Math 55b Homework 1

Due Wednesday February 2, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Extensions will be granted when circumstances genuinely warrant it, but should be requested ahead of the deadline (e-mail Prof. Auroux).
- Questions marked * are harder. It's perfectly ok to have no idea how to get started on those on your own.
- You are encouraged to discuss the homework problems with other students. Start using the `#homework` channel on Slack to ask questions, or `#studygroups` to connect with others and work on a problem together. You should also plan on attending one or more discussion section(s) and office hours in order to ask any questions!
- Handwritten submissions are welcome, but if you have spare time, you could learn how use LaTeX! Sign up for Overleaf at `https://www.overleaf.com/edu/harvard` and see the tutorials at `https://www.overleaf.com/learn`

Material covered: Metric spaces (handout; see also Rudin pp. 30-36, pp. 47-51, pp. 83-88). Topological spaces, bases, examples, closed sets and limit points, continuity (Munkres §12-18).

0. (a) If you haven't done so already, please post a short self-introduction on the `#general` Slack channel. Consider including your name, your physical location for the spring semester, what you did during winter break, possibly a hobby or a fun fact about you. (Even if you were in 55a last fall! This will be useful to students who weren't).

(b) Sometime over the weekend of January 28-30, please complete the week 1 feedback survey (in Canvas). This is important to help us assess how well the course structure, pacing, and our efforts at getting students to know each other are working. (There will be more surveys).

1. Exercise 2.4 in the handout on metric spaces.

2. Exercise 3.3 in the handout on metric spaces.

3. Exercise 4.4 in the handout on metric spaces.

4. Munkres exercise 17.6.

5. Munkres exercise 17.8.

6. Munkres exercise 17.11.

7. Munkres exercise 17.13.

8. (a) Let X be a topological space, and $Y \subset X$ any subset. Show that the subspace topology induced on Y is the coarsest topology on Y such that the inclusion map $Y \hookrightarrow X$ is continuous.

(b) Let X and Y be topological spaces, and let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the two projection maps. Show that the product topology on $X \times Y$ is the coarsest topology such that p_1 and p_2 are continuous.

9. Let $X = \mathbb{R}$, and define a topology on X by $\mathcal{T} = \{U \subset X \,|\, U = \emptyset \text{ or } \#(X \setminus U) < \infty\}$, that is, a nonempty subset is open if and only if its complement is a finite set. This is called the *finite complement topology* on $\mathbb{R}$ (and is also an instance of the Zariski topology used by algebraic geometers).

(a) Show that $\mathcal{T}$ is indeed a topology.

(b) In this topology, to what limit or limits does the sequence $p_n = 1/n$ converge?

10.* (optional, extra credit) Prove Kuratowski's theorem: given a topological space X, and starting a subset $A \subset X$, by repeatedly taking the closure and the complement one can obtain at most 14 different subsets of X. Also find a subset of $\mathbb{R}$ with its usual topology for which the upper bound of 14 is achieved.

Hint: denote by cl and $int : \mathcal{P}(X) \to \mathcal{P}(X)$ the maps which take a subset of X to its closure and its interior. First show that $\forall E \subset X$, $int(E) \subset int(cl(int(E)))$ and $cl(int(cl(E))) \subset cl(E)$, then deduce that $cl \circ int \circ cl \circ int = cl \circ int$ and $int \circ cl \circ int \circ cl = int \circ cl$.

11.* (NOT DUE YET) If you have time on your hands, I recommend getting a head start on the problem on the next page, which describes how to construct the *completion* of a metric space; that is, given a metric space X, how to construct a complete metric space in which X sits as a dense subset. (The completion of $\mathbb{Q}$ is one of the two main approaches for rigorously defining the real numbers.)

The problem is fairly long, but does not rely on any material beyond what we have seen about metric spaces; it will be due with HW 4 only, but I encourage you to get a head start on it.

12. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?

11.* (not due yet – due with HW 4) In this problem, we'll see how to construct the *completion* of a metric space; that is, given a metric space X, we'll construct a complete metric space X^* (i.e., every Cauchy sequence in X^* has a limit) in which X sits as a dense subset.

To start, let (X, d) be any metric space, and let $\mathcal{C}(X)$ denote the set of all Cauchy sequences $\{p_n\} = p_1, p_2, p_3 \dots$ in X. We define an equivalence relation $\sim$ on $\mathcal{C}(X)$ by

$$\{p_n\} \sim \{q_n\} \quad \text{iff} \quad d(p_n, q_n) \to 0, \quad \text{i.e.:} \quad \forall \epsilon > 0,\ \exists N : \forall n \geq N,\ d(p_n, q_n) < \epsilon.$$

We then define the set X^* to be the quotient $\mathcal{C}(X)/\sim$, that is, a point $P \in X^*$ is an equivalence class of Cauchy sequences in X. Finally, we define a distance function D on X^* by

$$D(\{p_n\}, \{q_n\}) \;=\; \lim_{n\to\infty} d(p_n, q_n)$$

We will take for granted the fact that $\mathbb{R}$ is complete, i.e. every Cauchy sequence in $\mathbb{R}$ (with its usual distance) has a limit.

(a) Show that $\sim$ is indeed an equivalence relation on $\mathcal{C}(X)$.

(b) Show that D is well defined and gives a metric on X^*.

(Hint: you need to check three things: (1) given two Cauchy sequences in X, the limit in the definition of $D(\{p_n\}, \{q_n\})$ exists; (2) this quantity does not depend on the choice of $\{p_n\}$ in its equivalence class; (3) D satisfies the axioms of a metric).

(c) Show that the metric space (X^*, D) is complete.

(Hint: given a Cauchy sequence $P_1, P_2, \dots$ in (X^*, D), the first step in showing that it converges to some limit $Q \in X^*$ is to construct Q. First choose a Cauchy sequence $p_{n,1}, p_{n,2}, \dots$ in X to represent each P_n, then construct a new sequence $q_1, q_2, \dots$ in X by choosing each $q_n = p_{n,k_n}$ for k_n sufficiently large, so all subsequent terms of the sequence $\{p_{n,k}\}$ are within distance $1/n$ of q_n. Use the triangle inequality to show that $\{q_n\}$ is a Cauchy sequence, defining a point $Q \in X^*$, and finally show that P_n converges to Q in (X^*, D). [Why can't we just choose $q_n = p_{n,n}$?])

(d) Show that the map $\iota : X \to X^*$ defined by $p \mapsto \{p, p, p, \dots\}$ is injective, and that for any $p, q \in X$ we have $D(\iota(p), \iota(q)) = d(p, q)$ (that is, ι is an *isometry*).

(e) Finally, show that the image $\iota(X) \subset X^*$ is dense.

Note that applying this construction to the metric space $\mathbb{Q}$ (with the usual distance) gives $\mathbb{R}$.

In fact, this is one of the ways in which $\mathbb{R}$ can be constructed! The real number field $\mathbb{R}$ is *characterized* by its properties (namely, $\mathbb{R}$ is an ordered field with the least upper bound property), but to prove the existence of such a field, one needs to actually construct it. The two standard approaches are via Dedekind cuts, see e.g. Rudin pp. 17-21, or by completion of $\mathbb{Q}$, see e.g. `http://www.math.ucsd.edu/~tkemp/140A/Construction.of.R.pdf`