141A MATHEMATICAL LOGIC I

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Review of equivalence relations

Recall that E is an **equivalence relation** on a set X if it satisfies the following properties:

(1) $x\ E\ x$ for any $x \in X$. (Reflexive)
(2) If $x\ E\ y$ and $y\ E\ z$ then $x\ E\ z$, for any x, y, z in X. (Transitivity)
(3) If $x\ E\ y$ then $y\ E\ x$ for any x, y in X. (Symmetric)

Note that these properties can be formalized as axioms, as in Exercise 3.39 in the notes. However that is *not* the point at the moment. We will want to deal with external equivalence relations. (That is, not necessarily part of some formal language, or definable in some structure.)

Example 0.1. Let $X = \mathbb{Z}$ and define $n\ E\ m \iff n - m$ is even. Show that E is an equivalence relation.

Given an equivalence relation E on X and $x \in X$, define $[x]_E = \{y \in X : x\ E\ y\}$.

Claim 0.2. For $x, y \in X$, $x\ E\ y$ if and only if $[x]_E = [y]_E$.

Proof. Assume that $x\ E\ y$. Given $z \in [x]_E$, $x\ E\ z$. By symmetry and transitivity, $y\ E\ z$, and so $z \in [y]_E$. We conclude that $[x]_E \subseteq [y]_E$. By symmetry we know that $y\ E\ x$ as well, so the same argument shows $[y]_E \subseteq [x]_E$ as well, and therefore $[x]_E = [y]_E$.

On the other hand, assume that $[x]_E = [y]_E$. Since $y \in [y]_E$, then $y \in [x]_E$, and so by definition $x\ E\ y$, as required. $\square$

Claim 0.3. For $x, y \in X$, if $[x]_E \cap [y]_E \neq \emptyset$ (there is something in the intersection of $[x]_E$ and $[y]_E$, then $x\ E\ y$ and so $[x]_E = [y]_E$.

Proof. Let z be in $[x]_E \cap [y]_E$. Then (using the symmetry condition) $x\ E\ z$ and $z\ E\ y$, and so $x\ E\ y$. $\square$

Corollary 0.4. For $x, y \in X$, $[x]_E$ and $[y]_E$ are disjoint if and only if x and y are *not* E-related.

Let $X/E = \{[x]_E : x \in X\}$ be the set of all E-equivalence classes. This is sometimes called the **quotient space**.

Corollary 0.5. X/E is a partition of X into disjoint (not empty) subsets.

Example 0.6. In the example above, $\mathbb{Z}/E = \{A, B\}$ where A is the set of even numbers and B is the set of odd numbers. Note that $A = [0]_E = [n]_E$ for any even number n, and $B = [1]_E = [n]_E$ for any odd number n.

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Example 0.7. Define E on $\mathbb{R}^2$ by

$$(a, b)\ E\ (c, d) \iff a^2 + b^2 = c^2 + d^2.$$

(1) Prove that E is an equivalence relation on $\mathbb{R}^2$.
(2) Describe the equivalence classes.

Example 0.8 (Group theory)**.** Let G be a group and $H \leq G$ a subgroup. Define E on G by $g\ E\ h \iff g^{-1}h \in H$, for any $g, h \in G$. The quotient space G/E is precisely the (left) cosets of H.

If in addition H is a normal subgroup of G, that is, $g^{-1}Hg = H$ for any $g \in G$, then there is a *natural group structure* on the quotient G/E defined by $[g]_E \cdot [h]_E = [g \cdot h]_E$. You need to use the normality assumption to show that this is well defined. With this operation the quotient space G/E is a group as well: the quotient group.

Example 0.9. Let $(A, <)$ be a linear order. Define E on A as follows. For $a, b \in A$, $a\ E\ b$ if

- either $a = b$; or
- $a < b$ and the set $\{c \in A :\ a < c < b\}$ is finite; or
- $b < a$ and $\{c \in A :\ b < c < a\}$ is finite.

Show that E is an equivalence relation.

Let X/E be the quotient space. There is a natural order on the quotient space making it a linear order as well. For equivalence classes $[x]_E$ and $[y]_E$, define $[x]_E < [y]_E$ if and only if $[x]_E \neq [y]_E$ and $x < y$.

Why is this well defined? The issue is that if $[x]_E = [a]_E$ and $[y]_E = [b]_E$, then the definition $[x]_E < [y]_E$ should also agree with $a < b$.

Indeed, assume $[x]_E \neq [y]_E$ and $x < y$, and $a \in [x]_E$ and $b \in [y]_E$. Then there are infinitely many elements between x and y. However there are only finitely many elements between x and a, and between y and b. Therefore, it cannot be that $b < a$. [Draw a picture.] Since $(A, <)$ is a linear order, it must be that $a < b$.

Similar arguments show that $(X/E, <)$ is in fact a linear order.

Some examples:

(1) Let $(A, <)$ be $(\mathbb{Z}, <)$. What is the quotient space?
(2) Let $(A, <)$ be $(\mathbb{Q}, <)$. What is the quotient space?
(3) Let $A = \mathbb{N} \cup \left\{N + \frac{1}{n} :\ N = 0, 1, 2, ... n = 1, 2, 3, ...\right\}$. Consider $(A, <)$ with the usual order. What is the quotient space?