose that $\mathcal{M} = (M, <_M)$ is a countable dense total order without endpoints. Then $\mathcal{M}$ and $(\mathbb{Q}, <)$ are isomorphic.*

Proof. Let $m_1, m_2, \ldots$ and $q_1, q_2, \ldots$ be respective enumerations of M and $\mathbb{Q}$.

Define a sequence of functions $\langle f_n : n \in \mathbb{N}\rangle$ by defining f_n by induction on n as follows such that:

(1.1) f_n has finite domain which is a subset of M, and the range of f_n is a subset of $\mathbb{Q}$.

(1.2) The domain of f_n is a subset of the domain of f_{n+1} and $f_{n+1}(a) = f_n(a)$ for all a in the domain of f_n.

(1.3) For all a, b in the domain of f_n, $a <_M b$ if and only if $f_n(a) < f_n(b)$.

Define f_0 to have domain $\{m_1\}$ and $f_0(m_1) = q_1$.

Having defined f_n, we define f_{n+1} in two steps, first defining f^0_{n+1} and then in the second step, defining f_{n+1} from f^0_{n+1}.

Note (and this is not important at all), if $n = 0$ then $f_1 = f_0$.

Step 1: The definition of f^0_{n+1}.

Let $a_1, \ldots, a_k$ be the domain of f_n in increasing order.

(2.1) For all a_i, let $f^0_{n+1}(a_i) = f_n(a_i)$. This ensures that f^0_{n+1} and f_n agree on the domain of f_n.

(2.2) Suppose m_{n+1} is in the domain of f_n. Then define $f^0_{n+1} = f_n$. Otherwise, if m_{n+1} is not in the domain of f_n then define $f^0_{n+1}(m_{n+1}) = q_j$ where j is the least integer such that:
   a) If $m_{n+1} <_M a_1$, then $q_j < f^0_{n+1}(a_1)$.
   b) If $a_i <_M m_{n+1} <_M a_{i+1}$ for some i, then $f^0_{n+1}(a_i) < q_j < f^0_{n+1}(a_{i+1})$.
   c) If $a_k <_M m_{n+1}$, then $f^0_{n+1}(a_k) < q_j$.

This preserves the order and ensures that m_{n+1} is in the domain of f^0_{n+1}.

Step 2: The definition of f_{n+1} from f^0_{n+1}

Let $b_1, \ldots, b_L$ be the domain of f^0_{n+1} in increasing order.

(3.1) For each a in the domain of f^0_{n+1} define $f_{n+1}(a) = f^0_{n+1}(a)$. This ensures that f_{n+1} and f^0_{n+1} agree on the domain of f_n, and so ensures that f_{n+1} and f_n agree on the domain of f_n.

(3.2) Suppose q_{n+1} is in the range of f^0_{n+1}. Then define $f_{n+1} = f^0_{n+1}$. Otherwise, if q_{n+1} is not in the range of f^0_{n+1} then q_{n+1} is in the domain of f_{n+1} and $f_{n+1}(m_j) = q_{n+1}$ where j is the least integer such that the following hold.
   a) If $q_{n+1} < f^0_{n+1}(b_1)$, then $m_j <_M b_1$.

b) If $f^0_{n+1}(b_i) < q_{n+1} < f^0_{n+1}(b_{i+1})$ for some i, then

$$b_i <_M m_j <_M b_{i+1}.$$

c) If $f^0_{n+1}(b_L) < q_{n+1}$, then $b_L <_M m_j$.

This preserves the order and ensures that q_{n+1} is in the range of f_{n+1}.

We note that since $(\mathbb{Q}, <)$ a dense linear order without endpoints, and since $\mathcal{M}$ is a dense linear order without endpoints, the extension of f_n to f_{n+1} exists as specified.

Let $f : \mathbb{Q} \to M$ be the function given by the union of the f_n; more precisely, for all $n \geq 1$, $f(m_n) = f_n(m_n)$.

We finish by proving

(4.1) f defines an isomorphism of $(\mathbb{Q}, <)$ with $(M, <)$.

From the definition, f preserves the order, more precisely for all $a, b \in M$, if $a <_M b$ then $f(a) < f(b)$. Further by the definition, for every $n \in \mathbb{N}$, m_{n+1} is in the domain of f_{n+1} and q_{n+1} is in the range of f_{n+1}. This proves (4.1) and hence the theorem. □

Remark 4.32 In fact, in the proof of Theorem 4.31, one can simply define $f_{n+1} = f^0_{n+1}$ at every stage, and so simply ignore Step 2, where f_{n+1} is defined from f^0_{n+1}. We leave as an amusing exercise showing that the final function f will still be a surjection.

The construction we give, emphasizes the back-and-forth nature of the construction which we shall use again, in more general context, in the proof of Theorem 6.13 on page 122 in Chapter 6. □

Finally we obtain the following version of Theorem 4.30 but for an *arbitrary* dense linear order $\mathcal{M} = (M, <)$ without endpoints. Here we refer to *intervals* of $(M, <)$, and *endpoints* of such intervals, in the obvious generalization of Definition 4.26 to the case of $(M, <)$.

As we have noted, there are examples of dense linear orders $(M, <)$ which are both substructures of $(\mathbb{R}, <)$ and for which there are *no* nontrivial automorphisms of $(M, <)$. For such structures there is *no* possible direct analysis of the definable sets using automorphisms.

Theorem 4.33 *Suppose that $\mathcal{M} = (M, <)$ is a dense total order without endpoints. Then the following conditions hold.*

(1) $\mathcal{M} \equiv (\mathbb{Q}, <)$.

(2) *If $X \subseteq M$ and $X \neq \emptyset$. Suppose $A \subseteq M$ is definable in the structure $\mathcal{M}$ with parameters from X, and $A \subset (a, b)$ for some $a < b$ in M. Then A is a finite union of intervals I such that the upper-endpoint of I and the lower-endpoint of I both exist and belong to X.*

Proof. By the Lowenheim-Skolem Theorem there exists an elementary substructure

$$(M_0, <_0) = \mathcal{M}_0 \preceq \mathcal{M}$$

such that M_0 is countable. But then, $\mathcal{M}_0 \cong (\mathbb{Q}, <)$ and so $\mathcal{M} \equiv (\mathbb{Q}, <)$. This proves (1).

We now prove (2). In fact (2) follows from (1) (why?) but we shall prove (2) more directly. Fix $X \subseteq M$ and $A \subseteq M$ such that A is definable in $\mathcal{M}$ with parameters from X. Let $\varphi(x_0, x_2, \ldots, x_n)$ be a formula and let $a_1, \ldots, a_n$ be elements of X such that

$$A = \{a \in M : \mathcal{M} \vDash \varphi[a, a_1, \ldots, a_n]\}.$$

We prove that A is a union of intervals with endpoints from $\{a_1, \ldots, a_n\}$.

Assume toward a contradiction that this fails. By Theorem 4.25, Choose $\mathcal{M}_0 = (M_0, I_0)$ so that $\{a_1, \ldots, a_n\} \subseteq M_0$ and so that $\mathcal{M}_0$ is a countable elementary substructure $\mathcal{M}$. Thus, since $\mathcal{M}_0 \preceq \mathcal{M}$,

$$A \cap M_0 = \{a \in M_0 : \mathcal{M}_0 \vDash \varphi[a, a_1, \ldots, a_n]\}$$

and $A \cap M_0$ is not a union of intervals of $\mathcal{M}_0$ with endpoints from $\{a_1, \ldots, a_n\}$. But $\mathcal{M}_0 \cong (\mathbb{Q}, <)$ and this contradicts Theorem 4.30. □

Thus we have managed to analyze the definable sets in an arbitrary structure $\mathcal{M} = (M, <)$ which is a dense order without endpoints. The analysis succeeds by using automorphisms of countable elementary substructures.

The analysis of the definable sets in familiar mathematical structures can be quite a complicated problem, and one whose resolution involves a deep understanding of those structures.

More interestingly, this offers an entirely new mathematical perspective on these structures. This perspective arguably originates only through the development of formal logic.

Theorem 4.34 (Tarski-Seidenberg) *Let $\mathcal{M}$ be the structure $\langle \mathbb{R}, +, \times, <, 0, 1\rangle$ Suppose $A \subseteq \mathbb{R}$ is definable from parameters in $\mathcal{M}$. Then A is a finite union of intervals.* □

What about expanded structures of the form

$$\langle \mathbb{R}, +, \times, <, F, 0, 1\rangle$$

where a single function $F : \mathbb{R} \to \mathbb{R}$ is added?

Lemma 4.35 *Let $\mathcal{M}$ be the structure*

$$\langle \mathbb{R}, +, \times, <, F, 0, 1\rangle$$

where $F(x) = \sin x$. Then there is a set $A \subseteq \mathbb{R}$ which is definable in $\mathcal{M}$ without parameters such that A is not a finite union of intervals.

Proof. Let A be the set of all $x \in \mathbb{R}$ such that $\sin x = 0$. □

In contrast to the simple counterexample provided by the lemma, there are the following remarkable theorems which were only proved just over 20 years ago. The first of these two theorems shows that the case of $F(x) = \sin x$ is actually quite subtle. The second theorem concerns the exponential function and this expansion has been extensively studied.

Theorem 4.36 (Wilkie:1996) *Let $\mathcal{M}$ be the structure*

$$\langle \mathbb{R}, +, \times, <, F, 0, 1 \rangle$$

where $F(x) = \sin(1/(1+x^2))$. Suppose $A \subseteq \mathbb{R}$ is definable from parameters in $\mathcal{M}$. Then A is a finite union of intervals. □

Theorem 4.37 (Wilkie:1996) *Let $\mathcal{M}$ be the structure*

$$\langle \mathbb{R}, +, \times, <, F, 0, 1 \rangle$$

where $F(x) = e^x$. Suppose $A \subseteq \mathbb{R}$ is definable from parameters in $\mathcal{M}$. Then A is a finite union of intervals. □

4.6.1 Exercises

(1) Suppose $\mathcal{A} = \{c_i \mid i \in \mathbb{N}\}$ that $\mathcal{M}$ is an infinite $\mathcal{L}_{\mathcal{A}}$-structure. Show that there is an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}_1$ such that $\mathcal{M}$ and $\mathcal{M}_1$ are elementarily equivalent and $\mathcal{M}_1$ has an element which is not the interpretation of any constant symbol.

Hint: This exercise is closely related to the next exercise.

(2) Suppose that $\mathcal{A} = \{P_i\}$ and $\pi(P_i) = 1$. Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures such that
(a) $I(P_i)$ and $M \setminus I(P_i)$ are each infinite.
(b) $J(P_i)$ and $N \setminus J(P_i)$ are each infinite.
Here $M \setminus I(P_i)$ denotes the set of all $a \in M$ such that $a \notin I(P_i)$, and similarly for $N \setminus J(P_i)$.
Show that $\mathcal{M} \equiv \mathcal{N}$.

(3) Consider the structure of the real numbers, $(\mathbb{R}, F, <)$, augmented with a function
$$F : \mathbb{R} \to \mathbb{R}.$$
Find a sentence θ such that
$$(\mathbb{R}, F, <) \vDash \theta$$
if and only if F is continuous everywhere.

(4) Consider the structure of the real numbers, $(\mathbb{R}, 0, 1, +, \times, F, <)$, augmented with a function
$$F : \mathbb{R} \to \mathbb{R}.$$
Find a sentence θ such that
$$(\mathbb{R}, 0, 1, +, \times, F, <) \vDash \theta$$
if and only if F is differentiable everywhere.

(5) Consider the structure of the real numbers, $(\mathbb{R}, F, <)$, augmented with a function
$$F : \mathbb{R} \to \mathbb{R}.$$
Show that there is no sentence θ such that
$$(\mathbb{R}, F, <) \vDash \theta$$
if and only if F is differentiable everywhere.

Hint: Suppose $G : \mathbb{R} \to \mathbb{R}$ is a surjection which is an increasing function (i.e. for all $a, b \in \mathbb{R}$, if $a < b$ then $G(a) < G(b)$). Show that for any sentence θ,
$$(\mathbb{R}, F, <) \vDash \theta$$
if and only if
$$(\mathbb{R}, H, <) \vDash \theta,$$
where H is the function such that for all $a \in \mathbb{R}$, $H(a) = G^{-1}(F(G(a)))$.

Extra Hint: Let F be the function $F(x) = 2x$, and let G be the function where $G(x) = x$ for $x \leq 1$, and $G(x) = 2x - 1$ for $x \geq 1$.

(6) Consider the structure of the real numbers, $(\mathbb{R}, 0, 1, +, \times, <)$. Let $\mathcal{N}$ be the substructure of $(\mathbb{R}, 0, 1, +, \times, <)$ given by the set of all $r \in \mathbb{R}$ such that the set $A = \{r\}$ is definable in $(\mathbb{R}, 0, 1, +, \times, <)$ without parameters.
Show that $\mathcal{N} \preceq (\mathbb{R}, 0, 1, +, \times, <)$.

Hint: Use Tarski's Criterion together with the Tarski-Seidenberg Theorem, Theorem 4.34.

(7) Consider the structure of the real numbers, $(\mathbb{R}, 0, 1, +, \times, <)$. Show that there are two countable structures $\mathcal{M}_1$ and $\mathcal{M}_2$ with the following properties.
- $\mathcal{M}_1 \preceq (\mathbb{R}, 0, 1, +, \times, <)$ and $\mathcal{M}_2 \preceq (\mathbb{R}, 0, 1, +, \times, <)$.
- $\mathcal{M}_1$ and $\mathcal{M}_2$ are not isomorphic.

Hint: Use the previous exercise and the Lowenheim-Skolem Theorem, Theorem 4.25.

5

The Gödel Completeness Theorem

5.1 The notion of proof

We shall now return to the general case of the language $\mathcal{L}$. Our goal is to prove the Gödel Completeness Theorem and this requires first defining the notion of proof. We begin with the notion of logical validity which we have already defined in Section 3.3 when we were analyzing substitution in $\mathcal{L}_{\mathcal{A}}$-formulas.

Definition 5.1 An $\mathcal{L}$-formula φ is *valid* if it is satisfied in every structure. That is, for all $\mathcal{M}$ and all ν, $(\mathcal{M}, \nu) \vDash \varphi$. We also say that φ is a *validity*.

An $\mathcal{L}$-formula φ is *satisfiable* if it is satisfied in some structure. That is, there exists an $\mathcal{M}$ and ν such that $(\mathcal{M}, \nu) \vDash \varphi$.

An $\mathcal{L}$-formula φ is *contradiction* if it is not satisfied in any structure. That is, there is no $\mathcal{M}$ and ν where $(\mathcal{M}, \nu) \vDash \varphi$. $\square$

Given a specific $\mathcal{L}$-sentence θ, the problem of verifying that θ is a validity looks apriori quite complicated since there are examples of sentences θ which are true in *every* finite structure but which are not valid. Such sentences θ exist in $\mathcal{L}_{\mathcal{A}}$ even if $\mathcal{A}$ just contains either a function symbol, or a predicate symbol of arity 2 (contains P_i for some i, such that $\pi(P_i) = 2$)

This is a significant change from the case of propositional formulas φ where one need only check all the truth assignments just restricted to the propositional symbols occurring in φ (an analog of just checking all finite structures).

Thus trying to characterize the set of all validities, which is evidently a fascinating set, looks quite difficult. In this chapter, we will give a syntactic characterization of this set, describing it in terms of pure logic. We will show that an $\mathcal{L}$-formula is valid if and only if it is provable. This is precisely the Gödel Completeness Theorem for the language $\mathcal{L}$.

The formal notion of proof involves specifying the *logical axioms*. Every logical axiom is valid, and there will be a straightforward algorithm to determine whether any given $\mathcal{L}$-formula is a logical axiom. This is just as was the case for the propositional language $\mathcal{L}_0$.

Some of the logical axioms involve the deduction of instances of a formula φ from the hypothesis $(\forall x_i \varphi)$. Others involve deducing that τ_1 has the property asserted by φ from the hypothesis that $(\tau_1 \doteq \tau_2)$ and τ_2 has that property.

We shall initially just define and analyze the formal notion of poof for our *entire* language $\mathcal{L}$, this is the language $\mathcal{L}_{\mathcal{A}}$ where the alphabet $\mathcal{A}$ contains *all* the constant, function, and predicate symbols. The generalization to the case of $\mathcal{L}_{\mathcal{A}}$ for an arbitrary alphabet $\mathcal{A}$ will follow rather easily, and our route for this leads naturally to the Craig Interpolation Theorem for the language $\mathcal{L}$.

Definition 5.2 The set of logical axioms, denoted Δ, is the smallest set Δ of $\mathcal{L}$-formulas such that the following hold.

(1) (*Instances of Propositional Tautologies*) Suppose that φ_1, φ_2 and φ_3 are $\mathcal{L}$-formulas. Then each of the following $\mathcal{L}$-formulas is in Δ.:
(Group I axioms)
(a) $((\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3)))$
(b) $(\varphi_1 \to \varphi_1)$
(c) $(\varphi_1 \to (\varphi_2 \to \varphi_1))$
(Group II axioms)
(a) $(\varphi_1 \to ((\neg\varphi_1) \to \varphi_2))$
(Group III axioms)
(a) $(((\neg\varphi_1) \to \varphi_1) \to \varphi_1)$
(Group IV axioms)
(a) $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$
(b) $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$

(2) Suppose that φ is an $\mathcal{L}$-formula, τ is a term, and that τ is free for x_i in φ. Then
$$((\forall x_i \varphi) \to \varphi(x_i; \tau)) \in \Delta.$$

(3) Suppose that φ_1 and φ_2 are $\mathcal{L}$-formulas. Then
$$((\forall x_i(\varphi_1 \to \varphi_2)) \to ((\forall x_i \varphi_1) \to (\forall x_i \varphi_2))) \in \Delta.$$

(4) Suppose that φ is an $\mathcal{L}$-formula and that x_i is not a free variable of φ. Then
$$(\varphi \to (\forall x_i \varphi)) \in \Delta.$$

(5) For every variable x_i, $(x_i \hat{=} x_i) \in \Delta$.

(6) Suppose that φ_1 and φ_2 are $\mathcal{L}$-formulas and that x_j is free for x_i in φ_1 and in φ_2.
$$\text{If } \varphi_2(x_i; x_j) = \varphi_1(x_i; x_j),$$
$$\text{then } ((x_i \hat{=} x_j) \to (\varphi_1 \to \varphi_2)) \in \Delta.$$

(7) Suppose that $\varphi \in \Delta$. Then $(\forall x_i \varphi) \in \Delta$. □

Definition 5.3 Suppose that Γ is a set of $\mathcal{L}$-formulas. A finite sequence $\langle \varphi_0, \ldots, \varphi_n \rangle$ is a Γ-*proof*, or a Γ-*deduction*, if the following hold.

(1) $\varphi_0 \in \Gamma \cup \Delta$,
(2) For each $i \leq n$, one of the following hold.
   (a) $\varphi_i \in \Gamma \cup \Delta$.
   (b) There exist $i_0 < i$ and $i_1 < i$ such that φ_{i_1} is equal to $(\varphi_{i_0} \rightarrow \varphi_i)$. This rule of inference is called *modus ponens*.

Suppose φ is an $\mathcal{L}$-formula. Then

$$\Gamma \vdash \varphi$$

or Γ *proves* φ, if and only if there exists a Γ-proof, $\langle \varphi_1, \ldots, \varphi_n \rangle$, with $\varphi_n = \varphi$. □

Definition 5.4 Suppose φ is an $\mathcal{L}$-formula. Then φ is *provable* if and only if $\Delta \vdash \varphi$. □

Note that an $\mathcal{L}$-formula φ is provable if and only if $\{\emptyset\} \vdash \varphi$.

5.2 Deduction and generalization theorems

We will now prove several basic results regarding the formal notion of proof.

Theorem 5.5 (Deduction) *Suppose that Γ is a set of $\mathcal{L}$-formulas and that φ_1 and φ_2 are $\mathcal{L}$-formulas. Then*

$$\Gamma \cup \{\varphi_1\} \vdash \varphi_2 \textit{ if and only if } \Gamma \vdash (\varphi_1 \rightarrow \varphi_2).$$

Proof. We first verify the implication from right to left. Suppose that

$$\langle \theta_1, \ldots, \theta_{n+1} \rangle$$

is a Γ-proof of $(\varphi_1 \rightarrow \varphi_2)$.

In particular, θ_{n+1} is equal to $(\varphi_1 \rightarrow \varphi_2)$. Then $\langle \theta_1, \ldots, \theta_n, (\varphi_1 \rightarrow \varphi_2), \varphi_1, \varphi_2 \rangle$ is a Γ-proof of φ_2 from Γ, as required.

For the implication from left to right, we proceed exactly as in the proof of Lemma 1.39, which is the Deduction Lemma for propositional logic. Note that we easily have inference; the analog of the Inference Lemma, Lemma 1.37, for $\mathcal{L}$.

Let

$$\langle \theta_1, \ldots, \theta_n \rangle$$

be a $(\Gamma \cup \{\varphi_1\})$-proof of φ_2. We prove by induction on $i \leq n$ that

$$\Gamma \vdash (\varphi_1 \rightarrow \theta_i).$$

First we consider the case $i = 1$. Either $\theta_1 \in \Gamma \cup \{\varphi_1\}$ or θ_1 is a logical axiom (possibly both). So there are three subcases of this case.

Subcase 1.1: $\theta_1 \in \Gamma$. So we must show that $\Gamma \vdash (\varphi_1 \to \theta_1)$. However

$$\Gamma \vdash (\theta_1 \to (\varphi_1 \to \theta_1))$$

since $(\theta_1 \to (\varphi_1 \to \theta_1))$ is a logical axiom. Further

$$\Gamma \vdash \theta_1$$

since $\theta_1 \in \Gamma$. Therefore by inference, $\Gamma \vdash (\varphi_1 \to \theta_1)$.

Subcase 1.2: $\theta_1 = \varphi_1$. Note that $(\varphi_1 \to \varphi_1)$ is a logical axiom and so

$$\Gamma \vdash (\varphi_1 \to \varphi_1).$$

Subcase 1.3: θ_1 is a logical axiom. This is just like subcase 1.1; $(\theta_1 \to (\varphi_1 \to \theta_1))$ is a logical axiom and so

$$\Gamma \vdash (\theta_1 \to (\varphi_1 \to \theta_1)).$$

Since θ_1 is a logical axiom, $\Gamma \vdash \theta_1$. Therefore by inference, $\Gamma \vdash (\varphi_1 \to \theta_1)$.

We now suppose that $k \le n$ and assume as an induction hypothesis that for all $i < k$,

$$\Gamma \vdash (\varphi_1 \to \theta_i).$$

There are two subcases.

Subcase 2.1: $\theta_k \in \Gamma \cup \{\varphi_1\}$ or θ_k is a logical axiom. But then exactly as in the case of θ_1, $\Gamma \vdash (\varphi_1 \to \theta_k)$.

Subcase 2.2: There exist $j_1 < k$ and $j_2 < k$ such that $\theta_{j_2} = (\theta_{j_1} \to \theta_k)$.

By the induction hypothesis; $\Gamma \vdash (\varphi_1 \to \theta_{j_1})$ and $\Gamma \vdash (\varphi_1 \to \theta_{j_2})$. Now we use the logical axiom

$$((\varphi_1 \to (\theta_{j_1} \to \theta_k)) \to ((\varphi_1 \to \theta_{j_1}) \to (\varphi_1 \to \theta_k))).$$

By the induction hypothesis,

$$\Gamma \vdash (\varphi_1 \to (\theta_{j_1} \to \theta_k)),$$

and so by inference,

$$\Gamma \vdash ((\varphi_1 \to \theta_{j_1}) \to (\varphi_1 \to \theta_k)).$$

Again by the induction hypothesis,

$$\Gamma \vdash (\varphi_1 \to \theta_{j_1}),$$

and so by inference one last time,

$$\Gamma \vdash (\varphi_1 \to \theta_k).$$

This completes the induction and so $\Gamma \vdash (\varphi_1 \to \varphi_2)$. Finally we note that only Group I logical axioms were used. $\square$

Theorem 5.6 (Generalization) *Suppose that* Γ *is a set of* $\mathcal{L}$*-formulas, that* φ *is an* $\mathcal{L}$*-formula, and that* $\Gamma \vdash \varphi$*. Suppose that* x_i *is a variable not free in any formula in* Γ*. Then* $\Gamma \vdash (\forall x_i \varphi)$*.*

Proof. Let $\langle \theta_1, \ldots, \theta_n \rangle$ be a Γ-proof of φ. We prove by induction on $j \leq n$ that $\Gamma \vdash (\forall x_i \theta_j)$.

First, suppose $j = 1$. Then either θ_1 is in Δ or θ_1 is in Γ.

If $\theta_1 \in \Delta$ then by Cause 7 in Definition 5.2, $(\forall x_i \theta_1) \in \Delta$ and so trivially $\Gamma \vdash (\forall x_i \theta_1)$. If $\theta_1 \in \Gamma$ then by Clause 4 in Definition 5.2, $(\theta_1 \to (\forall x_i \theta_1))$ is an element of Δ. Thus $\langle \theta_1, (\theta_1 \to (\forall x_i \theta_1)), (\forall x_i \theta_1) \rangle$ is a Γ-proof of $(\forall x_i \theta_1)$, as required.

Now suppose $j = k+1$ and $\Gamma \vdash (\forall x_i \theta_m)$ for all $m \leq k$.

If θ_{k+1} is in Δ, or if θ_{k+1} is in Γ, then exactly as above in the case where $j = 1$, $\Gamma \vdash (\forall x_i \theta_{k+1})$.

Finally suppose there are $m_1, m_2 < k+1$ and $\theta_{m_2} = (\theta_{m_1} \to \theta_{k+1})$. By induction hypothesis, Γ proves $(\forall x_i \theta_{m_1})$ and Γ proves $(\forall x_i (\theta_{m_1} \to \theta_{k+1}))$.

By Clause 3 of Definition 5.2,

$$((\forall x_i (\theta_{m_1} \to \theta_{k+1})) \to ((\forall x_i \theta_{m_1}) \to (\forall x_i \theta_{k+1})))$$

is an element of Δ.

Therefore by inference (twice), it follows that $\Gamma \vdash (\forall x_i \theta_{k+1})$, as required. $\square$

The next theorem requires two lemmas, and for the second lemma, as well as for the theorem, we need to extend our notation and definitions on substitution, for example Definition 2.28 and Definition 3.17, from the case of substituting terms for variables to the case of substituting terms for constants. First we recall from Definition 2.28 the basic notation which we repeat here.

Definition 5.7 Suppose that $\varphi(\vec{x})$ is an $\mathcal{L}$-formula.

(1) Suppose that x_i is a free variable of φ.
   (a) Suppose τ is a term. Then the term τ is *substitutable* for x_i if and only if every variable x_j of τ is free for x_i in φ.
   (b) If τ is substitutable for x_i in φ, then $\varphi(x_i; \tau)$ denotes the $\mathcal{L}$-formula obtained by substituting τ for each free occurrence of x_i in φ.

(2) Suppose that $\vec{x} = \langle x_{k_0}, \ldots, x_{k_n} \rangle$ is a sequence of variables with $k_0 < \cdots < k_n$ and $t\vec{au} = \langle \tau_0, \ldots, \tau_n \rangle$ is a sequence of terms. Then $\vec{\tau}$ is *substitutable* for $\vec{x}$ if for each $i \leq n$, if x_{k_i} is a free variable of φ then τ_i is substitutable for x_{k_i} in φ.

(3) If $\vec{\tau}$ is substitutable for $\vec{x}$ in φ then $\varphi(\vec{x}; \vec{\tau})$ denotes the $\mathcal{L}$-formula obtained by substituting τ_i for each free occurrence of x_{k_i} in φ, for all $i \leq n$. $\square$

We extend the notion of substitutability to case of substituting terms for constants.

Suppose φ is a formula and c_i is a constant. Then a variable x_k is *free* for c_i in φ if no occurrence of c_i in φ is within the scope of an occurrence of $\forall x_k$ in φ. This is defined exactly as for occurrences of variables, Definition 3.17.

Definition 5.8 Suppose that $\varphi(\vec{x})$ is an $\mathcal{L}$-formula.

(1) Suppose that c_i is a constant.

 (a) Suppose τ is a term. Then the term τ is *substitutable* for c_i if and only if every variable x_j of τ is free for c_i in φ.

 (b) If τ is substitutable for c_i in φ, then $\varphi(c_i; \tau)$ denotes the $\mathcal{L}$-formula obtained by substituting τ for each occurrence of c_i in φ.

(2) Suppose $\vec{c} = \langle c_{k_0}, \ldots, c_{k_n} \rangle$ is a sequence of constants and $\vec{\tau} = \langle \tau_0, \ldots, \tau_n \rangle$ is a sequence of terms. Then $\vec{\tau}$ is *substitutable* for $\vec{c}$ if for each $i \leq n$, τ_i is substitutable for c_{k_i} in φ.

(3) If $\vec{\tau}$ is substitutable for $\vec{c}$ in φ then $\varphi(\vec{c}; \vec{\tau})$ denotes the $\mathcal{L}$-formula obtained by substituting τ_i for each free occurrence of c_{k_i} in φ, for all $i \leq n$. $\square$

Lemma 5.9 *Suppose that φ is an $\mathcal{L}$-formula, x_i is free for x_j in φ, and x_i does not occur freely in $(\forall x_j \varphi)$. Then*

$$\emptyset \vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$$

Proof. Since x_i is free for x_j in φ, we may apply Clause 2 of Definition 5.2 to conclude that $((\forall x_j \varphi) \to \varphi(x_j; x_i))$ is an element of Δ. By the Deduction Theorem 5.5, $\{(\forall x_j \varphi)\} \vdash \varphi(x_j; x_i)$. Since x_i does not occur freely in $(\forall x_j \varphi)$, we can apply the Generalization Theorem 5.6 to conclude that $\{(\forall x_j \varphi)\} \vdash (\forall x_i \varphi(x_j; x_i))$. By the Deduction Theorem again, $\emptyset \vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$, as required. $\square$

Lemma 5.10 *Suppose that Γ is a set of $\mathcal{L}$-formulas and that the constant symbol c_i does not occur in any formula of Γ. Suppose that $\langle \theta_1, \ldots, \theta_m \rangle$ is a Γ-proof and that the variable x_j does not occur in any of the formulas $\theta_1, \ldots, \theta_m$. Then $\langle \theta_1(c_i; x_j), \ldots, \theta_m(c_i; x_j) \rangle$ is a Γ-proof.*

Proof. Note, if c_i does not occur in φ, then $\varphi(c_i; x_j) = \varphi$. By assumption c_i does not occur in any formula of Γ, so for each $\varphi \in \Gamma$, $\varphi(c_i; x_j) = \varphi$.

It can be verified by inspection of Definition 5.2 that if φ is a logical axiom and x_j does not occur in φ, then $\varphi(c_i; x_j)$ is a logical axiom.

Finally, if φ_1 and φ_2 are $\mathcal{L}$-formulas then

$$(\varphi_1 \to \varphi_2)(c_i; x_j) = (\varphi_1(c_i; x_j) \to \varphi_2(c_i; x_j)).$$

It follows by induction on $n \leq m$, that $\langle \theta_1(c_i; x_j), \ldots, \theta_n(c_i; x_j) \rangle$ is a proof from Γ. $\square$

Theorem 5.11 (Constants) *Suppose that Γ is a set of $\mathcal{L}$-formulas, that φ is an $\mathcal{L}$-formula, and that $\Gamma \vdash \varphi$. Suppose that c_i is a constant and that c_i does not occur in any formula of Γ. Let x_j be a variable which is substitutable for c_i in φ and which does not occur freely in φ. Then*

$$\Gamma \vdash (\forall x_j \varphi(c_i; x_j)).$$

Proof. Let $\langle\theta_0, \ldots, \theta_n\rangle$ be a Γ-proof of φ and let x_k be a variable such that x_k does not appear in any of the formulas $\theta_1, \ldots, \theta_n$.

Let Γ_0 be $\{\theta_1, \ldots, \theta_n\} \cap \Gamma$. By Lemma 5.10,

(1.1) $\langle\theta_1(c_i; x_k), \ldots, \theta_n(c_i; x_k)\rangle$ is a Γ_0-proof,

and so $\Gamma_0 \vdash \varphi(c_i; x_k)$.

Therefore by the Generalization Theorem, $\Gamma_0 \vdash (\forall x_k \varphi(c_i; x_k))$. Since x_j is substitutable for c_i in φ and does not occur freely in φ, x_j is substitutable for x_k and does not occur freely in $(\forall x_k \varphi(c_i; x_k))$. Thus by Lemma 5.9,

$$\emptyset \vdash \Big(\big(\forall x_k \varphi(c_i; x_k)\big) \to \big(\forall x_j \varphi(c_i; x_k)(x_k; x_j)\big)\Big).$$

Of course, $\varphi(c_i; x_k)(x_k; x_j)$ is equal to $\varphi(c_i; x_j)$ and so

$$\emptyset \vdash \Big(\big(\forall x_k \varphi(c_i; x_k)\big) \to \big(\forall x_j \varphi(c_i; x_j)\big)\Big).$$

Finally,

$$\Gamma_0 \vdash (\forall x_k \varphi(c_i; x_k)) \text{ and}$$
$$\Gamma_0 \vdash \Big(\big(\forall x_k \varphi(c_i; x_k)\big) \to \big(\forall x_j \varphi(c_i; x_j)\big)\Big)$$

and so $\Gamma_0 \vdash (\forall x_j \varphi(c_i; x_j))$. But $\Gamma_0 \subseteq \Gamma$, and so trivially, $\Gamma \vdash (\forall x_j \varphi(c_i; x_j))$. □

5.3 Soundness

Now that we have a defined a notion of proof, we will define two important properties, one syntactic and one semantic, of a set of formulas Γ that will be at the center of this chapter's results.

Definition 5.12 Suppose that Γ is a set of $\mathcal{L}$-formulas.

(1) Γ is *consistent* if and only if for every φ, if $\Gamma \vdash \varphi$, then $\Gamma \nvdash (\neg\varphi)$.
(2) Γ is *satisfiable* if and only if there exists a structure $\mathcal{M}$ and an $\mathcal{M}$-assignment ν such that $(\mathcal{M}, \nu) \models \Gamma$. □

Examining the definitions and results up to this point about first-order logic, it is natural to ask whether there is a connection between the provability and validity of a statement. If there is a statement that is satisfied by a mathematical structure, we know that we cannot prove its negation. Conversely, if there is a proof of a statement in that structure, then that structure must make that statement true.

The important connection between provability and validity is established in the Gödel Completeness Theorem, stated below. Our main goal is to prove this theorem.

Theorem 5.13 (Gödel Completeness) *Suppose Γ is a set of $\mathcal{L}$-formulas. Then Γ is consistent if and only if Γ is satisfiable.* □

The implication from satisfiability to consistency can be expressed heuristically: if $(\mathcal{M}, \nu)$ satisfies Γ, then $(\mathcal{M}, \nu)$ satisfies all of the deductive consequences of Γ. We check this implication in the following theorem.

Theorem 5.14 (Soundness) *Suppose that Γ is a set of $\mathcal{L}$-formulas, that φ is an $\mathcal{L}$-formula, and that $\Gamma \vdash \varphi$. Suppose that $\mathcal{M}$ is a structure and that ν is an $\mathcal{M}$-assignment such that $(\mathcal{M}, \nu) \models \Gamma$. Then $(\mathcal{M}, \nu) \models \varphi$.*

Proof. We show by induction on n that if $\langle \varphi_1, \dots, \varphi_n \rangle$ is a deduction from Γ, then for each i less than or equal to n, $\mathcal{M} \models \varphi_i$. We assume that the claim holds for every i less than n, and we check that it holds for n.

If $\varphi_n \in \Gamma$, then since $(\mathcal{M}, \nu) \models \Gamma$, $(\mathcal{M}, \nu) \models \varphi_n$.

If there are i and j less than n such that φ_j is equal to $(\varphi_i \to \varphi_n)$, then by induction $(\mathcal{M}, \nu) \models \varphi_i$ and $(\mathcal{M}, \nu) \models (\varphi_i \to \varphi_n)$. By the definition of satisfaction $(\mathcal{M}, \nu) \models \varphi_n$.

It remains to consider the case in which $\varphi_n \in \Delta$. For this, we must consider each of the clauses (1)–(7) in Definition 5.2.

Clause (1). φ_n is a Group I, Group II, Group III, or a Group IV axiom.

Then $(\mathcal{M}, \nu) \models \varphi_n$ by the definition of satisfaction for the logical connectives. (See Exercise 3 on page 55.)

Clause (2). $\varphi_n = ((\forall x_i \varphi) \to \varphi(x_i; \tau))$ has the form

$$((\forall x_i \varphi) \to \varphi(x_i; \tau)),$$

where τ is substitutable for x_i in φ.

If $(\mathcal{M}, \nu) \not\models (\forall x_i \varphi)$, then trivially $(\mathcal{M}, \nu) \models \varphi_n$. Therefore we can reduce to the case that $(\mathcal{M}, \nu) \models (\forall x_i \varphi)$.

Then for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \varphi)$, $(\mathcal{M}, \mu) \models \varphi$. In particular, if μ agrees with ν on all of the variables except for x_j and $\mu(x_j) = \overline{\nu}(\tau)$, then $(\mathcal{M}, \mu) \models \varphi$. By the Substitution Theorem, Theorem 3.18,

$$(\mathcal{M}, \nu) \models \varphi(x_i; \tau),$$

and so $(\mathcal{M}, \nu) \models \varphi_n$.

Clause (3). φ_n has the form

$$((\forall x_i (\psi_1 \to \psi_2)) \to ((\forall x_i \psi_1) \to (\forall x_i \psi_2))).$$

If $(\mathcal{M}, \nu) \not\models (\forall x_i (\psi_1 \to \psi_2))$ or if $(\mathcal{M}, \nu) \not\models (\forall x_i \psi_1)$, then $(\mathcal{M}, \nu) \models \varphi_n$.

Therefore we can reduce to the case that

(1.1) $(\mathcal{M}, \nu) \not\models (\forall x_i (\psi_1 \to \psi_2))$
(1.2) $(\mathcal{M}, \nu) \models (\forall x_i \psi_1)$.

Therefore, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i (\psi_1 \to \psi_2))$:

(2.1) $(\mathcal{M}, \mu) \models (\psi_1 \to \psi_2)$

(2.2) $(\mathcal{M}, \mu) \vDash \psi_1$.

Consequently, for every such μ, $(\mathcal{M}, \mu) \vDash \psi_2$. Since every free variable of $(\forall x_i \psi_2)$ is also free in $(\forall x_i(\psi_1 \to \psi_2))$, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \psi_2)$, $(\mathcal{M}, \mu) \vDash \psi_2$. It follows that $(\mathcal{M}, \nu) \vDash (\forall x_i \psi_2)$, and hence that $(\mathcal{M}, \nu) \vDash \varphi_n$.

Clause (4). φ_n has the form

$$(\psi \to (\forall x_i \psi))$$

where x_i is not free in ψ. If $(\mathcal{M}, \nu) \nvDash \psi$, then $(\mathcal{M}, \nu) \vDash \varphi_n$.

Therefore we can reduce to the case that $(\mathcal{M}, \nu) \vDash \psi$. Then by Theorem 3.12, for every $\mathcal{M}$-assignment μ, if ν and μ agree on the free variables of ψ, then $(\mathcal{M}, \mu) \vDash \psi$.

Since x_i is not free in ψ, the variables which occur freely ψ also occur freely in $(\forall x_i \psi)$. Thus, if ν and μ agree on the free variables of $(\forall x_i \psi)$, then $(\mathcal{M}, \mu) \vDash \psi$. It follows that $(\mathcal{M}, \nu) \vDash (\forall x_i \psi)$.

Clause (5). φ_n has the form $(x_i \mathrel{\hat{=}} x_i)$. Then it is immediate that $(\mathcal{M}, \nu) \vDash \varphi_n$.

Clause (6). φ_n has the form

$$((x_i \mathrel{\hat{=}} x_j) \to (\psi_1 \to \psi_2)),$$

where ψ_1 and ψ_2 are $\mathcal{L}$-formulas such that

(3.1) x_j is substitutable for x_i in ψ_1
(3.2) x_j is substitutable for x_i in ψ_2
(3.3) $\psi_2(x_i; x_j) = \psi_1(x_i; x_j)$.

If $(\mathcal{M}, \nu) \nvDash (x_i \mathrel{\hat{=}} x_j)$ or $(\mathcal{M}, \nu) \nvDash \psi_1$, then $(\mathcal{M}, \nu) \vDash \varphi_n$. Thus, we can reduce to the case that $(\mathcal{M}, \nu) \vDash (x_i \mathrel{\hat{=}} x_j)$ and $(\mathcal{M}, \nu) \vDash \psi_1$.

Since $(\mathcal{M}, \nu) \vDash (x_i \mathrel{\hat{=}} x_j)$ and since $\nu(x_i) = \overline{\nu}(\langle x_j \rangle)$, we can apply the Substitution Theorem, Theorem 3.18, to the $\mathcal{L}$-formula obtained by substituting the term $\langle x_j \rangle$ for the variable x_i in ψ_1. Thus, $(\mathcal{M}, \nu) \vDash \psi_1(x_i; x_j)$.

Since $\psi_2(x_i; x_j) = \psi_1(x_i; x_j)$, necessarily $(\mathcal{M}, \nu) \vDash \psi_2(x_i; x_j)$. Again noting that $\nu(x_i) = \overline{\nu}(\langle x_j \rangle)$, we can apply Theorem 3.18 once more and conclude from $(\mathcal{M}, \nu) \vDash \psi_2(x_i; x_j)$ that $(\mathcal{M}, \nu) \vDash \psi_2$. Consequently, $(\mathcal{M}, \nu) \vDash \varphi_n$.

Clause (7). φ_n has the form $(\forall x_i \psi)$, where $\psi \in \Delta$. By induction (since ψ has shorter length that φ_n), for every $\mathcal{M}$-assignment μ, $(\mathcal{M}, \mu) \vDash \psi$. Consequently, $(\mathcal{M}, \nu) \vDash (\forall x_i \psi)$, as required. □

Corollary 5.15 *Suppose that* Γ *is a set of* $\mathcal{L}$*-formulas. If* Γ *is satisfiable, then* Γ *is consistent.*

Proof. Let Γ be a set of $\mathcal{L}$-formulas that is satisfiable, that is, there is a $(\mathcal{M}, \nu)$ such that $(\mathcal{M}, \nu) \vDash \Gamma$. Suppose, towards contradiction, that Γ is inconsistent. Then there is a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash (\neg\varphi)$. By Theorem

5.12, $(\mathcal{M}, \nu)$ also satisfies all logical consequences of Γ. So, $(\mathcal{M}, \nu) \models \varphi$ and $(\mathcal{M}, \nu) \models (\neg\varphi)$. By the definition of satisfaction for negation, we have that $(\mathcal{M}, \nu) \not\models \varphi$. But we have that $(\mathcal{M}, \nu) \models \varphi$, so we have our contradiction. Therefore, Γ is consistent. $\square$

The above corollary proves half of the completeness theorem. The other direction, consistency implies satisfiability, requires more machinery.

Definition 5.16 Suppose that Γ is a consistent set of $\mathcal{L}$-formulas. Γ is *maximally consistent* if and only if for any $\mathcal{L}$-formula φ, either $\varphi \in \Gamma$ or $\Gamma \cup \{\varphi\}$ is not consistent. $\square$

Lemma 5.17 *Suppose that Γ is an maximally consistent set of $\mathcal{L}$-formulas. Then for each $\mathcal{L}$-formula φ, either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$.*

Proof. Suppose that $(\neg\varphi)$ does not belong to Γ. By the maximality of Γ, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent.

By the exercise at the end of the previous section, for any $\mathcal{L}$-formula θ,

$$\Gamma \cup \{(\neg\varphi)\} \vdash \theta.$$

Consequently, letting θ be $(\neg(x_1 \hat{=} x_1))$, $\Gamma \cup \{(\neg\varphi)\} \vdash (\neg(x_1 \hat{=} x_1))$. By the Deduction Theorem 5.5,

$$\Gamma \vdash ((\neg\varphi) \to (\neg(x_1 \hat{=} x_1))).$$

Applying Clause (1) in the definition of Δ,

$$\Gamma \vdash (((\neg\varphi) \to (\neg(x_1 \hat{=} x_1))) \to ((x_1 \hat{=} x_1) \to \varphi)).$$

Two applications of inference (modus ponens) yield $\Gamma \vdash \varphi$. Now, since $\Gamma \vdash \varphi$, any deduction from $\Gamma \cup \{\varphi\}$ can be converted into a deduction from Γ by replacing each instance of φ with a deduction of φ from Γ. Thus, for each θ, if $\Gamma \cup \{\varphi\} \vdash \theta$ then $\Gamma \vdash \theta$. Since Γ is consistent, $\Gamma \cup \{\varphi\}$ is also consistent.

Finally, since no proper superset of Γ is consistent, $\varphi \in \Gamma$, as required. $\square$

5.3.1 Exercises

(1) Show that for every pair of $\mathcal{L}$-formulas φ and ψ, $\{\varphi, (\neg\varphi)\} \vdash \psi$.

(2) Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is not consistent. Show that $\Gamma \vdash \varphi$. (This is a technical formulation of the legitimacy of proofs by contradiction.)

(3) Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and ν is an $\mathcal{M}$-assignment. Let

$$\Gamma = \{\varphi : (\mathcal{M}, \nu) \models \varphi\}.$$

Show that Γ is maximally consistent.

5.4 A substitution on constants

Our proof that if Γ is consistent then Γ is satisfiable will require that there are infinitely many constants c_i which do not occur in any formula of Γ. We now verify that one can easily reduce to this case.

Suppose that

$$\rho : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

is a function which is 1-to-1.

For each formula φ of $\mathcal{L}$ let φ^ρ be the $\mathcal{L}$-formula obtained where for each constant c_i which occurs in φ, every occurrence of c_i in φ is replaced by the constant $\rho(c_i)$.

If Γ is a set of $\mathcal{L}$-formulas then Γ^ρ denotes the set

$$\{\varphi^\rho \mid \varphi \in \Gamma\}.$$

Thus Γ^ρ is also set of $\mathcal{L}$-formulas.

With these definitions we have the following lemmas.

Lemma 5.18 *Suppose that*

$$\rho : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

is a function which is 1*-to-*1*. For all formulas φ, ψ of $\mathcal{L}$, for all variables x_i:*
(1) $(\forall x_i \varphi)^\rho = (\forall x_i \varphi^\rho)$.
(2) $(\neg\varphi)^\rho = (\neg\varphi^\rho)$.
(3) $(\varphi \to \psi)^\rho = (\varphi^\rho \to \psi^\rho)$. $\square$

Lemma 5.19 *Suppose that*

$$\rho : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

is a function which is 1*-to-*1*. Suppose Γ is a set of formulas of $\mathcal{L}$ and φ is a formula of $\mathcal{L}$. Suppose that $\Gamma \vdash \varphi$. Then $\Gamma^\rho \vdash \varphi^\rho$.*

Proof. Note that if $\varphi \in \Delta$ then $\varphi^\rho \in \Delta$. Thus if $\langle \varphi_0, \ldots, \varphi_n \rangle$ is a Γ-proof then by Lemma 5.18(3), it follows easily that

$$\langle (\varphi_0)^\rho, \ldots, (\varphi_n)^\rho \rangle$$

is a Γ^ρ-proof. $\square$

Lemma 5.20 *Suppose that*

$$\rho : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

is a function which is 1*-to-*1*. Suppose Γ is a set of formulas of $\mathcal{L}$ and φ is a formula of $\mathcal{L}$. Suppose that $\Gamma^\rho \vdash \varphi^\rho$. Then $\Gamma \vdash \varphi$.*

Proof. The problem is that ρ may not be a surjection and so we cannot use Lemma 5.19 with ρ replaced by its inverse.

However since proofs are finite, we can reduce to the case that Γ is finite. Let $\mathcal{C}$ be the set of constants which occur in some formula of

$$\Gamma \cup \{\varphi\}.$$

We have that

$$\rho : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

is 1-to-1. Therefore since $\mathcal{C}$ is finite, there is a 1-to-1

$$e : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

such that for all $c_i \in \mathcal{C}$, $e(\rho(c_i)) = c_i$.

For all $\psi \in \Gamma$,

$$(\psi^\rho)^e = \psi$$

and so $(\Gamma^\rho)^e = \Gamma$. Similarly, $(\varphi^\rho)^e = \varphi$. Here of course for each formula ψ and each set of formulas Σ, ψ^e and Σ^e are defined as above but with $\rho = e$.

Therefore by replacing ρ with e, replacing Γ with Γ^ρ, and replacing φ with φ^ρ; by Lemma 5.19, it follows that $\Gamma \vdash \varphi$. □

Combining Lemma 5.19 and Lemma 5.20, we obtain the following equivalence.

Lemma 5.21 *Suppose that*

$$\rho : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

is a function which is 1*-to-*1*. Suppose* Γ *is a set of formulas of* $\mathcal{L}$ *and* φ *is a formula of* $\mathcal{L}$*. Then the following are equivalent.*

(1) $\Gamma \vdash \varphi$.

(2) $\Gamma^\rho \vdash \varphi^\rho$. □

Finally we note the following lemma.

Lemma 5.22 *Suppose that*

$$\rho : \{c_i : i \in \mathbb{N}\} \to \{c_i : i \in \mathbb{N}\}$$

is a function which is 1*-to-*1*. Suppose* Γ *is a set of formulas of* $\mathcal{L}$ *and that* Γ^ρ *is satisfiable. Then* Γ *is satisfiable.*

Proof. Let $\mathcal{M} = (M, I)$ be the $\mathcal{L}$-structure, and let ν be an $\mathcal{M}$-assignment such that

$$(\mathcal{M}, \nu) \vDash \Gamma^\rho.$$

Let $\mathcal{M}^\rho = (M, I^\rho)$ be the $\mathcal{L}$-structure obtained from $\mathcal{M}$ and ρ, where for all $i \in \mathbb{N}$,

(1.1) $I^\rho(F_i) = I(F_i)$ and $I^\rho(P_i) = I(P_i)$.
(1.2) $I^\rho(c_i) = I(\rho(c_i))$.

Then by Lemma 5.18, and by Theorem 3.19 on page 54, which connected the satisfaction relation with the interpretation of constants, $(\mathcal{M}^\rho, \nu) \models \Gamma$. □

Remark 5.23 We are headed toward proving the Gödel Completeness Theorem and by Soundness, we have reduced proving this theorem to just proving that if Γ is a consistent set of $\mathcal{L}$-formulas, then Γ is satisfiable.

The key point is that by Lemma 5.21 and by Lemma 5.22, it suffices to restrict to the special case that there are infinitely many constants c_i such that c_i does not occur in any formula of Γ.

This is precisely the special case for which we will prove that if Γ is consistent then Γ is satisfiable. □

5.5 The Henkin property

Now we show that if a set of $\mathcal{L}$-formulas Γ is consistent, then there exists a model which satisfies all formulas in Γ. In order to prove this direction of the completeness theorem, we will build a model using the formulas in a given consistent set Γ.

In this section, we will cover two key ideas that will help us construct the needed model.

Definition 5.24 We use the notation $(\exists x_i \varphi)$ to represent the $\mathcal{L}$-formula $(\neg(\forall x_i(\neg\varphi)))$. □

By inspection of Definition 3.10, $(\mathcal{M}, \nu) \models (\exists x_i \varphi)$ the following lemma is immediate.

Lemma 5.25 *Suppose φ is an $\mathcal{L}$-formula, x_i is a variable, $\mathcal{M}$ is an $\mathcal{L}$-structure, and that ν is an $\mathcal{M}$-assignment. Then the following are equivalent.*

(1) $(\mathcal{M}, \nu) \models (\exists x_i \varphi)$.
(2) *There is an $\mathcal{M}$-assignment μ such that*
  (a) $\mu(x_k) = \nu(x_k)$ *for all $k \in \mathbb{N}$ such that $k \neq i$,*
  (b) $(\mathcal{M}, \mu) \models \varphi$. □

Definition 5.26 A set of $\mathcal{L}$-formulas Γ has the *Henkin Property* if and only if for each $\mathcal{L}$-formula φ and for each variable x_i, if $(\exists x_i \varphi) \in \Gamma$ then there exists a constant c_j such that $\varphi(x_i; c_j) \in \Gamma$. □

The following application of Tarski's Theorem motivates the definition of the Henkin property. This theorem also shows that if Γ is a set of $\mathcal{L}$-formulas which is both maximally consistent and has the Henkin property, then Γ is uniquely determined by the set $\Sigma \subset \Gamma$ of all the sentences in Γ.

Theorem 5.27 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}$-structure and ν is an $\mathcal{M}$-assignment such that*

$$\{\nu(x_i) : i \in \mathbb{N}\} \subseteq \{I(c_i) : i \in \mathbb{N}\}.$$

Let $\Gamma = \{\varphi : (\mathcal{M}, \nu) \models \varphi\}$ and let Σ be the set of all sentences θ such that $\mathcal{M} \models \theta$. Then the following are equivalent.

(1) *Γ has the Henkin property.*
(2) *Σ has the Henkin property.*
(3) *There exists an elementary substructure $(M_0, I_0) \preceq \mathcal{M}$ such that*

$$M_0 = \{I(c_i) : i \in \mathbb{N}\}.$$

Proof. (1) trivially implies (2). Therefore it suffices that (2) implies (3) and that (3) implies (1).

We first assume (2) and prove (3). To prove (3) it suffices to show that M_0 satisfies Tarski's Criterion.

Let $m_1, \ldots, m_n$ be elements of M_0 and suppose that $A \subseteq M$ is definable in $\mathcal{M}$ from these elements as follows.

$$a \in A \ \leftrightarrow \ \mathcal{M} \models \varphi[a, m_1, \ldots, m_n]$$

We must show that $A \cap M_0$ is not empty.

Since each element of M_0 is in the range of I applied to the set of constant symbols, we fix $\vec{c} = \langle c_{i_1}, \ldots, c_{i_n} \rangle$ so that for each $j \leq n$, $I(c_{i_j}) = m_j$. By the Substitution Theorem, Theorem 3.18, for all $a \in M$,

$$a \in A \ \leftrightarrow \ \mathcal{M} \models \varphi(\vec{x}; \vec{c})[a].$$

Since A is not empty,

$$(\mathcal{M}, \nu) \models (\exists x_0 \varphi(\vec{x}; \vec{c})).$$

Then $(\exists x_0 \varphi(\vec{x}; \vec{c}))$ is an element of Σ, and so by the Henkin property, there is a c_{i_0} such that

$$\varphi(\vec{x}; \vec{c})(x_0; c_{i_0}) \in \Sigma.$$

Note that

$$\varphi(\vec{x}; \vec{c})(x_0; c_{i_0}) = \varphi(x_0, \ldots, x_n; c_{i_0}, \ldots, c_{i_n}).$$

Consequently,

$$\mathcal{M} \models \varphi(x_0, \ldots, x_n; c_{i_0}, c_{i_1}, \ldots, c_{i_n}),$$

and so

$$\mathcal{M} \models \varphi[I(c_{i_0}), I(c_{i_1}), \ldots, I(c_{i_n})].$$

By the above, each $I(c_{i_j})$ is equal to m_j, so

$$\mathcal{M} \models \varphi[I(c_{i_0}), m_1, \ldots, m_n].$$

$I(c_{i_0})$ is the desired element of $M_0 \cap A$.

We now assume (3) and prove (1). Let $\mathcal{M}_0 = (M_0, I_0)$. Fix $(\exists x_i \varphi) \in \Gamma$. Thus by the definition of Γ,

$$(\mathcal{M}, \nu) \vDash (\exists x_i \varphi),$$

Since $\mathcal{M}_0 \preceq \mathcal{M}$ and since ν is an $\mathcal{M}_0$-assignment,

$$(\mathcal{M}_0, \nu) \vDash (\exists x_i \varphi),$$

By Lemma 5.25, there exists an $\mathcal{M}_0$-assignment μ such that

(1.1) $(\mathcal{M}_0, \mu) \vDash \varphi$,
(1.2) $\mu(x_k) = \nu(x_k)$ for all $k \in \mathbb{N}$ such that $k \neq i$.

Fix a constant c_m such that $I_0(c_m) = I(c_m) = \mu(x_i)$. The constant c_m exists because μ is an $\mathcal{M}_0$-assignment.

Again since $\mathcal{M}_0 \preceq \mathcal{M}$, and since μ is an $\mathcal{M}$-assignment,

$$(\mathcal{M}, \mu) \vDash \varphi$$

By Substitution Theorem, Theorem 3.18, and since $\mu(x_i) = I(c_m)$,

$$(\mathcal{M}, \mu) \vDash \varphi(x_i; c_m).$$

Finally x_i is not a free variable of $\varphi(x_i, c_m)$ and $\mu(x_k) = \nu(x_k)$ for all $k \in \mathbb{N}$ such that $k \neq i$. Therefore by Theorem 3.12,

$$(\mathcal{M}, \nu) \vDash \varphi(x_i; c_m)$$

and so $\varphi(x_i; c_m) \in \Gamma$. □

This is a another variation of the previous theorem. The proof uses the following lemma which is left to the exercises. For each $\mathcal{L}$-formula φ, let $\mathcal{A}_\varphi$ be the alphabet consisting of the predicate symbols, function symbols, and constant symbols occurring in φ.

Lemma 5.28 *Suppose $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}$-structures such that $M = N$. Suppose ν is an $\mathcal{M}$-assignment and that φ is an $\mathcal{L}$-formula such that*

$$I \restriction \mathcal{A}_\varphi = J \restriction \mathcal{A}_\varphi.$$

Then $(\mathcal{M}, \nu) \vDash \varphi$ if and only if $(\mathcal{N}, \nu) \vDash \varphi$. □

Theorem 5.29 *Suppose Γ is a set of $\mathcal{L}$-formulas such that Γ is satisfiable and such that there are infinitely many constants c_i such that c_i does not occur in any formula of Γ. Then there is an $\mathcal{L}$-structure $\mathcal{M} = (M, I)$ and an $\mathcal{M}$-assignment ν such that the following hold.*

(1) $(\mathcal{M}, \nu) \vDash \Gamma$.
(2) $M = \{I(c_i) \mid i \in \mathbb{N}\}$.

Proof. Since Γ is satisfiable, there exists an $\mathcal{L}$-structure $\mathcal{M} = (M, I)$ and an $\mathcal{M}$-assignment ν such that
$$(\mathcal{M}, \nu) \models \Gamma.$$
By the Lowenheim-Skolem Theorem, we can assume that M is countable. Let $\mathcal{C}$ be the set of contants c_i such that c_i does not occur in any formula of Γ. Since M is countable and since $\mathcal{C}$ is infinite, there exists a surjection
$$e : \mathcal{C} \to M.$$
Define an interpretation map J as follows. For each function symbol F_i, $J(F_i) = I(F_i)$, for each predicate symbol P_i, $J(P_i) = I(P_i)$, and for each constant symbol c_i
$$J(c_i) = \begin{cases} I(c_i), & \text{if } c_i \notin \mathcal{C}; \\ e(c_i), & \text{if } c_i \in \mathcal{C}. \end{cases}$$
Thus $\mathcal{N} = (M, J)$ is an $\mathcal{L}$-structure, ν is an $\mathcal{N}$-assignment, and by Lemma 5.28
$$(\mathcal{N}, \nu) \models \Gamma.$$
Further $M = \{I(c_i) \mid i \in \mathbb{N}\}$. This proves the theorem. □

Theorem 5.27 and Theorem 5.29 show that if Γ is a set of $\mathcal{L}$-formulas such that Γ is satisfiable and such that there are infinitely many constants c_i such that c_i does not occur in any formula of Γ, then there exists a set Σ of $\mathcal{L}$-formulas such that

(1) $\Gamma \subseteq \Sigma$.
(2) Σ is satisfiable and Σ has the Henkin property.
(3) Σ is maximally consistent.

One can naturally define when a set Γ of $\mathcal{L}_{\mathcal{A}}$ has the Henkin Property whenever $\mathcal{A}$ is an alphabet which contains all the constant symbols.

Definition 5.30 Suppose $\mathcal{A}$ is an alphabet which contains all the constant symbols. A set of $\mathcal{L}_{\mathcal{A}}$-formulas Γ has the *Henkin Property* if and only if for each $\mathcal{L}_{\mathcal{A}}$-formula φ and for each variable x_i, if $(\exists x_i \varphi) \in \Gamma$ then there exists a constant c_j such that $\varphi(x_i; c_j) \in \Gamma$. □

5.5.1 Exercises

(1) Suppose that Γ is a maximally consistent set of $\mathcal{L}$-formulas. Define the relation $\sim_\Gamma$ between terms by $\tau_1 \sim_\Gamma \tau_2$ if and only if $(\tau_1 \hat{=} \tau_2) \in \Gamma$.

    Show that $\sim_\Gamma$ is an equivalence relation. More precisely, show that relation $\sim_\Gamma$ is reflexive, symmetric, and transitive.

(2) Let $\mathcal{A} = \{c_i : i \in \mathbb{N}\}$. Give an example of a consistent theory T_1 in the language $\mathcal{L}_{\mathcal{A}}$ such that every model of T_1 is infinite.

(3) Let $\mathcal{A} = \{c_i : i \in \mathbb{N}\}$. Give an example of a consistent theory T_2 in the language $\mathcal{L}_\mathcal{A}$ such that every model of T_1 is finite.

(4) Suppose $\mathcal{M} = (M, I)$ is an $\mathcal{L}$-structure and that

$$T_\mathcal{M} = \{\varphi : \varphi \text{ is a sentence of } \mathcal{L} \text{ and } \mathcal{M} \models \varphi\}$$

Suppose that M is infinite. Can $T_\mathcal{M}$ have a finite model?

(5) Prove Lemma 5.28.

Hint: Suppose φ is an $\mathcal{L}$-formula. Show by induction on the length of φ that for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi$$

if and only

$$(\mathcal{N}, \nu) \models \varphi$$

Note: Lemma 5.28 is the version of Theorem 1.22 for the language $\mathcal{L}$.

(6) Let $\mathcal{A} = \{c_i : i \in \mathbb{N}\}$. Suppose $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure. Which sets $X \subseteq M$ are definable in $\mathcal{M}$ **without** parameters and why?

Hint: Use Lemma 5.28 (and Theorem 4.5).

(7) Let $\mathcal{A} = \{c_i : i \in \mathbb{N}\}$. Give an example of an $\mathcal{L}_\mathcal{A}$-structure $\mathcal{M}$ such that

$$T_\mathcal{M} = \{\varphi : \varphi \text{ is a sentence of } \mathcal{L}_\mathcal{A} \text{ and } \mathcal{M} \models \varphi\}$$

does not have the Henkin property.

Hint: Use Exercise 6.

(8) Let $\mathcal{A} = \{c_i : i \in \mathbb{N}\}$. Suppose $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and that $\{I(c_i) : i \in \mathbb{N}\}$ is infinite.
Show that

$$T_\mathcal{M} = \{\varphi : \varphi \text{ is a sentence of } \mathcal{L}_\mathcal{A} \text{ and } \mathcal{M} \models \varphi\}$$

has the Henkin property.

Hint: Use Exercise 6.

5.6 The Gödel Completeness Theorem

We now carry out the proof of the last half of the Gödel Completeness Theorem: if a set of $\mathcal{L}$-formulas Γ is consistent, then it is satisfiable. We will do this in two parts.

First, we will show that we can extend any consistent set Γ to a consistent set Σ with the Henkin property, assuming that there are infinitely many constants, c_i, which do not occur in any formula of Γ. Then second, we will show that any such set, Σ, is satisfiable.

This will trivially suffice because any model which satisfies Σ, necessarily satisfies Γ, since $\Gamma \subseteq \Sigma$.

Theorem 5.31 *Suppose that Γ is a consistent set of $\mathcal{L}$-formulas and that there are infinitely many constants, c_i, which do not occur in any formula of Γ. Then there is a set of formulas Σ such that*

(1) $\Gamma \subseteq \Sigma$,

(2) Σ *is maximally consistent,*

(3) Σ *has the Henkin property.*

Proof. Let $\langle c_{n_i} : i \in \mathbb{N} \rangle$ enumerate the constants which do not occur in any formula of Γ. Let $\langle \varphi_i : i \in \mathbb{N} \rangle$ be an enumeration of all $\mathcal{L}$-formulas which satisfies

(1.1) for each formula φ there exist distinct positive integers i_0 and i_1 such that $\varphi = \varphi_{i_0} = \varphi_{i_1}$,

(1.2) no constant in the set, $\{c_{n_k} : k \geq i\}$, occurs in φ_i.

Define by induction on $i \in \mathbb{N}$ an increasing sequence $\langle \Sigma_i : i \in \mathbb{N} \rangle$ of sets of formulas as follows.

(2.1) $\Sigma_0 = \Gamma$.

(2.2) a) If $\varphi_i \notin \Sigma_i$ and if $\Sigma_i \cup \{\varphi_i\}$ is consistent, then $\Sigma_{i+1} = \Sigma_i \cup \{\varphi_i\}$.

b) If $\varphi_i \in \Sigma_i$ and if φ_i is an existential formula $\varphi_i = (\exists x_j \psi)$, then $\Sigma_{i+1} = \Sigma_i \cup \{\varphi(x_j; c_{n_i})\}$.

c) Otherwise, $\Sigma_{i+1} = \Sigma_i$.

We prove by indiction on i that for each i, the following properties hold.

(3.1) $\Gamma \subseteq \Sigma_i$.

(3.2) $\Sigma_i \subseteq \Sigma_{i+1}$.

(3.3) Σ_i is consistent.

(3.4) No constant in the set, $\{c_{n_k} : k \geq i\}$, occurs in any formula of Σ_i.

(3.5) $\varphi_i \in \Sigma_{i+1}$ or $\Sigma_i \cup \{\varphi_i\}$ is not consistent.

All these claims are immediate by induction except possibly (3.3). The only subtle part is to show that if Σ_i is consistent then Σ_{i+1} is consistent when it is defined by means of case 2(b). We give the argument for this case below.

Suppose that (3.1)–(3.5) hold for i and that $\Sigma_{i+1} = \Sigma_i \cup \{\psi(x_j; c_{n_j})\}$ as specified in case 2(b). Suppose for a contradiction that Σ_{i+1} is not consistent. Then $\Sigma_i \cup \{\psi(x_j; c_{n_i})\} \vdash (\neg(x_1 \hat{=} x_1))$, as every formula can be derived from an inconsistent set. By the Deduction Theorem, $\Sigma_i \vdash (\psi(x_j; c_{n_i}) \rightarrow (\neg(x_1 \hat{=} x_1)))$ and so $\Sigma_i \vdash (\neg\psi(x_j; c_{n_i}))$. By the Theorem on Constants, Theorem 5.11, $\Sigma_i \vdash (\forall x_j(\neg\psi(x_; c_{n_i})))(c_{n_i}; x_j)$. Making the substitution, $\Sigma_i \vdash (\forall x_j(\neg\psi))$. But since case 2(b) applied, $(\exists x_j \psi) \in \Sigma_i$ and so $(\neg(\forall x_j(\neg\psi))) \in \Sigma_i$.

Thus, Σ_i is not consistent, contrary to assumption. Therefore Σ_{i+1} is consistent, as required. This proves (3.3).

Let $\Sigma = \cup\{\Sigma_i : i \in \mathbb{N}\}$. By (3.1)–(3.4), $\Gamma \subseteq \Sigma$ and Σ is consistent. Further, since every $\mathcal{L}$-formula appears in the list $\langle \varphi_i : i \in \mathbb{N}\rangle$, by (3.3) and (3.5), Σ is maximally consistent.

Finally, we verify that Σ has the Henkin property. Suppose that $(\exists x_j \psi) \in \Sigma$. By our choice of the enumeration $\langle \varphi_i : i \in \mathbb{N}\rangle$, there exist $i_0 < i_1$ such that $\varphi_{i_0} = \varphi_{i_1} = (\exists x_j \psi)$. Since $(\exists x_j \psi) \in \Sigma$, $\Sigma_{i_0} \cup \{(\exists x_j \psi)\}$ is consistent. By case 2(a) in the definition of Σ_{i+1} from Σ_i, $(\exists x_j \psi) \in \Sigma_{i_0+1} \subseteq \Sigma_{i_1}$. Thus, case 2(b) applies in the definition of Σ_{i_1+1}, and so $\psi(x_j; c_{n_{i_1}}) \in \Sigma_{i_1+1}$, as required to verify the Henkin property. □

We prove as Theorem 5.33, that if Γ is a set of $\mathcal{L}$-formulas which is both maximally consistent and has the Henkin property, then Γ is satisfiable. The following lemma will be useful for that proof, and it is this lemma which verifies we have enough logical axioms involving $\hat{=}$.

Lemma 5.32 *Suppose that φ is an $\mathcal{L}$-formula with no quantifiers and that $\vec{\tau} = \langle \tau_0, \ldots, \tau_n\rangle$ and $\vec{\sigma} = \langle \sigma_0, \ldots, \sigma_n\rangle$ are sequences of terms. Then for any sequence of (distinct) variables $\vec{x} = \langle x_{m_0}, \ldots, x_{m_n}\rangle$,*

$$\{(\tau_i \hat{=} \sigma_i) : i \leq n\} \cup \{\varphi(\vec{x}; \vec{\tau})\} \vdash \varphi(\vec{x}; \vec{\sigma}).$$

Proof. We prove Lemma 5.32 by induction on n. Suppose that it holds for all $\mathcal{L}$-formulas with no quantifiers and for all sequences $\vec{\tau}$ and $\vec{\sigma}$ of length m such that $m \leq n$.

Suppose that $\langle \tau_0, \ldots, \tau_n\rangle$ and $\langle \sigma_0, \ldots, \sigma_n\rangle$ are sequences of terms, φ is an $\mathcal{L}$-formula with no quantifiers, and $\langle x_{m_0}, \ldots, x_{m_n}\rangle$ is a sequence of variables.

Since we are substituting terms for *all* of the occurrences of x_{m_j} in φ, and this is for all $j \leq n$; we may assume that none of these variables occur in any of the terms τ_i or σ_i, for any $i \leq n$.

Let $\vec{\tau} = \langle \tau_i : i < n\rangle$, $\vec{\sigma} = \langle \sigma_i : i < n\rangle$, and let $\vec{x} = \langle x_{m_i} : i < n\rangle$. By the induction hypothesis for any quantifier free formula ψ:

$$\{(\tau_i \hat{=} \sigma_i) : i < n\} \cup \{\psi(\vec{x}; \vec{\tau})\} \vdash \psi(\vec{x}; \vec{\sigma})$$

Therefore sine φ is quantifier free, by the Deduction Theorem,

$$\{(\tau_i \hat{=} \sigma_i) : i < n\} \vdash (\varphi(\vec{x}; \vec{\tau}) \rightarrow \varphi(\vec{x}; \vec{\sigma}))$$

Since x_{m_n} does not appear in any of the formulas $(\tau_i \hat{=} \sigma_i)$ where $i < n$,

$$\{(\tau_i \hat{=} \sigma_i) : i < n\} \vdash \Big(\forall x_{m_n}\big(\varphi(\vec{x}; \vec{\tau}) \rightarrow \varphi(\vec{x}; \vec{\sigma})\big)\Big)$$

Then Clause (2) applies and so

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i < n\} \vdash \Big(\varphi(\vec{x};\vec{\tau})(x_{m_m};\sigma_n) \to \varphi(\vec{x};\vec{\sigma})(x_{m_n};\sigma_n)\Big)$$

where here and below we use the notation, $\varphi(\vec{x};\vec{\sigma})(x_{m_n};\sigma_n)$ to denote the formula $\psi(x_{m_n};\sigma_n)$, where $\psi = \varphi(\vec{x};\vec{\sigma})$, etc.

Using the fact that none of the variables x_{m_i}, for $i \leq n$, appear in any of the terms $\tau_0, \ldots, \tau_n$ and $\sigma_0, \ldots, \sigma_n$, we obtain from the above:

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i < n\} \vdash \Big(\varphi(\vec{x}, x_{m_n};\vec{\tau},\sigma_n) \to \varphi(\vec{x}, x_{m_n};\vec{\sigma},\sigma_n)\Big)$$

Let x_k be a variable which does not appear in any of the τ_i's or σ_i's and does not appear in φ. By Clause (6) in Definition 5.2, and since

$$\Big(\varphi(\vec{x};\vec{\tau})\Big)(x_{m_n};x_k) = \Big(\varphi(\vec{x};\vec{\tau})(x_{m_n};x_k)\Big)(x_{m_n};x_k),$$

we have:

$$\emptyset \vdash \Big((x_{m_n} \mathrel{\hat{=}} x_k) \to \big(\varphi(\vec{x};\vec{\tau}) \to \varphi(\vec{x};\vec{\tau})(x_{m_n};x_k)\big)\Big).$$

By Clause (7) and then Clause (2) in the definition of the logical axioms,

$$\emptyset \vdash \Big((x_{m_n} \mathrel{\hat{=}} x_k) \to \big(\varphi(\vec{x};\vec{\tau}) \to \varphi(\vec{x};\vec{\tau})(x_{m_n};x_k)\big)\Big)(x_{m_n}, x_k;\tau_n,\sigma_n)$$

Making the substitution indicated by $(x_{m_n}, x_k;\tau_n,\sigma_n)$:

$$\emptyset \vdash \Big((\tau_n \mathrel{\hat{=}} \sigma_n) \to \big(\varphi(\vec{x}, x_{m_n};\vec{\tau},\tau_n) \to \varphi(\vec{x}, x_{m_n};\vec{\tau},\sigma_n)\big)\Big)$$

By the Deduction Theorem,

$$\{(\tau_n \mathrel{\hat{=}} \sigma_n)\} \vdash (\varphi(\vec{x}, x_{m_n};\vec{\tau},\tau_n) \to \varphi(\vec{x}, x_{m_n};\vec{\tau},\sigma_n)).$$

and so

$$\{(\tau_n \mathrel{\hat{=}} \sigma_n)\} \ \cup\ \{\varphi(\vec{x}, x_{m_n};\vec{\tau},\tau_n)\} \vdash \varphi(\vec{x}, x_{m_n};\vec{\tau},\sigma_n)$$

But then since we have shown above that

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i < n\} \vdash \Big(\varphi(\vec{x};\vec{\tau})(x_{m_m};\sigma_n) \to \varphi(\vec{x};\vec{\sigma})(x_{m_n};\sigma_n)\Big)$$

it follows that

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i \leq n\} \ \cup\ \{\varphi(\vec{x}, x_{m_n};\vec{\tau},\tau_n)\} \vdash \varphi(\vec{x}, x_{m_n};\vec{\sigma},\sigma_n)$$

as required. □

Theorem 5.33 *Suppose that* Γ *is a maximally consistent set of* $\mathcal{L}$*-formulas with the Henkin property. Then* Γ *is satisfiable.*

Proof. Our proof is divided into two parts. First we must define a model $\mathcal{M}$ and an assignment ν for that model. Then we must verify that $(\mathcal{M}, \nu)$ satisfies Γ. The model $\mathcal{M}$ is called the *Henkin model* for Γ and it is uniquely determined up to isomorphism by Γ. Similarly ν is uniquely determined by Γ, given $\mathcal{M}$.

Defining $\mathcal{M}$ and ν. Define a relation $\sim_\Gamma$ on the set of all constant symbols by $c_i \sim_\Gamma c_j$ if and only if $(c_i \mathbin{\hat{=}} c_j) \in \Gamma$. As was stated in first problem of the exercises on page 94, $\sim_\Gamma$ is an equivalence relation on the set of all constant symbols. More precisely, for all i, j, and k;

(1.1) $c_i \sim_\Gamma c_i$,
(1.2) if $c_i \sim_\Gamma c_j$ then $c_j \sim_\Gamma c_i$,
(1.3) if $c_i \sim_\Gamma c_j$ and $c_j \sim_\Gamma c_k$ then $c_i \sim_\Gamma c_k$.

For each $i \in \mathbb{N}$, let

$$[c_i]_\Gamma = \{c_j : j \in \mathbb{N} \text{ and } c_i \sim_\Gamma c_j\}.$$

$[c_i]_\Gamma$ is the equivalence class of c_i under the equivalence relation $\sim_\Gamma$. This set of equivalence classes is the universe of our model. Let

$$M = \{[c_i]_\Gamma : i \in \mathbb{N}\}.$$

We now define our $\mathcal{M}$-assignment ν. For each variable x_i, choose j so that $(x_i \mathbin{\hat{=}} c_j) \in \Gamma$ and let $\nu(x_i) = [c_j]_\Gamma$.

To see that ν is well defined, we must show that for each x_i there is at least one c_j such that $(x_i \mathbin{\hat{=}} c_j) \in \Gamma$. Further, we must show that for any two constant symbols c_{j_1} and c_{j_2}, if $(x_i \mathbin{\hat{=}} c_{j_1}) \in \Gamma$ and $(x_i \mathbin{\hat{=}} c_{j_2}) \in \Gamma$ then $c_{j_1} \sim_\Gamma c_{j_2}$.

For the first of these claims, consider the formula $(\exists x_{i+1}(x_i \mathbin{\hat{=}} x_{i+1}))$. If it is not an element of Γ, then by Lemma 5.17 $(\forall x_{i+1}(\neg(x_i \mathbin{\hat{=}} x_{i+1})))$ is an element of Γ. But then x_i is substitutable for x_{i+1} in $(\neg(x_i \mathbin{\hat{=}} x_{i+1}))$, and so $\Gamma \vdash (\neg(x_i \mathbin{\hat{=}} x_i))$. But $(x_i \mathbin{\hat{=}} x_i) \in \Delta$ and so $\Gamma \vdash (x_i \mathbin{\hat{=}} x_i)$. Thus, Γ is not consistent, contrary to assumption. Thus, $(\exists x_{i+1}(x_i \mathbin{\hat{=}} x_{i+1})) \in \Gamma$. Since Γ has the Henkin property, there exists a constant c_j such that $(x_i \mathbin{\hat{=}} c_j) \in \Gamma$. Thus, for each x_i, there is a c_j as required by the definition of ν.

The second claim follows from Lemma 5.32. Thus, ν is well defined.

We next define the interpretation map I.

(2.1) Suppose c_i is a constant.

$$I(c_i) = [c_i]_\Gamma.$$

(2.2) Suppose that P_i is a predicate symbol and that $n = \pi(P_i)$. Then

$$I(P_i) = \{\langle [c_{k_1}]_\Gamma, \ldots, [c_{k_n}]_\Gamma \rangle \in M^n : P_i(c_{k_1}, \ldots, c_{k_n}) \in \Gamma\}.$$

(2.3) Suppose that F_i is a function symbol and that $n = \pi(F_i)$. Then

$$I(F_i)([c_{k_1}]_\Gamma, \ldots, [c_{k_n}]_\Gamma) = [c_{k_{n+1}}]_\Gamma$$

if and only if

$$\big(F_i(c_{k_1}, \ldots, c_{k_n}) \mathbin{\hat{=}} c_{k_{n+1}}\big) \in \Gamma.$$

The proofs that $I(P_i)$ and $I(F_i)$ are well defined are analogous to the proof that ν is well defined.

Claim 5.34 *For any term τ, $\overline{\nu}(\tau) = [c_i]_\Gamma$ if and only if $(\tau \mathrel{\hat{=}} c_i) \in \Gamma$.*

Proof. We prove Claim 5.34 by induction on the length of τ. If τ has length 1 then for some $k \in \mathbb{N}$, $\tau = \langle x_k \rangle$ or $\tau = \langle c_k \rangle$. If $\tau = \langle x_k \rangle$ then $\overline{\nu}(\tau) = \nu(x_k)$ and the claim follows by the fact that ν is well defined. If $\tau = \langle c_k \rangle$ then $\overline{\nu}(\tau) = I(c_k) = [c_k]_\Gamma$ and the claim follows from the definition of $\sim_\Gamma$.

Now suppose that τ has length $n > 1$ and that:

Induction Hypothesis: *If σ is a term of length less than n, then for all constants c_k, $\overline{\nu}(\sigma) = [c_k]_\Gamma$ if and only if $(\sigma \mathrel{\hat{=}} c_k) \in \Gamma$.*

Since τ has length > 1, there is a sequence of terms $\vec{\sigma} = \langle \sigma_1, \ldots, \sigma_m \rangle$ and a function symbol F_i such that $\tau = F_i(\vec{\sigma})$, where $|\vec{\sigma}| = \pi(F_i)$. By the definition of $\overline{\nu}$,

$$\overline{\nu}(\tau) = I(F_i)(\overline{\nu}(\vec{\tau})).$$

Let $\langle c_{j_1}, \ldots, c_{j_m} \rangle$ be a sequence of constants such that $\overline{\nu}(\tau_k) = [c_{j_k}]_\Gamma$ for all $k = 1, \ldots, m$. Thus,

$$\overline{\nu}(\tau) = I(F_i)\big([c_{j_1}]_\Gamma, \ldots, [c_{j_m}]_\Gamma\big).$$

By the definition of $I(F_i)$, for each constant symbol c_s,

$$I(F_i)\big([c_{j_1}]_\Gamma, \ldots, [c_{j_m}]_\Gamma\big) = [c_s]_\Gamma$$

if and only if

$$\big(F_i(c_{j_1}, \ldots, c_{j_m}) \mathrel{\hat{=}} c_s\big) \in \Gamma.$$

Consequently, $\overline{\nu}(\tau) = [c_s]_\Gamma$ if and only if $(F_i(c_{j_1}, \ldots, c_{j_m}) \mathrel{\hat{=}} c_s) \in \Gamma$.

By the induction hypothesis, $(\tau_k \mathrel{\hat{=}} c_{j_k})$ is an element of Γ, for all $k = 1, \ldots, m$. Therefore, by Lemma 5.32, and letting $\vec{c} = \langle c_{j_1}, \ldots, c_{j_m} \rangle$,

$$(F_i(\vec{\tau}) \mathrel{\hat{=}} F_i(\vec{c})) \in \Gamma.$$

We can conclude that

$$(F_i(\vec{c}) \mathrel{\hat{=}} c_s) \in \Gamma \text{ if and only if } (\tau \mathrel{\hat{=}} c_s) \in \Gamma.$$

Consequently, $\overline{\nu}(\tau) = [c_s]_\Gamma$ if and only if $(\tau \mathrel{\hat{=}} c_s) \in \Gamma$. This completes the inductive step, and this finishes the proof Claim 5.34.

Verifying that $(\mathcal{M}, \nu)$ satisfies Γ. We now prove that for each formula φ, $\varphi \in \Gamma$ if and only if $(\mathcal{M}, \nu) \vDash \varphi$.

We first reduce to the case in which φ is a sentence. Suppose that φ is a formula, x_i is a variable and that $\nu(x_i) = [c_j]_\Gamma$. Suppose that x_k is a variable not occurring in φ. Thus, $((x_i \mathrel{\hat{=}} x_k) \to (\varphi \to \varphi(x_i; x_k)))$ is a logical axiom, and by application of Clause (7) and then Clause (2),

$$\emptyset \vdash ((x_i \mathrel{\hat{=}} c_j) \to (\varphi \to \varphi(x_i; c_j))),$$

since $\varphi(x_i; x_k)(x_k; c_j) = \varphi(x_i; c_j)$. It follows that if $\nu(x_i) = [c_j]_\Gamma$, then $\Gamma \vdash (\varphi \to \varphi(x_i; c_j))$. In a similar way, if $\nu(x_i) = [c_j]_\Gamma$, then $\Gamma \vdash (\varphi(x_i; c_j) \to \varphi)$: use the above argument for $(\neg\varphi)$ and then apply Clause (1). Consequently, $\Gamma \vdash (\varphi \leftrightarrow \varphi(x_i; c_j))$ and so $(\varphi \leftrightarrow \varphi(x_i; c_j)) \in \Gamma$.

Let n be large enough so that all the free variables of φ belong to $\{x_0, \ldots, x_n\}$. For each $k \leq n$ let m_k be such that $\nu(x_k) = c_{m_k}$. By the above analysis, if $\vec{x} = \langle x_0 \ldots, x_n \rangle$ and $\vec{c} = \langle c_{k_0}, \ldots, c_{k_n} \rangle$ then $\varphi(\vec{x}; \vec{c}))$ is a sentence and

$$(\varphi \leftrightarrow \varphi(\vec{x}; \vec{c})) \in \Gamma.$$

Thus, for each formula φ there exists a sentence φ^* such that $(\varphi \leftrightarrow \varphi^*) \in \Gamma$.

Claim 5.35 *For every sentence φ, $\varphi \in \Gamma$ if and only if $\mathcal{M} \vDash \varphi$.*

Proof. We proceed by induction on the length of φ and this is the only time we will prove something about sentences (as opposed to something about formulas) by induction on length.

We first suppose that φ is a sentence and that φ is an atomic formula. There are two subcases.

First, φ could be of the form $(\tau_1 \mathrel{\hat{=}} \tau_2)$, where τ_1 and τ_2 are terms. Let c_{i_1} be a constant such that $\overline{\nu}(\tau_1) = [c_{i_1}]_\Gamma$ and let c_{i_2} be a constant such that $\overline{\nu}(\tau_2) = [c_{i_2}]_\Gamma$. By Claim 5.34, $(\tau_1 \mathrel{\hat{=}} c_{i_1})$ and $(\tau_2 \mathrel{\hat{=}} c_{i_2})$ are elements of Γ. Lemma 5.32 applies, and so $(\tau_1 \mathrel{\hat{=}} \tau_2) \in \Gamma$ if and only if $(c_{i_1} \mathrel{\hat{=}} c_{i_2})$ is an element of Γ. Further $(c_{i_1} \mathrel{\hat{=}} c_{i_2})$ is an element of Γ if and only if $[c_{i_1}]_\Gamma = [c_{i_2}]_\Gamma$. Consequently, $(\tau_1 \mathrel{\hat{=}} \tau_2) \in \Gamma$ if and only if $\mathcal{M} \vDash (\tau_1 \mathrel{\hat{=}} \tau_2)$, as required.

The second subcase is that $\varphi = P_i(\vec{\tau})$. Let $n = \pi(P_i)$. By definition,

$$(\mathcal{M}, \nu) \vDash \varphi \text{ if and only if } \overline{\nu}(\vec{\tau}) \in I(P_i).$$

For each $k = 1, \ldots, n$, let c_{j_k} be a constant such that $\overline{\nu}(\tau_k) = [c_{j_k}]_\Gamma$. Thus,

$$\overline{\nu}(\vec{\tau}) \in I(P_i) \text{ if and only if } \langle [c_{j_1}]_\Gamma, \ldots, [c_{j_n}]_\Gamma \rangle \in I(P_i).$$

By the definition of $I(P_i)$,

$$\langle [c_{j_1}]_\Gamma, \ldots, [c_{j_n}]_\Gamma \rangle \in I(P_i) \text{ if and only if } P_i(c_{j_1}, \ldots, c_{j_n}) \in \Gamma.$$

By Claim 5.34, $(\tau_k \mathrel{\hat{=}} c_{j_k}) \in \Gamma$ for each $k = 1, \ldots, n$ and so we can apply Lemma 5.32 to conclude that

$P_i(c_{j_1}, \ldots, c_{j_n}) \in \Gamma$ if and only if $P_i(\vec{\tau}) \in \Gamma$.

Thus, $\mathcal{M} \models \varphi$ if and only if $\varphi \in \Gamma$. This finishes the case in which φ is an atomic formula.

We now suppose that the length of φ is N, φ is not an atomic formula and that the following condition holds.

Induction Hypothesis: *Suppose that ψ is a sentence of length less than N. Then $\mathcal{M} \models \psi$ if and only if $\psi \in \Gamma$.*

There are three subcases.

Negation. Suppose that $\varphi = (\neg\psi)$. Since φ is a sentence, so is ψ. Consequently, the induction hypothesis applies, and so $\mathcal{M} \models \psi$ if and only if $\psi \in \Gamma$. Therefore $\mathcal{M} \models (\neg\psi)$ if and only if $\mathcal{M} \not\models \psi$, if and only if $\psi \notin \Gamma$, if and only if $(\neg\psi) \in \Gamma$ (since Γ is maximally consistent).

Implication. Suppose that $\varphi = (\psi_1 \to \psi_2)$. Again, the induction hypothesis applies and we obtain both: $\mathcal{M} \models \psi_1$ if and only if $\psi_1 \in \Gamma$ and $\mathcal{M} \models \psi_2$ if and only if $\psi_2 \in \Gamma$. By definition, $\mathcal{M} \models \varphi$ if and only if either $\mathcal{M} \not\models \psi_1$ or $\mathcal{M} \models \psi_2$. Since Γ is maximally consistent, $\varphi \in \Gamma$ if and only if either $(\neg\psi_1) \in \Gamma$ or $\psi_2 \in \Gamma$. Thus, $\mathcal{M} \models \varphi$ if and only if $\varphi \in \Gamma$.

Quantification. Suppose that $\varphi = (\forall x_i \psi)$. By the Henkin property,

$$(\exists x_i(\neg\psi)) \in \Gamma$$

if and only if for some constant c_j,

$$(\neg\psi)(x_i; c_j) \in \Gamma.$$

By a straightforward deduction, $(\forall x_i \psi) \in \Gamma$ if and only if $(\exists x_i(\neg\psi)) \notin \Gamma$, if and only if for every constant c_j, $(\neg\psi)(x_i; c_j) \notin \Gamma$. But $(\neg\psi)(x_i; c_j)$ is equal to $(\neg\psi(x_i; c_j))$, and so $(\forall x_i\psi) \in \Gamma$ if and only if for every constant c_j, $\psi(x_i; c_j) \in \Gamma$.

By definition, $\mathcal{M} \models \varphi$ if and only if for all $\mathcal{M}$-assignments μ, $(\mathcal{M}, \mu) \models \psi$. (Since φ is a sentence, every $\mathcal{M}$-assignment agrees with every other one on the free variables of φ.)

Suppose that μ is an $\mathcal{M}$-assignment. Let c_j be a constant such that $\mu(x_i) = [c_j]_\Gamma$. Then, since x_i is the only free variable of ψ and since $I(c_j) = \mu(x_i)$, it follows that $(\mathcal{M}, \mu) \models \psi$ if and only if $\mathcal{M} \models \psi(x_i; c_j)$. Thus, the condition:

For all $\mathcal{M}$-assignments μ, $(\mathcal{M}, \mu) \models \psi$

is equivalent to the condition:

For all constants c_j, $\mathcal{M} \models \psi(x_i; c_j)$.

By the induction hypothesis, for each constant c_j, $\mathcal{M} \models \psi(x_i; c_j)$ if and only if $\psi(x_i; c_j) \in \Gamma$.

Thus, $\varphi \in \Gamma$ if and only if for each constant c_j, $\psi(x_i; c_j) \in \Gamma$, if and only if for each constant c_j, $\mathcal{M} \models \psi(x_i; c_j)$, if and only if $\mathcal{M} \models \varphi$.

This completes the final case, which proves Claim 5.35.

Finally by Claim 5.35, Γ is satisfiable. □

Now, we finally restate the completeness theorem and provide its proof.

Theorem 5.36 (Gödel Completeness Theorem) *For any set of $\mathcal{L}$-formulas Γ, the following conditions are equivalent.*

(1) Γ *is consistent.*

(2) Γ *is satisfiable.*

Proof. By the Soundness Theorem, Theorem 5.14, if Γ is satisfiable then it is consistent. Therefore we have only to show that if Γ is consistent then Γ is satisfiable.

By Lemma 5.21 and Lemma 5.22, we can reduce to the case that there are infinitely many constants which do not occur in any formula of Γ. But then by Theorem 5.31, Γ can be extended to a maximally consistent set Σ with the Henkin property and by Theorem 5.33, Σ is satisfiable. This trivially implies that Γ is satisfiable since $\Gamma \subseteq \Sigma$. □

The Gödel Completeness Theorem is often succinctly reformulated as follows where $\Gamma \vDash \varphi$ is used to express the condition that $\Gamma \cup \{(\neg\varphi)\}$ is not satisfiable.

Theorem 5.37 *For any set of $\mathcal{L}$-formulas Γ and any $\mathcal{L}$-formula φ,*

$$\Gamma \vDash \varphi \text{ if and only if } \Gamma \vdash \varphi. \quad \square$$

Corollary 5.38 *An $\mathcal{L}$-formula is valid if and only if it is provable.*

Proof. From Theorem 5.37, take Γ to be Δ and refer to Definition 5.4 □

5.7 The Craig Interpolation Theorem

The Completeness Theorem states that if φ is satisfied whenever Γ is satisfied then there is a proof of φ from Γ. In this section, we generalize the Completeness Theorem to the restricted languages $\mathcal{L}_\mathcal{A}$.

Of course one can fairly easily convince oneself that the proof of the Completeness Theorem can be adapted to the languages $\mathcal{L}_\mathcal{A}$ (at least for those alphabets $\mathcal{A}$ with infinitely many constant symbols). In fact, for the cases where $\mathcal{A}$ has no function symbols, the proof is easier since all terms are then trivial (and one can deduce the Gödel Completeness Theorem for $\mathcal{L}$ from that for $\mathcal{L}_\mathcal{A}$).

We shall take a slightly different approach for it will reveal some additional interesting features of our formal notion of proof, this approach culminates with the statement and proof of the Craig Interpretation Theorem for the language $\mathcal{L}$. This completes a journey which began with the version of this theorem that we proved for the propositional language $\mathcal{L}_0$.

Definition 5.39 Suppose that Γ is a set of $\mathcal{L}_\mathcal{A}$-formulas and that φ is an $\mathcal{L}_\mathcal{A}$-formula. Then $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$ if and only if there exists a proof $\langle \varphi_1, \ldots, \varphi_n \rangle$ of φ from Γ such that for each $i \leq n$, φ_i is an $\mathcal{L}_\mathcal{A}$-formula. □

Eliminating predicate symbols. Suppose that φ is a formula and that P_i is a predicate symbol. Let $[\varphi]_{P_i}$ denote the formula defined as follows by induction on the length of φ:

(1) If φ is an atomic formula, then

$$[\varphi]_{P_i} = \begin{cases} (\tau_1 \mathrel{\hat{=}} \tau_1), & \text{if } \varphi = P_i(\tau_1 \ldots \tau_n), \text{ where } n = \pi(P_i); \\ \varphi, & \text{otherwise.} \end{cases}$$

(2) $[(\neg\psi)]_{P_i} = (\neg[\psi]_{P_i})$.
(3) $[(\psi_1 \rightarrow \psi_2)]_{P_i} = ([\psi_1]_{P_i} \rightarrow [\psi_2]_{P_i})$.
(4) $[(\forall x_i \psi)]_{P_i} = (\forall x_i [\psi]_{P_i})$.

Thus $[\varphi]_{P_i}$ is obtained from φ by replacing every instance of P_i by a trivial formula.

Lemma 5.40 (Predicates) *Suppose that Γ is a set of formulas and that P_i is a predicate symbol which does not occur in any formula of Γ. Suppose that $\langle \psi_1, \ldots, \psi_m \rangle$ is a proof from Γ. Then $\langle [\psi_1]_{P_i}, \ldots, [\psi_m]_{P_i} \rangle$ is a proof from Γ.*

Proof. Note that if P_i does not occur in φ, then $[\varphi]_{P_i} = \varphi$. Thus, for each $\varphi \in \Gamma$, $[\varphi]_{P_i} = \varphi$.

By inspection of Definition 5.2, if φ is a logical axiom then $[\varphi]_{P_i}$ is a logical axiom.

Finally, if φ_1 and φ_2 are formulas then

$$[(\varphi_1 \rightarrow \varphi_2)]_{P_i} = ([\varphi_1]_{P_i} \rightarrow [\varphi_2]_{P_i}).$$

It follows by induction on $n \leq m$, that $\langle [\psi_1]_{P_i}, \ldots, [\psi_n]_{P_i} \rangle$ is a proof from Γ. □

Eliminating function symbols. Suppose that τ is a term and that F_i is a function symbol. Let $[\tau]_{F_i}$ denote the term defined by induction on the length of τ as follows.

(1) $[x_j]_{F_i} = x_j$, $[c_j]_{F_i} = c_j$;
(2) Suppose $\tau = F_j(\tau_1 \ldots \tau_m)$, where $m = \pi(F_j)$. Then

$$[\tau]_{F_i} = \begin{cases} F_j([\tau_1]_{F_i}, \ldots, [\tau_m]_{F_i}), & \text{if } F_i \neq F_j; \\ [\tau_1]_{F_i}, & \text{otherwise.} \end{cases}$$

Let $[\varphi]_{F_i}$ denote the formula defined as follows by induction on the length of φ.

(1) If φ is an atomic formula and $\varphi = (\tau_1 \mathrel{\hat{=}} \tau_2)$, then $[\varphi]_{F_i} = ([\tau_1]_{F_i} \mathrel{\hat{=}} [\tau_2]_{F_i})$.
(2) If φ is an atomic formula and $\varphi = P_j(\tau_1 \ldots \tau_n)$, where $n = \pi(P_j)$, then $[\varphi]_{F_i} = P_j([\tau_1]_{F_i}, \ldots, [\tau_n]_{F_i})$.
(3) $[(\neg\psi)]_{F_i} = (\neg[\psi]_{F_i})$.

(4) $[\psi_1 \to \psi_2]_{F_i} = ([\psi_1]_{F_i} \to [\psi_2]_{F_i})$.
(5) $[(\forall x_i \psi)]_{F_i} = (\forall x_i [\psi]_{F_i})$.

Thus $[\varphi]_{F_i}$ is obtained from φ by replacing those terms which express application of F_i by simpler terms which do not refer to F_i. Thus in essence, we are in effect trivializing F_i.

Lemma 5.41 (Functions) *Suppose that Γ is a set of formulas and that F_i is a function symbol which does not occur in any formula of Γ. Suppose that $\langle \psi_1, \ldots, \psi_m \rangle$ is a proof from Γ. Then $\langle [\psi_1]_{F_i}, \ldots, [\psi_m]_{F_i} \rangle$ is a proof from Γ.*

Proof. The proof is quite similar to that of the Lemma on Predicates, Lemma 5.40.

Note that if F_i does not occur in φ then $[\varphi]_{F_i} = \varphi$. Thus for each $\varphi \in \Gamma$, $[\varphi]_{F_i} = \varphi$. By inspection of Definition 5.2, if φ is a logical axiom then $[\varphi]_{F_i}$ is a logical axiom. Finally, if φ_1 and φ_2 are formulas, then $[(\varphi_1 \to \varphi_2)]_{F_i} = ([\varphi_1]_{F_i} \to [\varphi_2]_{F_i})$.

It follows by induction on $n \leq m$, that $\langle [\psi_1]_{F_i}, \ldots, [\psi_n]_{F_i} \rangle$ is a proof from Γ. □

As a corollary we obtain the following theorem which confirms that the potentially two different notions of proof for $\mathcal{L}_{\mathcal{A}}$ are the same.

This of course would have to be the case by the Gödel Completeness Theorem for $\mathcal{L}_{\mathcal{A}}$, which is Theorem 5.43 below.

The key point which underlies of all of this, is that logical implication, this is the relation that $\Gamma \vDash \varphi$, is obviously necessarily the same where $\Gamma \cup \{\varphi\}$ is a set of $\mathcal{L}_{\mathcal{A}}$-formulas, whether it is defined as we have defined it or specialized to the language $\mathcal{L}_{\mathcal{A}}$.

Theorem 5.42 *Suppose that Γ is a set of $\mathcal{L}_{\mathcal{A}}$-formulas and that φ is an $\mathcal{L}_{\mathcal{A}}$-formula. Then*

$\Gamma \vdash \varphi$ *if and only if* $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}}} \varphi$.

Proof. The implication from right to left is immediate, so we have only to prove that if $\Gamma \vdash \varphi$ then $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}}} \varphi$. Let $\langle \psi_1, \ldots, \psi_m \rangle$ be a proof from Γ of φ.

Let $\langle \mathcal{A}_k : k \leq n \rangle$ be an increasing sequences of sets such that for all $k < n$,

(1.1) $\mathcal{A}_k$ is an alphabet.
(1.2) $\mathcal{A}_{k+1} \backslash \mathcal{A}_k$ contains at most one element,
(1.3) $\mathcal{A}_0 = \mathcal{A}$,
(1.4) For each $j \leq m$, ψ_j is an $\mathcal{L}_{\mathcal{A}_n}$-formula.

Thus by condition (1.4), $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}_n}} \varphi$. We prove:

Claim: For each $k < n$, if $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}_{k+1}}} \varphi$ then $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}_k}} \varphi$.

We prove the claim by just proving directly (and *not* by induction) that the conclusion of the claim holds for each $k < n$. Fix $k < n$. The claim (for k) follows from Lemma 5.10 if $A_{\mathcal{L}_{\mathcal{A}_{k+1}}} \backslash A_{\mathcal{L}_{\mathcal{A}_k}}$ contains only a constant symbol; it follows from Lemma 5.40 if $A_{\mathcal{L}_{\mathcal{A}_{k+1}}} \backslash A_{\mathcal{L}_{\mathcal{A}_k}}$ contains only a predicate symbol; and it follows from Lemma 5.41 if $A_{\mathcal{L}_{\mathcal{A}_{k+1}}} \backslash A_{\mathcal{L}_{\mathcal{A}_k}}$ contains only a predicate symbol. This proves the claim.

Thus (by reverse induction), $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$, since $\mathcal{A}_0 = \mathcal{A}$. □

Theorem 5.43 (Gödel Completeness Theorem for $\mathcal{L}_\mathcal{A}$) *Suppose that Γ is a set of $\mathcal{L}_\mathcal{A}$-formulas and that φ is an $\mathcal{L}_\mathcal{A}$-formula. Then*

$\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$ *if and only if* $\Gamma \models \varphi$.

Proof. By Theorem 5.37, $\Gamma \models \varphi$ if and only if $\Gamma \vdash \varphi$. By Theorem 5.42, $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$. Thus, $\Gamma \models \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$, as required. □

The three lemmas on Functions, Predicates, and Constants (these are Lemma 5.41, Lemma 5.40, and Lemma 5.10) can be generalized and combined into a single theorem, the Craig Interpolation Theorem.

For the proof of the Craig Interpolation Theorem it is convenient to use the following definitions. Recall that a theory is a set of sentences, see page 41.

Definition 5.44 Suppose $\mathcal{A}$ is an alphabet, $\Gamma \subset \mathcal{L}_\mathcal{A}$ is a theory, and Γ is consistent. Then Γ is $\mathcal{L}_\mathcal{A}$-*maximally consistent* if for all sentences $\varphi \in \mathcal{L}_\mathcal{A}$, either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$. □

Definition 5.45 Suppose $\mathcal{A}$ is an alphabet and $\Gamma \subset \mathcal{L}_\mathcal{A}$. Suppose that $\mathcal{C}$ is a nonempty set of constants and that $\mathcal{C} \subseteq \mathcal{A}$. Then Γ has the $\mathcal{C}$-*Henkin property* if for each formula $(\exists x_k \varphi) \in \Gamma$ there exists $c_m \in \mathcal{C}$ such that $\varphi(x_k; c_m) \in \Gamma$. □

The following variations on Generalization (Theorem 5.6) will also be useful.

Lemma 5.46 *Suppose $\Gamma \subset \mathcal{L}_\mathcal{A}$ is a set of sentences, φ is a sentence, θ is a formula, and*

$$\Gamma \vdash (\varphi \to \theta).$$

Suppose x_k is a variable. Then

$$\Gamma \vdash (\varphi \to (\forall x_k \theta)).$$

Proof. Thus $\Gamma \cup \{\varphi\} \vdash \theta$. Every formula in $\Gamma \cup \{\varphi\}$ is a sentence and so by Generalization, $\Gamma \cup \{\varphi\} \vdash (\forall x_k \theta)$. Finally by Deduction,

$$\Gamma \vdash (\varphi \to (\forall x_k \theta)).$$ □

Lemma 5.47 *Suppose $\Gamma \subset \mathcal{L}_\mathcal{A}$ is a set of sentences, φ is a sentence, θ is a formula, and*

$$\Gamma \vdash (\theta \to \varphi).$$

Suppose x_k is a variable. Then

$$\Gamma \vdash ((\forall x_k \theta) \to \varphi)$$

Proof. Since every formula in Γ is a sentence, by Generalization

$$\Gamma \vdash (\forall x_k(\theta \to \varphi)).$$

But both

(1.1) $((\forall x_k(\theta \to \varphi)) \to ((\forall x_k \theta) \to (\forall x_k \varphi)))$
(1.2) $((\forall x_k \varphi) \to \varphi$

are logical axioms. Thus by Inference (twice),

$$\Gamma \vdash ((\forall x_k \theta) \to \varphi) \qquad \square$$

The following lemma, which is left to the exercises, motivates the key strategy in the proof of Theorem 5.49

Lemma 5.48 *Suppose that T_1 and T_2 are theories of $\mathcal{L}_{\mathcal{A}}$ such that $T_1 \cup T_2$ has no models. Then there is a sentence θ such that $T_1 \vdash \theta$ and such that $T_2 \vdash (\neg\theta)$.*$\square$

Suppose that Σ is a theory. Let $\mathcal{A}_\Sigma$ be the set of constant symbols, predicate symbols and function symbols which occur in some sentence of Σ. Thus $\mathcal{A}_\Sigma$ is the minimum alphabet (under set inclusion), $\mathcal{A}$, such that each sentence of Σ is in the language of $\mathcal{L}_{\mathcal{A}}$.

Theorem 5.49 (Craig Interpolation Theorem: Version I) *Suppose Γ is a set of sentences, φ_1 and φ_2 are sentences, and that*

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Let

$$\mathcal{A}_* = \left(\mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}}\right) \cup \mathcal{A}_\Gamma$$

Then there is a sentence ψ, called the interpolant, such that the following hold.

(1) *ψ is a sentence in the language $\mathcal{L}_{\mathcal{A}_*}$.*
(2) *$\Gamma \vdash (\varphi_1 \to \psi)$.*
(3) *$\Gamma \vdash (\psi \to \varphi_2)$.*

Proof. We assume toward a contradiction that there is no interpolant ψ with the properties (1)-(3). By Lemma 5.21 and by Lemma 5.22, we can reduce to the case that there is an infinite set $\mathcal{C}$ of constants c_m such that c_m does not occur in $\mathcal{A}_\Gamma \cup \mathcal{A}_{\{\varphi_1\}} \cup \mathcal{A}_{\{\varphi_2\}}$.

Let

$$\mathcal{A}_*^{'} = \left(\mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}}\right) \cup \mathcal{A}_\Gamma \cup \mathcal{C}$$

We will prove the Craig Interpolation Theorem using a method similar to how we proved the completeness theorem, but instead of using the consistency of a set of formulas as a guide on how to build our extensions, we use the notion of inseparability instead. This we define below.

Suppose T and U are theories in $\mathcal{L}_{\mathcal{A}_*^{'}}$. Then:

(1.1) Suppose that $\theta \in \mathcal{L}_{\mathcal{A}'_*}$ and θ is a sentence. Then θ *separates* T and U if and only if $T \vdash \theta$ and $U \vdash (\neg\theta)$.

(1.2) T and U are *inseparable* if and only if there is no sentence in the language of $\mathcal{L}_{\mathcal{A}'_*}$ which separates them.

We prove that there is a model $\mathcal{M}$ such that

(2.1) $\mathcal{M} \vDash \Gamma$,

(2.2) $\mathcal{M} \vDash (\varphi_1 \wedge (\neg\varphi_2))$.

This by the Soundness Theorem will contradict that

$$\Gamma \vdash (\varphi_1 \rightarrow \varphi_2).$$

We first prove:

(3.1) $\Gamma \cup \{\varphi_1\}$ is consistent.

(3.2) $\Gamma \cup \{(\neg\varphi_2)\}$ is consistent.

Assume $\Gamma \cup \{\varphi_1\}$ is not consistent. Choose $\psi \in \mathcal{L}_{\mathcal{A}_*}$ such that ψ is a contradiction. Thus $\Gamma \cup \{\varphi_1\} \vdash \psi$ and so by the Deduction Theorem,

$$\Gamma \vdash (\varphi_1 \rightarrow \psi).$$

Since ψ is a contradiction, necessarily

$$\Gamma \vdash (\psi \rightarrow \varphi_2)$$

and this contradicts there is no interpolant ψ with the properties (1)-(3).

Similarly, assume $\Gamma \cup \{(\neg\varphi_2)\}$ is not consistent. Then $\Gamma \vdash \varphi_2$. But $\Gamma \vdash (\varphi_1 \rightarrow \varphi_2)$ and so necessarily $\Gamma \vdash \varphi_1$. Choose $\psi \in \mathcal{L}_{\mathcal{A}_*}$ such that such that $\Gamma \vdash \psi$. Then $\Gamma \vdash (\varphi_1 \rightarrow \psi)$ and $\Gamma \vdash (\psi \rightarrow \varphi_2)$. This again contradicts there is no interpolant ψ with the properties (1)-(3).

Recall that $\mathcal{A}_{\{\varphi_1\}}$ is the set of all the nonlogical symbols which occur in φ_1; similarly for $\mathcal{A}_{\{\varphi_2\}}$. Further $\mathcal{A}_* = (\mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}}) \cup \mathcal{A}_\Gamma$ and $\mathcal{A}'_* = \mathcal{A}_* \cup \mathcal{C}$. Let

- ${\mathcal{A}_1}' = \mathcal{A}_{\{\varphi_1\}} \cup \mathcal{A}_\Gamma \cup \mathcal{C}$ and $\mathcal{L}'_1 = \mathcal{L}_{\mathcal{A}'_1}$
- ${\mathcal{A}_2}' = \mathcal{A}_{\{\varphi_2\}} \cup \mathcal{A}_\Gamma \cup \mathcal{C}$ and $\mathcal{L}'_2 = \mathcal{L}_{\mathcal{A}'_2}$
- $\mathcal{L}'_* = \mathcal{L}_{\mathcal{A}'_*}$.

First we prove:

(4.1) The theories $\{\varphi_1\} \cup \Gamma$ and $\{(\neg\varphi_2)\} \cup \Gamma$ are inseparable.

Assume toward a contradiction that they are separable. Then there exists a sentence $\theta \in \mathcal{L}_{\mathcal{A}'_*}$ such that $\Gamma \cup \{\varphi_1\} \vdash \theta$ and $\Gamma \cup \{(\neg\varphi_2)\} \vdash (\neg\theta)$. Thus by the Deduction Theorem, we get that

$$\Gamma \vdash (\varphi_1 \rightarrow \theta) \text{ and } \Gamma \vdash ((\neg\varphi_2) \rightarrow \neg\theta)$$

which implies that

$$\Gamma \vdash (\varphi_1 \rightarrow \theta) \text{ and } \Gamma \vdash (\theta \rightarrow \varphi_2).$$

If θ contains no constants from $\mathcal{C}$ then $\theta \in \mathcal{L}_{\mathcal{A}_*}$ and this contradicts our assumption that no interpolant exists. Let $c_{n_1}, \ldots, c_{n_k}$ be the constants of $\mathcal{C}$ which occur in θ and let $x_{m_1}, \ldots, x_{m_k}$ be distinct variables which do not occur in φ_1, φ_2, or θ. Let

$$\theta^* = \theta(c_{n_1}, \ldots, c_{n_k}; x_{m_1}, \ldots, x_{m_k})$$

be the formula obtained from θ by substituting x_{m_i} for each occurrence of c_{n_i}, for each $i = 1, \ldots, k$. Each variable x_{m_i} is free for c_{n_i} in θ (since it does not occur in in θ).

Since no $c_m \in \mathcal{C}$ occurs in any formula of Γ and since every formula of Γ is a sentence:

$$\Gamma \vdash (\varphi_1 \to \theta^*) \text{ and } \Gamma \vdash (\theta^* \to \varphi_2).$$

Now let

$$\theta^{**} = (\forall x_{m_1} \ldots (\forall x_{m_k} \theta^*) \ldots).$$

Since every formula in $\Gamma \cup \{\varphi_1, \varphi_2\}$ is a sentence, it follows by repeated applications of Lemma 5.46 and Lemma 5.47 that:

$$\Gamma \vdash (\varphi_1 \to \theta^{**}) \text{ and } \Gamma \vdash (\theta^{**} \to \varphi_2).$$

But $\theta^{**} \in \mathcal{L}_{\mathcal{A}_*}$ and θ^{**} is a sentence. This again contradicts our assumption that no interpolant exists.

This proves (4.1).

Both $\mathcal{L}_1^{'}$ and $\mathcal{L}_2^{'}$ are countable languages, and so we can enumerate their sentences:

$$\sigma_0, \sigma_1, \sigma_2, \ldots$$

$$\delta_0, \delta_1, \delta_2, \ldots$$

where for each i, the σ_i's and the δ_i's are in the languages $\mathcal{L}_1^{'}$ and $\mathcal{L}_2^{'}$, respectively.

We now inductively construct two increasing sequences of theories via the following steps:

(5.1) $T_0 = \{\varphi_1\} \cup \Gamma$

(5.2) $U_0 = \{(\neg\varphi_2)\} \cup \Gamma$

(5.3) If $T_i \cup \{\sigma_i\}$ and U_i are inseparable, then $\sigma_i \in T_{i+1}$.

(5.4) If T_{i+1} and $U_i \cup \{\delta_i\}$ are inseparable, then $\delta_i \in U_{i+1}$.

(5.5) If $\sigma_i = (\exists x_k \sigma)$ and $\sigma_i \in T_{i+1}$, then $\sigma(x_k; c_m) \in T_{i+1}$ for some $c_m \in \mathcal{C}$ such that c_m does not occur in σ_i and such that c_m does not occur in any formula of T_i.

(5.6) If $\delta_i = (\exists x_k \delta)$ and $\delta_i \in U_{i+1}$, then $\delta(x_k; c_m) \in U_{i+1}$ for some $c_m \in \mathcal{C}$ that such c_m does not occur in δ_i and such that c_m does not occur in any formula of U_i.

We then define the two sets:

$$T_\omega = \bigcup_{i<\omega} T_i \text{ and } U_\omega = \bigcup_{i<\omega} U_i$$

We have proved that $\{\varphi_1\} \cup \Gamma$ and $\{(\neg\varphi_2)\} \cup \Gamma$ are inseparable. Further by (5.3) and (5.4), inseparability is preserved at each step of our construction. Thus by induction it follows easily that T_ω and U_ω are inseparable theories.

Note that by (5.5) and (5.6), T_ω and U_ω each have the $\mathcal{C}$-Henkin property.

We claim that T_ω and U_ω are both consistent sets of sentences. To see this assume toward a contradiction that T_ω is inconsistent. Fix $\theta \in \mathcal{L}_{\mathcal{A}_*}$ such that $(\neg\theta) \in U_\omega$. Then

$$U_\omega \vdash (\neg\theta)$$

and since T_ω is inconsistent, $T_\omega \vdash \theta$. Thus there exists a sentence which separates T_ω and U_ω, contradicting their inseparability. Similarly, if U_ω is inconsistent then there exists a sentence which separates T_ω and U_ω, again contradicting their inseparability.

We continue by proving a series of claims.

Claim 1: T_ω is $\mathcal{L}_1'$-maximally consistent.

Proof of Claim 1: Suppose that T_ω is not $\mathcal{L}_1'$-maximally consistent, then there is some $\sigma \in \mathcal{L}_1'$ such that $\sigma \notin T_\omega$ but $T_\omega \cup \{\sigma\}$ is consistent. Then for some $n < \omega$, $T_n \cup \{\sigma\}$ and U_n are separable, so there exists a sentence $\theta_1 \in \mathcal{L}_{\mathcal{A}_*}$ such that $T_n \cup \{\sigma\} \vdash \theta_1$ and $U_n \vdash (\neg\theta_1)$.

Furthermore, since $\sigma \notin T_\omega$, $T_\omega \cup \{(\neg\sigma)\}$ is also consistent. Then for some $m < \omega, T_m \cup \{(\neg\sigma)\}$ and U_m are separable, so there exists a sentence $\theta_2 \in \mathcal{L}_{\mathcal{A}_*}$ such that $T_m \cup \{(\neg\sigma)\} \vdash \theta_2$ and $U_m \vdash (\neg\theta_2)$.

We then get that

(6.1) $T_\omega \vdash (\theta_1 \vee \theta_2)$ (since $T_n \cup \{\sigma\} \vdash \theta_1$ and $T_m \cup \{(\neg\sigma)\} \vdash \theta_2$)
(6.2) $U_\omega \vdash (\,(\neg\theta_1) \wedge (\neg\theta_2)\,)$ (since $U_n \vdash (\neg\theta_1)$ and $U_m \vdash (\neg\theta_2)$.

But $(\,(\neg\theta_1) \wedge (\neg\theta_2)\,)$ is equivalent to $(\neg(\theta_1 \vee \theta_2))$, and so we have a contradiction since T_ω and U_ω are inseparable.

This proves Claim 1 and the same argument proves:

Claim 2: U_ω is $\mathcal{L}_2'$-maximally consistent.

Both Claim 1 and Claim 2 would hold if we defined separability by requiring the witness sentence θ actually be in $\mathcal{L}_\mathcal{A}$ where

$$\mathcal{A} = \left(\mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}}\right) \cup \mathcal{C}.$$

It is for the next claim that we need to use the definition we have given where the sentence θ is only required to be in $\mathcal{L}_{\mathcal{A}_*'}$.

Claim 3: $T_\omega \cap U_\omega$ is $\mathcal{L}_*'$-maximally consistent.

Proof of Claim 3: Suppose not, then there is some sentence $\sigma \in \mathcal{L}_*'$ such that $\sigma \notin T_\omega \cap U_\omega$ and such that $(T_\omega \cap U_\omega) \cup \{\sigma\}$ is consistent.

First we consider the case where $\sigma \in T_\omega$ but $\sigma \notin U_\omega$. The case where σ is instead in U_ω instead of T_ω is symmetric. Since U_ω is $\mathcal{L}_2'$-maximally consistent, then $(\neg\sigma) \in U_\omega$. But then we have that $T_\omega \vdash \sigma$ and $U_\omega \vdash (\neg\sigma)$, contradicting their inseparability.

Finally suppose that $\sigma \notin T_\omega$ and $\sigma \notin U_\omega$. Since both theories are maximally consistent, we have that $(\neg\sigma) \in T_\omega$ and $(\neg\sigma) \in U_\omega$. Thus $(\neg\sigma) \in T_\omega \cap U_\omega$. But then this implies that $(T_\omega \cap U_\omega) \cup \{\sigma\}$ is inconsistent, which contradicts our assumption that $(T_\omega \cap U_\omega) \cup \{\sigma\}$ is consistent. Thus, $T_\omega \cap U_\omega$ must be a $\mathcal{L}_*^{'}$-maximally consistent.

Claim 4: There exists a model $\mathcal{M}$ of $T_\omega \cup U_\omega$.

Proof of Claim 4: Let $\mathcal{M}_1 = (M_1, I_1)$ be the Henkin model of T_ω and let $\mathcal{M}_2 = (M_2, I_2)$ be the Henkin model of U_ω. Thus since both T_ω and U_ω have the $\mathcal{C}$-Henkin property:

(7.1) $M_1 = \{I_1(c_m) \mid c_m \in \mathcal{C}\}$.
(7.2) $M_2 = \{I_2(c_m) \mid c_m \in \mathcal{C}\}$.

Now we come to the key point. Since $T_\omega \cap U_\omega$ is $\mathcal{L}_*^{'}$-maximally consistent, we can reduce to the case that

(8.1) $M_1 = M_2 = \{I_1(c_m) \mid c_m \in \mathcal{C}\} = \{I_2(c_m) \mid c_m \in \mathcal{C}\}$.
(8.2) For all $c_m \in \mathcal{C}$, $I_1(c_m) = I_2(c_m)$.

Thus we can "merge" the models $\mathcal{M}_1$ and $\mathcal{M}_2$ into a single model $\mathcal{M} = (M, I)$ which is an $\mathcal{L}_\mathcal{A}$-structure where

$$\mathcal{A} = \mathcal{A}_{\{\varphi_1\}} \cup \mathcal{A}_{\{\varphi_1\}} \cup \mathcal{A}_\Gamma \cup \mathcal{C}$$

by simply defining

(9.1) $M = M_1 = M_2$,
(9.2) $I = I_1 \cup I_2$.

Thus:

$$\mathcal{M} \vDash T_\omega \cup U_\omega,$$

and this proves Claim 4.

Finally:

(10.1) $\Gamma \cup \{\varphi_1\} \subseteq T_\omega$,
(10.2) $\Gamma \cup \{(\neg\varphi_2)\} \subseteq U_\omega$,

and so by Claim 4, necessarily $\Gamma \cup \{\varphi_1\} \cup \{(\neg\varphi_2)\}$ is satisfiable. This contradicts that $\Gamma \vdash (\varphi_1 \to \varphi_2)$. $\square$

The following variation of the Craig Interpolation Theorem is an immediate special case of Theorem 5.49.

Theorem 5.50 (Craig Interpolation Theorem: Version II) *Suppose that φ_1 and φ_2 are sentences and that*

$$\emptyset \vdash (\varphi_1 \to \varphi_2).$$

Let

$$\mathcal{A}_* = \mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}}.$$

Then there is a sentence ψ, called the interpolant, such that the following hold.

(1) *ψ is a formula in the language $\mathcal{L}_{\mathcal{A}_*}$.*
(2) $\emptyset \vdash (\varphi_1 \to \psi)$.
(3) $\emptyset \vdash (\psi \to \varphi_2)$. □

5.7.1 Exercises

(1) Prove Lemma 5.48.
(2) Consider the construction of the theories T_ω and U_ω as in the proof for the Craig Interpolation Theorem (Version I). Show that the reducts (see Definition 3.4 on page 44) of their Henkin models to the alphabet, $(\mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}}) \cup \mathcal{A}_\Gamma$, are isomorphic.

5.8 The Compactness Theorem

In the previous section, we established the result that a given formula φ is valid if and only if it is provable. That is, every validity has a proof. Gödel's Completeness Theorem also has another very important consequence: the Compactness Theorem.

Compactness is used throughout model theory as a method to construct models for certain theories.

Theorem 5.51 (The Compactness Theorem) *Suppose that Γ is a set of formulas and that for every finite subset Γ_0 of Γ, Γ_0 is satisfiable. Then Γ is satisfiable.*

Proof. By contrapositive, suppose that Γ is not satisfiable. Then, by the Completeness Theorem, Γ is not consistent, and so it proves any formula. Let $\langle \psi_1, \ldots, \psi_n \rangle$ be a deduction from Γ such that ψ_n is equal to $(\neg(x_1 \hat{=} x_1))$. Let Γ_0 be $\Gamma \cap \{\psi_1, \ldots, \psi_n\}$. Note that Γ_0 is a finite subset of Γ. Then $\langle \psi_1, \ldots, \psi_n \rangle$ is a deduction from Γ_0 of $(\neg(x_1 \hat{=} x_1))$. By the Soundness Theorem, Γ_0 is not satisfiable since $(\neg(x_1 \hat{=} x_1))$ is not satisfied by any model.

This contradicts our assumption that every finite subset of Γ is satisfiable, and the theorem follows. □

5.9 Applications of the Compactness Theorem

5.9.1 Satisfiability in finite structures

Suppose that Γ is a theory. Recall that an immediate corollary of the Lowenheim-Skolem Theorem is that if Γ is satisfiable, then there is an $\mathcal{M}$ such that the universe of $\mathcal{M}$ is countable and $\mathcal{M} \vDash \Gamma$.

But now, we consider the problem of determining whether Γ has an infinite model or whether Γ has only infinite models. We can prove two relevant results using compactness. For this we restrict to the case of $\mathcal{L}$, but everything applies to the case of $\mathcal{L}_{\mathcal{A}}$ where $\mathcal{A}$ is an alphabet.

Theorem 5.52 *Suppose that Γ is a theory such that for every $n \in \mathbb{N}$, there is an $\mathcal{M}$ such that $(\mathcal{M}, \nu) \models \Gamma$ and the universe of $\mathcal{M}$ has at least n elements. Then there is an $\mathcal{M}$ such that the universe of $\mathcal{M}$ is infinite and $\mathcal{M} \models \Gamma$.*

Proof. Consider the set of formulas Δ defined as follows:

$$\Delta = \Gamma \cup \{(\neg(x_i \hat{=} x_j)) : i \text{ and } j \text{ are distinct natural numbers}\}$$

Suppose that Δ_0 is a finite subset of Δ, and let n be the size of Δ_0. By assumption, choose $\mathcal{M}$ so that $\mathcal{M} \models \Gamma$ and so that the universe of $\mathcal{M}$ has at least $2n$ elements.

Γ is a theory, so no variable occurs freely in any element of Γ. The elements of $\Delta_0 \setminus \Gamma$ are formulas of the form $(\neg(x_i \hat{=} x_j))$. Each of these has at most two freely occurring variables. So, there are at most $2n$ variables which occur freely in Δ_0. Choose an $\mathcal{M}$-assignment ν such that for any pair of distinct variables which occur freely in Δ_0, ν assigns these variables to distinct element of the universe of $\mathcal{M}$. Since $\mathcal{M} \models \Gamma$, we get that $(\mathcal{M}, \nu) \models \Delta_0 \cap \Gamma$. By the choice of ν, $(\mathcal{M}, \nu) \models \Delta_0 \setminus \Gamma$. Thus, $(\mathcal{M}, \nu) \models \Delta_0$.

Since Δ_0 was an arbitrary finite subset of Δ, every finite subset of Δ is satisfiable. By the Compactness Theorem, Δ is satisfiable. Suppose that $(\mathcal{M}^*, \nu^*) \models \Delta$, and hence $\mathcal{M}^* \models \Gamma$. Then for all $i \neq j$, $\nu(x_i) \neq \nu(x_j)$ and so the universe of $\mathcal{M}^*$ is infinite. □

Theorem 5.53 *Suppose that φ is an $\mathcal{L}$-sentence such that for every $\mathcal{L}$-structure $\mathcal{M}$, if the universe of $\mathcal{M}$ is infinite, then $\mathcal{M} \models \varphi$. Then there is an n such that for every $\mathcal{L}$-structure $\mathcal{M}$, if the universe of $\mathcal{M}$ has at least n elements, then $\mathcal{M} \models \varphi$.*

Proof. For the sake of a contradiction, suppose that for every n there is an $\mathcal{M}$ such that the universe of $\mathcal{M}$ has at least n elements and $\mathcal{M} \models (\neg\varphi)$. By Theorem 5.52, there is an $\mathcal{L}$-structure $\mathcal{M}^*$ such that $\mathcal{M}^*$ has an infinite universe and $\mathcal{M}^* \models (\neg\varphi)$. The existence of $\mathcal{M}^*$ contradicts our assumption on φ, proving the theorem. □

5.9.2 Wellordered sets

Definition 5.54 Suppose that $<$ is a total ordering of a set X. Then $<$ is a *wellorder* of X if and only if there is no infinite sequence $\langle a_n : n \in \mathbb{N}\rangle$ such that for each n, $a_{n+1} < a_n$. □

Example 5.55 $(\mathbb{N}, <)$ is a wellorder and $(\mathbb{Q}, <)$ is not a wellorder. □

Fix an alphabet $\mathcal{A}_0 = \{P_i\}$ such that $\pi(P_i) = 2$. Thus if $<$ is a total ordering on a set X, then $(X, <)$ defines an $\mathcal{L}_{\mathcal{A}_0}$-structure $\mathcal{M} = (M, I)$ where $M = X$ and $I(P_i) = <$. With this notation, we have the following theorem.

Theorem 5.56 *Suppose that Γ is a theory in the language $\mathcal{L}_{\mathcal{A}_0}$ such that there is an infinite set X and a total ordering $<$ on X such that*

$$(X, <) \models \Gamma.$$

Then there is an infinite set $\hat{X}$ and a total ordering $\hat{<}$ on $\hat{X}$ such that
(1) $(\hat{X}, \hat{<}) \models \Gamma$.
(2) $\hat{<}$ *is not a wellorder of* $\hat{X}$.

Proof. Let Γ_{TOTAL} be the finite set of sentences of $\mathcal{L}_{\mathcal{A}_0}$ which axiomatize the properties of a total order. Let Δ be the following set of formulas.

$$\Delta = \Gamma \cup \{x_j < x_i : i < j \text{ in } \mathbb{N}\} \cup \Gamma_{\mathrm{TOTAL}}$$

Letting $\mathcal{M}_0 = (X, <)$, since X is infinite, for any finite set $\Delta_0 \subset \Delta$, there exists an $\mathcal{M}_0$-assignment ν_0 such that

$$(\mathcal{M}_0, \nu_0) \models \Delta_0.$$

Thus by the Compactness Theorem, Δ is satisfiable. Suppose that

$$(\mathcal{M}, \nu) \models \Delta.$$

Then $\mathcal{M}$ is a total order which satisfies Γ and which is not a wellorder. $\square$

5.9.3 Exercises

(1) Suppose that T is a theory of $\mathcal{L}_{\mathcal{A}}$ and that there is an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{N} = (N, J)$ such that
- $\mathcal{N} \models T$
- N is infinite.

Show that there is an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M} = (M, I)$ and an element a of M such that
(a) $\mathcal{M} \models T$
(b) a is not definable in $\mathcal{M}$ without parameters.

(2) Suppose that $\mathcal{M}$ is an infinite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-structure $\mathcal{M}_1$ such that $\mathcal{M}$ and $\mathcal{M}_1$ are elementarily equivalent and $\mathcal{M}_1$ has an element which is not the interpretation of any constant symbol.

(3) Suppose that T is a theory of $\mathcal{L}_{\mathcal{A}}$.
Define T to be *finitely axiomatizable* if and only if there is a finite theory $\Gamma \subset \mathcal{L}_{\mathcal{A}}$ such that for all $\mathcal{M}$, $\mathcal{M} \models \Gamma$ if and only if $\mathcal{M} \models T$.

Suppose that T is not satisfiable. Show that T Is finitely axiomatizable.

(4) Suppose $\mathcal{A} = \{P_i\}$ is an alphabet with just one symbol P_i and $\pi(P_i) = 2$. Let $\mathcal{M} = (M, I)$ be the $\mathcal{L}_{\mathcal{A}}$-structure where
- $M = \mathbb{R}$
- $I(P_i) = \{(a, b) \mid a < b\}$.

Let $T_{\mathcal{M}} = \{\varphi \mid \ \varphi \text{ is an } \mathcal{L}_{\mathcal{A}}\text{-sentence and } \mathcal{M} \vDash \varphi\}$.
Show that $T_{\mathcal{M}}$ is finitely axiomatizable.

Hint: Use the results of Section 4.5 and Section 4.6.

(5) Suppose that T_1 and T_2 are sets of sentences of $\mathcal{L}_{\mathcal{A}}$ such that for every $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$, $\mathcal{M} \vDash T_1$ if and only if $\mathcal{M} \nvDash T_2$.
Show that T_1 and T_2 are finitely axiomatizable.

Hint: Note that if a theory T is not finitely axiomatizable, *and* if T is satisfiable, then for each finite set $\Sigma \subseteq T$, there must exist a model $\mathcal{M} \vDash \Sigma$ such that $\mathcal{M} \nvDash T$.

6

Advanced Topics

6.1 Types

When we consider various mathematical structures, we can ask questions regarding the connection between their properties and the formulas we can write down to express them.

Given a particular structure, we could ask if the theory of a that structure is finitely axiomatizable (see Exercise (3) on page 114). A classic counterexample is a field with characteristic zero.

We can establish such results and much more, using *types*. In general, the analysis of types helps one understand the limits of first order languages.

Definition 6.1 Suppose $n \in \mathbb{N}$. An $(n+1)$-*type* of the language $\mathcal{L}_{\mathcal{A}}$ is a set Γ of $\mathcal{L}_{\mathcal{A}}$-formulas such that

(1) Γ is consistent,
(2) (maximality) Suppose $\varphi = \varphi(x_0, \ldots, x_n)$ is an $\mathcal{L}_{\mathcal{A}}$-formula such that every free variable of φ is includes in $\{x_0, \ldots, x_n\}$. Then either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$,
(3) Every variable which occurs freely in an element of Γ belongs to the set $\{x_0, \ldots, x_n\}$. □

We write $\Gamma(x_0, \ldots, x_n)$ to indicate that Γ is an $(n+1)$-type.

Example 6.2 If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and $m_0, \ldots, m_n$ are elements of M, then

$$\Gamma(x_0, \ldots, x_n) = \{\varphi(x_0, \ldots, x_n) \in \mathcal{L}_{\mathcal{A}} : \mathcal{M} \models \varphi[m_0, \ldots, m_n]\}$$

is an $(n+1)$-type, called the *type of* $(m_0, \ldots, m_n)$ *in* $\mathcal{M}$. □

Definition 6.3 Suppose $\mathcal{M}$ is an $\mathcal{L}_{\mathcal{A}}$-structure. If $\Gamma = \Gamma(x_0, \ldots, x_n)$ is the type of some $(m_0, \ldots, m_n)$ in $\mathcal{M}$, then we say that $\mathcal{M}$ *realizes* Γ. Otherwise, $\mathcal{M}$ *omits* Γ. □

Similarly, we can speak of $\mathcal{M}$'s realizing or omitting a set of formulas even when that set is not a type, that is when that set is not maximally consistent.

Theorem 6.4 *Let T be a theory of $\mathcal{L}_{\mathcal{A}}$, $n \in \mathbb{N}$, and that $\Gamma \subseteq \mathcal{L}_{\mathcal{A}}$ is an $(n+1)$-type extending T. The following are equivalent.*

(1) *There is an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $\mathcal{M}$ realizes Γ.*
(2) *For every finite subset Γ_0 of Γ, there is an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $\mathcal{M}$ realizes Γ_0.*
(3) *The set*

$$T \cup \left\{ (\exists x_0 \exists x_1 \ldots \exists x_n (\varphi_1 \wedge \cdots \wedge \varphi_k)) : \begin{array}{c} k \in \mathbb{N} \text{ and} \\ \varphi_1, \ldots, \varphi_k \in \Gamma \end{array} \right\}$$

is consistent.

Proof.

(1) $\Longrightarrow$ (2):

Clearly, (1) implies (2). Any model of T which realizes Γ also realizes every finite subset of Γ.

(2) $\Longrightarrow$ (3):

Suppose that $\varphi_1, \ldots, \varphi_k$ are elements of Γ. By (2), let $\mathcal{M} = (M, I)$ be a model which realizes $\{\varphi_1, \ldots, \varphi_k\}$. Thus, we may fix $m_0, \ldots, m_n$ in M so that for all $i \leq k$, $\mathcal{M} \models \varphi_i[m_0, \ldots, m_n]$. So, if ν is an assignment such that for all $i \leq n$, $\nu(x_i) = m_i$, then

$$(\mathcal{M}, \nu) \models (\varphi_1 \wedge \cdots \wedge \varphi_k).$$

Consequently, $\mathcal{M} \models (\exists x_0 \exists x_1 \ldots \exists x_n (\varphi_1 \wedge \cdots \wedge \varphi_k))$. If $k > k_1$, then

$$(\exists x_0 \exists x_1 \ldots \exists x_n (\varphi_1 \wedge \cdots \wedge \varphi_k))$$

implies $(\exists x_0 \exists x_1 \ldots \exists x_n (\varphi_1 \wedge \cdots \wedge \varphi_{k_1})$. Claim (3) follows.

(3) $\Longrightarrow$ (1):

By the Completeness Theorem 5.13, there are $\mathcal{M}$ and ν such that

$$(\mathcal{M}, \nu) \models T \cup \Gamma.$$

Letting m_i denote $\nu(x_i)$, we can rewrite this condition by saying that for each $\varphi \in \Gamma$, $\mathcal{M} \models \varphi[m_0, \ldots, m_n]$. Thus, $\mathcal{M}$ realizes Γ, as required. □

Definition 6.5 A theory T of $\mathcal{L}_{\mathcal{A}}$ is $\mathcal{L}_{\mathcal{A}}$-*complete* if and only if for every sentence $\varphi \in \mathcal{L}_{\mathcal{A}}$, either $\varphi \in T$ or $(\neg\varphi) \in T$. □

Note that a consistent theory $T \subset \mathcal{L}_{\mathcal{A}}$ is $\mathcal{L}_{\mathcal{A}}$-complete if and only if T is $\mathcal{L}_{\mathcal{A}}$-maximally consistent, as defined in Definition 5.44.

Definition 6.6 Suppose that T is an $\mathcal{L}_{\mathcal{A}}$-complete theory in the language $\mathcal{L}_{\mathcal{A}}$ and that Γ is an $(n+1)$-type in the language $\mathcal{L}_{\mathcal{A}}$ extending T. Γ is a *principal* type if and only if there is a formula φ in Γ such that for all $\psi(x_0, \dots, x_n)$,

$$\psi \in \Gamma \Leftrightarrow T \cup \{\varphi\} \vdash \psi.$$

Lemma 6.7 *Suppose that T is an $\mathcal{L}_{\mathcal{A}}$-complete consistent theory, and suppose that $\Gamma = \Gamma(x_0, \dots, x_n)$ is an $(n+1)$-type of $\mathcal{L}_{\mathcal{A}}$ which contains T. If T has a model which omits Γ, then Γ is not a principal type.*

Proof. For the sake of a contradiction, suppose that T does not locally omit Γ. Then there is a $\psi(x_0, \dots, x_n)$ such that for all $\varphi \in \Gamma$, $T \vdash (\psi \rightarrow \varphi)$ and such that $T \cup \psi$ is consistent. Since T is consistent, it must be the case that $(\forall x_0 \dots \forall x_n(\neg\psi)) \notin T$. Since T is complete, $(\neg(\forall x_1 \dots \forall x_n(\neg\psi))) \in T$. In other words, $(\exists x_0 \dots \exists x_n \psi) \in T$. Now suppose that $\mathcal{M} = (M, I)$ is given so that $\mathcal{M} \models T$. Since $\mathcal{M}$ satisfies $(\exists x_0 \dots \exists x_n \psi)$, we may fix $m_0, \dots, m_n$ in M so that $\mathcal{M}$ satisfies $\psi[m_0, \dots, m_n]$. As every element of Γ can be deduced from ψ, it follows that for every $\varphi \in \Gamma$, $\mathcal{M} \models \varphi[m_0, \dots, m_n]$. So, $\mathcal{M}$ realizes Γ. Since $\mathcal{M}$ was arbitrary, there is no model of T which omits Γ. This contradicts our assumption on T, and the lemma follows. □

Theorem 6.8 (Omitting Types) *Suppose that $T \subset \mathcal{L}_{\mathcal{A}}$ is an $\mathcal{L}_{\mathcal{A}}$-complete consistent theory. Suppose that $n \in \mathbb{N}$ and that $\Gamma = \Gamma(x_0, \dots, x_n)$ is an $(n+1)$-type of $\mathcal{L}_{\mathcal{A}}$ which contains T and which is not a principal type.*

Then T has a model which omits Γ.

Proof. To keep our notation simple, assume that $\Gamma = \Gamma(x_0)$ is a 1-type.

As in the proof of the Gödel Completeness Theorem (Theorem 5.13) and the proof of the Craig Interpolation Theorem (Theorem 5.49), by Lemma 5.21 and by Lemma 5.22, we can reduce to the case that there is an infinite set of constant symbols which do not appear in $\mathcal{A}$. Let $\{c_{i_k} : k \in \mathbb{N}\}$ be this set and let

$$\mathcal{A}^* = \mathcal{A} \cup \{c_{i_k} \mid k \in \mathbb{N}\}.$$

We proceed as in the proof of the Completeness Theorem to construct a maximally consistent set T_∞ of $\mathcal{L}_{\mathcal{A}^*}$-formulas such that $T \subset T_\infty$, T_∞ has the Henkin property, and such that model given by T_∞ has the required properties.

Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be an enumeration of all $\mathcal{L}_{\mathcal{A}^*}$-formulas, presented so that for each k, c_{i_k} does not appear in $\varphi_0, \varphi_1, \dots, \varphi_k$. We construct a sequence of sets

$$T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

so that the following properties hold.

(1.1) For each k, T_{k+1} is a finite consistent extension of T.

(1.2) For each k, $\varphi_k \in T_{k+1}$ or $(\neg\varphi_k) \in T_{k+1}$.

(1.3) For each k, if $\varphi_k \in T_k$ and φ_k is of the form $(\exists x_j \psi)$, then $\psi(x_j; c_{i_k}) \in T_{k+1}$.

Let $T_0 = \{\varphi_0\}$. Given T_k, construct T_{k+1} by extending T_k in three steps as follows.

A. First, if $T_k \cup \{\varphi_k\}$ is consistent, then add φ_k to T_k. Otherwise, add $(\neg\varphi_k)$ to T_k. Call the result T^a_{k+1}.

B. If φ_k is of the form $(\exists x_j \psi)$ and $\varphi_k \in T^a_{k+1}$, then add $\psi(x_j; c_{i_k})$ to T^a_{k+1}. Call the result T^b_{k+1}.

Steps A and B are essentially identical with the steps that we took in the proof of Theorem 5.31. The only change here, is that now we are working with the language $\mathcal{L}_{\mathcal{A}^*}$ instead of the entire language $\mathcal{L}$.

By the argument given there, if T_k is consistent then so is T^b_{k+1}. Further, in each of these steps we add at most one formula to T_k. Consequently, if T_k is a finite extension of T, then so is T^b_{k+1}.

C. Let T^b_{k+1} be $T \cup \{\psi_1, \ldots, \psi_r\}$. Fix $n_0 \in \mathbb{N}$ such that $n_0 > k$ and such that all the variables which occur in any of the formulas, $\psi_1, \ldots, \psi_r$, are included in the set $\{x_0, \ldots, x_{n_0}\}$.

For each ψ_m where $m \in \{1, \ldots, r\}$, let

$$\hat{\psi}_m = \psi_m(x_0, \ldots, x_k; x_{n_0+1}, \ldots, x_{n_0+k+1}).$$

Thus $x_0, \ldots, x_k$ do not occur in any of the formulas in $\{\hat{\psi}_1, \ldots, \hat{\psi}_r\}$, and moreover the variables which do occur in one of the formulas in $\{\hat{\psi}_1, \ldots, \hat{\psi}_r\}$, are included in $x_{k+1}, \ldots, x_{n_0+k+1}$.

Consider the formula $\theta(x_0)$:

- $\exists x_1 \ldots \exists x_k \exists x_{k+1} \ldots \exists x_{n_0+k+1} (\hat{\psi}_1 \wedge \cdots \wedge \hat{\psi}_r)[c_{i_0} \ldots, c_{i_k}, ; x_k \ldots, x_0])$.

Note that x_0 is the only variable which might occur freely in θ and that $\theta \in \mathcal{L}_{\mathcal{A}}$. In particular, θ contains none of the constants from $\{c_{i_0}, \ldots, c_{i_k}\}$. The actual order of substitutions used to define $\theta(x_0)$ is not important except x_0 must be substituted for c_{i_k}.

Since Γ is not a principal type (and since Γ contains T), there is a formula $\sigma \in \Gamma$ such that

$$T \cup \{\theta\} \nvdash \sigma.$$

Thus, $T \cup \{\theta\} \cup \{(\neg\sigma)\}$ is consistent. We add $(\neg\sigma(x_0; c_{i_k}))$ to T^b_{k+1} and let T_{k+1} be the result, finishing the definition of T_{k+1}.

The consistency of T_{k+1} follows from Theorem 5.11, the theorem on constants, and the fact that x_0 was substituted for c_{i_k} in defining $\theta(x_0)$.

Let T_∞ be the union of the T_k. T_∞ is complete, consistent, and has the Henkin property. Thus, the structure built from constants using T_∞ satisfies T_∞.

Every element of this model is the interpretation of some constant symbol $c_n \in \mathcal{A}^*$. If $c_n \in \mathcal{A}$, then since Γ is a not a principal type, there is an element σ of Γ such that $(\neg\sigma)[x_0; c_n]$ is an element of T and hence of T_∞.

If $c_n \notin \mathcal{A}$, then for some k, c_n is equal to c_{i_k} and we ensured the existence of such a σ during the third step in the definition of T_{k+1} of the construction.

Thus, the structure built from constants using T_∞ omits the type Γ, as required. $\square$

6.1.1 ω-categorical theories

We consider a very interesting collection of theories. These are the $\mathcal{L}_{\mathcal{A}}$-complete consistent theories T in the language $\mathcal{L}_{\mathcal{A}}$ which have only one countable model up to isomorphism.

Lemma 6.9 *Suppose that T is an $\mathcal{L}_{\mathcal{A}}$-complete, consistent, theory. Suppose that $n \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) *There are only finitely many $(n+1)$-types which contain T.*

(2) *Every $(n+1)$-type containing T is principal.*

Proof. We first show that if there are infinitely many principal $(n+1)$-types which contain T, then there is a non-principal $(n+1)$-type which contains T.

Let $\langle \Gamma_k : k \in \mathbb{N} \rangle$ enumerate all principal $(n+1)$-types which contain T. For each $k \in \mathbb{N}$ let $\varphi_k(x_0, \dots, x_n)$ be a formula in Γ_k such that

$$\Gamma_k = \{\psi(x_0, \dots, x_n) \mid T \vdash (\varphi_k \to \psi)\}.$$

Thus for each $k \in \mathbb{N}$, if $i \neq k$ then $(\neg\varphi_k) \in \Gamma_i$.

Let

$$\Sigma = T \cup \{(\neg\varphi_k) \mid k \in \mathbb{N}\}.$$

Suppose $S \subset \Sigma$ is finite. Then $S \subset \Gamma_i$ for all sufficiently large $i \in \mathbb{N}$. Therefore Σ is consistent. Let Γ be an $(n+1)$-type which contains Σ. Then $\Gamma \neq \Gamma_k$ for each $k \in \mathbb{N}$ and so Γ is not a principal type.

Thus if if there are infinitely many principal $(n+1)$-types which contain T, then there is a non-principal $(n+1)$-type which contains T.

Finally suppose there are only finitely many $(n+1)$-types which contain T. We prove that every $(n+1)$-type which contains T is principal.

Let $\langle \Gamma_1, \dots, \Gamma_k \rangle$ enumerate all the $(n+1)$-types which contain T. For each $i, j \leq k$, if $i \neq j$ let $\varphi_i^j(x_0, \dots, x_n) \in \Gamma_i$ be such that $\varphi_i^j \notin \Gamma_j$; and if $i = j$ let φ_i^i be the formula $(x_0 \hat{=} x_0)$ (i. e. a formula in Γ_i).

For each $i \leq k$ let

$$\varphi_i = (\varphi_i^1 \wedge \cdots \wedge \varphi_i^k)$$

It follows that for each $i \leq k$, $\varphi_i \in \Gamma_i$ and for each $j \neq i$, $(\neg\varphi_i) \in \Gamma_j$.

Fix $i \leq k$. We claim that

$$\Gamma_i = \{\varphi(x_0, \dots, x_n) \mid T \vdash (\varphi_i \to \varphi)\}.$$

Suppose not. Then there is a formula $\varphi \in \Gamma_i$ such that $T \cup \{(\neg(\varphi_i \to \varphi))\}$ is consistent. Let Γ be an $(n+1)$-type which contains $T \cup \{(\neg(\varphi_i \to \varphi))\}$. Since

$$\vdash (\neg(\varphi_i \to \varphi)) \to (\neg\varphi),$$

necessarily, $(\neg\varphi) \in \Gamma$. Similarly $\varphi_i \in \Gamma$ since,

$$\vdash (\neg(\varphi_i \to \varphi)) \to \varphi_i.$$

This is a contradiction. Since $\varphi_i \in \Gamma$, $\Gamma \neq \Gamma_j$ for all $j \leq k$ with $j \neq i$. So $\Gamma = \Gamma_i$. But $(\neg\varphi) \in \Gamma$ and $\varphi \in \Gamma_i$.

Thus for each $i \leq k$,

$$\Gamma_i = \{\varphi(x_0, \ldots, x_n) \mid T \vdash (\varphi_i \to \varphi)\},$$

and so every $(n+1)$-type which contains T is principal. □

Definition 6.10 Suppose that T is an $\mathcal{L}_\mathcal{A}$-complete, consistent, theory. T is ω-*categorical* if and only if any two countable models of T are isomorphic. □

Example 6.11 Let $\mathcal{A} = \{P_i\}$ where $\pi(P_i) = 2$. Let T be the set of all $\mathcal{L}_\mathcal{A}$-sentences φ such that

$$(\mathbb{Q}, <) \vDash \varphi.$$

Then by Cantor's theorem, Theorem 4.31, T is ω-categorical. □

Lemma 6.12 *Suppose that T is an $\mathcal{L}_\mathcal{A}$-complete, consistent, theory. Suppose $n \in \mathbb{N}$ and that there is a non-principal $(n+1)$-type which contains T.*

Then T is not ω-categorical.

Proof. Let Γ be a non-principal $(n+1)$-type which contains T. By the Completeness Theorem, there is a countable structure $\mathcal{M}$ such that

$$\mathcal{M} \vDash T,$$

and such that $\mathcal{M}$ realizes Γ.

By the Omitting Types Theorem, Theorem 6.8, there is a countable structure $\mathcal{N}$ such that

$$\mathcal{N} \vDash T$$

and such that $\mathcal{N}$ omits Γ. Clearly $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic. Thus T is not ω-categorical. □

Theorem 6.13 (Ryll-Nardzewski) *Suppose that T is an $\mathcal{L}_\mathcal{A}$-complete, consistent, theory. The following conditions are equivalent.*

(1) *T is ω-categorical.*
(2) *For all $n \in \mathbb{N}$, there are finitely many $(n+1)$-types.*
(3) *For every type Γ extending T, Γ is principal.*

Proof. By Lemma 6.9 and Lemma 6.12, if T is ω-categorical then both (2) and (3) must hold.

We now suppose that (3) holds and prove that T is ω-categorical.

Let $\mathcal{M}_1 = (M_1, I_1)$ and $\mathcal{M}_2 = (M_2, I_2)$ be countable models of T. We prove $\mathcal{M}_1 \cong \mathcal{M}_2$. For this we build an isomorphism f between $\mathcal{M}_1$ and $\mathcal{M}_2$ by a back-and-forth construction reminiscent of the proof that any two countable dense linear orders without endpoints are isomorphic, see Theorem 4.31.

Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be enumerations of M_1 and M_2, respectively. We define $f : M_1 \to M_2$ by defining f on a finite set

$$\{a_{i_1}, \ldots, a_{i_k}\}$$

which includes a_n in its domain and which includes b_n in its range, by induction on n.

We first prove the following claim.

Claim: Suppose $a_{i_0}, \ldots, a_{i_k}$ and $b_{j_0}, \ldots, b_{j_k}$ are given such that $\Gamma_1 = \Gamma_2$ where

- Γ_1 is the type $(a_{i_0}, \ldots, a_{i_k})$ realizes in $\mathcal{M}_1$,
- Γ_2 is the type $(b_{j_0}, \ldots, b_{j_k})$ realizes in $\mathcal{M}_2$.

Then

(1) For any $a \in M_1$, there exists $b \in M_2$ such that $(a_{i_0}, \ldots, a_{i_k}, a)$ realizes the same type in $\mathcal{M}_1$ as $(b_{j_0}, \ldots, b_{j_k}, b)$ does in $\mathcal{M}_2$.

(2) For any $b \in M_2$, there exists $a \in M_1$ such that $(a_{i_0}, \ldots, a_{i_k}, a)$ realizes the same type in $\mathcal{M}_1$ as $(b_{j_0}, \ldots, b_{j_k}, b)$ does in $\mathcal{M}_2$.

Proof. Let

(1.1) Γ be the type of $(a_{i_0}, \ldots, a_{i_k})$ in $\mathcal{M}_1$

(1.2) Γ^+ be the type of $(a_{i_0}, \ldots, a_{i_k}, a)$ in $\mathcal{M}_1$.

By assumption, every type extending T is principal. Fix

$$\psi(x_0, \ldots, x_{k+1}) \in \Gamma^+$$

such that every element of Γ^+ is a consequence of $T \cup \{\psi\}$. Then

$$\mathcal{M}_1 \models (\exists x_{k+1}\psi)[a_{i_1}, \ldots, a_{i_k}]$$

and so $(\exists x_{k+1}\psi) \in \Gamma$. Since Γ is also the type of $(b_{j_0}, \ldots, b_{j_k})$ in $\mathcal{M}_2$,

$$\mathcal{M}_2 \models (\exists x_{k+1}\psi)[b_{j_0}, \ldots, b_{j_k}].$$

Let b be an element of $\mathcal{M}_2$ such that $\mathcal{M}_2 \models \psi[b_{j_0}, \ldots, b_{j_k}, b]$. Since Γ^+ is the set of consequence of $T \cup \{\psi\}$, $(b_{j_0}, \ldots, b_{j_k}, b)$ realizes Γ^+ in $\mathcal{M}_2$, as required to prove the (1) of claim.

The same argument proves (2) of the claim. □

Note that by the proof of the claim we also get the following simple version of that claim.

(2.1) For each a in $\mathcal{M}_1$ there exists $b \in \mathcal{M}_2$ such that the 1-type realized in $\mathcal{M}_1$ by a is the same as the 1-type realized by b in $\mathcal{M}_2$.

Now we can define the isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$. First we can define f on a set $\{a_{i_0}, a_{i_1}\}$ which includes a_0 (so $i_0 = 0$) and which includes b_0 in its range. This is done in two steps. First, choose b such that a_0 realizes the same 1-type $\mathcal{M}_1$ that b realizes in $\mathcal{M}_2$, and set $f(a_0) = b$. By (2.1), b exists.

Second, choose a such that $\langle a_0, a\rangle$ realizes the same 2-type in $\mathcal{M}_1$ that $\langle f(a_0), b_0\rangle$ realizes in $\mathcal{M}_2$. Define $f(a) = b_0$. By the claim, a exists.

At the inductive stage $n+1$, assume that there is a finite subset $\{a_{i_1}, \ldots, a_{i_k}\}$ of M_1 on which we have defined f such that $\Gamma_0 = \Gamma_1$ where

- Γ_1 is the type realized in $\mathcal{M}_1$ by $(a_{i_1}, \ldots, a_{i_k})$.
- Γ_2 is the type realized in $\mathcal{M}_2$ by $(f(a_{i_1}), \ldots, f(a_{i_k}))$.

By using the claim twice, we can first extend the definition of f to include a_{n+1} in its domain, and then extend further to include b_{n+1} in its range, in each case preserving the property that the resulting sequences realize the same type in their respective models.

Let f be the function defined in the limit. M_1 is the domain of f and M_2 is its range. Additionally, f is an elementary function, since for each $k \in \mathbb{N}$ and for each formula $\varphi(x_0, \ldots, x_n)$,

$$\begin{aligned}
&\mathcal{M}_1 \models \varphi[a_0, \ldots, a_n] \\
&\quad \leftrightarrow\ \varphi \text{ is in the type of } \langle a_0, \ldots, a_n\rangle \text{ in } \mathcal{M}_1 \qquad \text{(by definition)} \\
&\quad \leftrightarrow\ \varphi \text{ is in the type of } \langle f(a_0), \ldots, f(a_n)\rangle \text{ in } \mathcal{M}_2 \quad \text{(by construction of } f\text{)} \\
&\quad \leftrightarrow\ \mathcal{M}_2 \models \varphi[f(a_0), \ldots, f(a_n)] \qquad \text{(by definition)}
\end{aligned}$$

Of course, any elementary surjection is an isomorphism, and so f is an isomorphism. This verifies $\mathcal{M}_1 \cong \mathcal{M}_2$, which proves (1). □

6.1.2 Exercises

(1) In the proof of Theorem 6.8, complete the argument that T_{k+1} is consistent from the assumption that T^b_{k+1} is consistent.

(2) Suppose that $\mathcal{A}$ is finite and that $\mathcal{M}$ is a finite $\mathcal{L}$-structure. Show that every type realized in $\mathcal{M}$ is principal.

(3) Suppose that T is a consistent theory such that (for some n) every n-type consistent with T is principal. Show that T has only finitely many consistent completions.

(4) Suppose that T is ω-categorical and M is a countable infinite model of T. Show that M has a nontrivial automorphism.

Hint: Use Theorem 6.13(2) and the proof that (2) implies (1) in the proof of Theorem 6.13.

6.1.3 A theory with uncountably many models

In contrast with ω-categorical theories, there are natural theories which have not just infinitely many non-isomorphic countable models, but uncountably many. In fact, the theory of elementary arithmetic is one such.

Definition 6.14 Let $Th(\mathbb{N})$ denote the set of first order sentences satisfied by the natural numbers with constants for 0 and 1 and with binary function symbols for addition and multiplication. □

Lemma 6.15 *There are uncountably many 1-types extending $Th(\mathbb{N})$.*

Proof. Let p_i denote the ith prime number greater than or equal to 2, and let τ_i denote the term

$$\tau_i = \underbrace{1+1+\ldots+1}_{p_i \text{ many 1's}}$$

More precisely, $\tau_i = \sigma_{p_i}$ where the terms σ_i, for $i \geq 1$, are defined by induction on i as follows. $\sigma_1 = c$ and $\sigma_{i+1} = F(\sigma_i, c)$; where c is the constant interpreted by 1 and F is the function symbol interpreted by the function $+(a,b) = a+b$.

For each $X \subseteq \mathbb{N}$, let G_X be the set of formulas

$$\begin{aligned} &Th(\mathbb{N}) \cup \{\tau_i \text{ is a factor of } x_1 : i \in X\} \\ &\quad \cup \{\tau_i \text{ is not a factor of } x_1 : i \notin X\} \end{aligned}$$

For each X, let Γ_X be a type extending G_X. Since each of these types are consistent, no two of them are equal. Since there are uncountably many subsets of $\mathbb{N}$, the lemma follows. □

Corollary 6.16 *There are uncountably many distinct isomorphism types of countable models of $Th(\mathbb{N})$.*

Proof. Suppose that $\{\mathcal{M}_i : i \in \mathbb{N}\}$ is a countable set of countable models of $Th(\mathbb{N})$. We must show that there is a countable model of $Th(\mathbb{N})$ which is not isomorphic to any of the models $\mathcal{M}_i$, for any $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$, the model $\mathcal{M}_i$ realizes only countably many 1-types. Consequently, the set of 1-types Γ such that Γ is realized in at least one of the models $\mathcal{M}_i$, is a countable set. Since there are uncountably many 1-types extending $Th(\mathbb{N})$, there is a 1-type Δ which is not realized in any of the models $\mathcal{M}_i$, for any $i \in \mathbb{N}$.

By Theorem 6.4, there is a countable $\mathcal{M}$ of $Th(\mathbb{N})$ such that $\mathcal{M}$ realizes Δ. Thus for each $i \in \mathbb{N}$, $\mathcal{M}$ is not isomorphic to $\mathcal{M}_i$. □

6.2 The countable spectrum

Definition 6.17 Suppose $T \subset \mathcal{L}_\mathcal{A}$ is an $\mathcal{L}_\mathcal{A}$-complete consistent theory. The *countable spectrum* of T is the number of countable models of T up to isomorphism. □

Example 6.18 Suppose $T \subset \mathcal{L}_\mathcal{A}$ is an $\mathcal{L}_\mathcal{A}$-complete consistent theory which is ω-categorical. Then the countable spectrum of T is 1. □

By Problem 4 on page 61, we have the following lemma.

Lemma 6.19 *Suppose $T \subset \mathcal{L}_{\mathcal{A}}$ is an $\mathcal{L}_{\mathcal{A}}$-complete consistent theory and there is a finite $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ such that*

$$\mathcal{M} \vDash T$$

Then T is ω-categorical. □

Thus if the countable spectrum of T is not 1 then every model of T is infinite. Therefore the countable spectrum problem (of determining the possibilities) reduces to the cases of $\mathcal{L}_{\mathcal{A}}$-complete consistent theories with *no* finite models.

There are natural examples of $\mathcal{L}_{\mathcal{A}}$-complete consistent theories whose countable spectrum is exactly $\aleph_0$.

Example 6.20 Consider this structure: the field of complex numbers,

$$\mathbf{C} = (\mathbb{C}, 0, 1, i, +, \times).$$

Let T be the set of sentences φ such that

$$\mathbf{C} \vDash \varphi.$$

Then the countable spectrum of T is $\aleph_0$. □

What about the cases where the countable spectrum is finite? Here there is a surprise. There are examples of $\mathcal{L}_{\mathcal{A}}$-complete consistent theories where the countable spectrum is n for various finite n. In fact there are examples for any value of n *except* $n = 2$. Remarkably that value is forbidden.

Theorem 6.21 (Vaught) *Suppose $T \subset \mathcal{L}_{\mathcal{A}}$ is an $\mathcal{L}_{\mathcal{A}}$-complete consistent theory. Then the countable spectrum of T is not 2.* □

6.3 Vaught's conjecture

After 126 pages of definitions and theorems, it might seem that first order logic is well understood. However, there are fundamental questions which remain unsolved, despite decades of research. For example, the following is still open.

Conjecture 6.22 (Vaught) *Suppose that T is a complete consistent theory and that the countable spectrum of T is infinite. Show that one of the following two conditions holds.*

(1) *There is a countable set of models $\{\mathcal{M}_i : i \in \mathbb{N}\}$ such that for every countable $\mathcal{M}$, if $\mathcal{M} \vDash T$, then there is an i such that $\mathcal{M}$ is isomorphic to $\mathcal{M}_i$.*
  - *The countable spectrum of T is $\aleph_0$.*

(2) *There is a set of models $\{\mathcal{M}_X : X \in \mathbb{R}\}$ such that each $\mathcal{M}_X$ satisfies T, and if $X \neq Y$, then $\mathcal{M}_X$ is not isomorphic to $\mathcal{M}_Y$.*
  - *The countable spectrum of T is $2^{\aleph_0}$.* □

Index

Sets	
$\emptyset$	Empty set, $\{\}$
M, N	Arbitrary Sets
$\mathbb{N}$	Natural numbers, $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	Integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	Rationals numbers, $\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$
$\mathbb{R}$	Reals numbers

Logical Symbols (Syntax)	
A_i	Propositional Symbols
φ, ψ	Formulas
$\mathcal{L}_0$	Propositional Language
x_i	Variable Symbol
c_i	Constant Symbol
F_i	Function Symbol
P_i	Predicate Symbol
τ	Term
$\mathcal{A}$	Alphabet
$\mathcal{L}_{\mathcal{A}}$	Language (with alphabet $\mathcal{A}$)

Logical Connectives	
$\neg$	Negation
$\rightarrow$	Implies
$\wedge, \vee$	And, Or (Conjunction, Disjunction)
$\leftrightarrow$	Biconditional "if and only if"
$\forall$	Universal Quantifier
$\exists$	Existential Quantifier
$\vdash$	Proves
$\models$	Satisfies/Models
$\cong$	Isomorphic
$\equiv$	Elementary Equivalence
$\subseteq$	Subset/Substructure
$\preceq$	Elementary Substructure

Logical Symbols (Semantics)	
I, J	Interpretation Function
ν, μ, ρ	Truth Assignment, $\mathcal{M}$-Assignment
$\mathcal{M}, \mathcal{N}$	Structures