Remember:
- Please wear masks at all times. This is really important.
- If sick or in isolation/quarantine, please don't come to class! Instead, watch the lecture on Zoom.

- Outside of lecture:
  - → Office hours & discussion sections
  - → Canvas (notes, assignments, ...)
  - → Slack (please join + introduce yourself in #general)
  - → e-mail

Course staff:  Prof. Denis AUROUX, auroux@math.harvard.edu — office hours Mondays & Wednesdays 12:15-1:15 in SC 530

CAs: Oliver Cheng  | Edis Memis  | Dora Woodruff  | Eric Yan 

- Office hours & sections: to be announced on Canvas.

- See course information & syllabus on Canvas (more logistics, policies, exams)
- Homework due Wednesdays on Canvas. HW 1 (due Feb. 2) is posted.
  Handwritten submissions are fine, or try LaTeX / Overleaf
  Collaboration encouraged (but write your own solution!). Ask CAs for hints if needed!
  Use slack (#studygroups, #homework). List your collaborators.
- Feedback survey to be completed this weekend.

- Please be civil & respectful of each other, and the rest of the math/Harvard community, at all times.

Course Content: first half = topology + real analysis.

1. Point-set topology: topological spaces (incl. some pieces of analysis)
2. Intro to algebraic topology: fundamental groups.
3. A bit more real analysis.

Then move on to complex analysis.

Books you should have:
- → Munkres, Topology, 2nd ed.
- → Ahlfors, Complex analysis, 3rd ed.
- → recommended: Rudin, Principles of Mathematical Analysis

What is topology? Unlike geometry, which concerns quantitative information about spaces (distances, volumes, ...), topology concerns itself with qualitative properties that are invariant under continuous deformation.

Eg: is it connected? (a single piece) simply connected?  vs. 

Point-set topology also gives a language (topological spaces, open & closed sets, compactness ...) both for algebraic topology (associate alg. invariants to spaces, eg fundamental group) and for analysis.

Ex: extreme value theorem says: $f:[a,b]\to\mathbb{R}$ continuous $\Rightarrow$ f achieves its max and min at some points of $[a,b]$.

This is in fact true for any continuous $f: X\to\mathbb{R}$ whenever X is a compact topological space, and is a special instance of:

Theorem: If $f: X\to Y$ continuous mapping between topological spaces, & X compact, then $f(X)$ is compact.

Since the general notion of topological space is quite abstract, let's start with a more familiar class of examples: METRIC SPACES

Def: A metric space (X,d) is a set X together with a distance function $d: X\times X\to\mathbb{R}_{\geq 0}$ s.t.

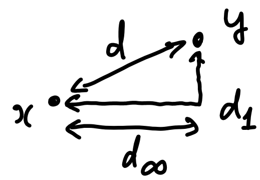
1) For $p,q\in X$, $d(p,q)=0 \Leftrightarrow p=q$
2) —"—, $d(p,q)=d(q,p)$
3) For $p,q,r\in X$, $d(p,r)\leq d(p,q)+d(q,r)$ (triangle inequality)

Ex: $X=\mathbb{R}^n$ with Euclidean distance $d(x,y)=\left(\sum_{i=1}^{n}(y_i-x_i)^2\right)^{1/2}$.

Ex: If $Y\subset X$ then $(Y,d_{|Y})$ is a metric space. ("induced metric").

Ex: different metrics on $\mathbb{R}^n$: $d_1(x,y)=\sum_{i=1}^{n}|y_i-x_i|$

$d_\infty(x,y)=\max(|y_i-x_i|)$

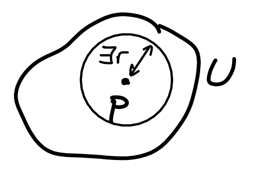


(Exercise: check $(\mathbb{R}^n,d_1)$ & $(\mathbb{R}^n,d_\infty)$ are metric spaces. What do balls look like?)

Def: • (X,d) metric space, $p\in X$, $r>0$; the open ball of radius r around p (or neighborhood) is $B_r(p)=\{q\in X \mid d(p,q)<r\}$.

Here's a more general notion:

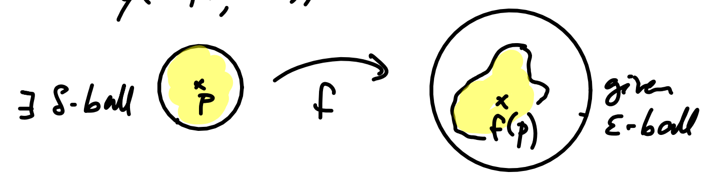
Def: • $U\subset X$ is open if $\forall p\in U$, $\exists r>0$ st. $B_r(p)\subset U$.



Prop: (HW!) open balls are open; so are arbitrary unions & finite intersections of open sets.

- In fact: open sets are unions of open balls! $\left(U = \bigcup_{p \in U} B_{r(p)}(p)\right)$
- This is useful to a general discussion of continuity:

Def: (X, d_X), (Y, d_Y) metric spaces. $f: X \to Y$ is continuous if $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ st. $d_X(p,x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon$.



Theorem: $f: X \to Y$ is continuous iff $\forall U \subset Y$ open, $f^{-1}(U) \subset X$ is open.

Pf:
- assume f continuous, let $U \subset Y$ open, let $p \in f^{-1}(U)$, ie. $f(p) \in U$.
  Since U is open, $\exists \varepsilon > 0$ st. $B_\varepsilon(f(p)) \subset U$.
  By continuity, $\exists \delta > 0$ st. $d_X(p,x) < \delta \Rightarrow f(x) \in B_\varepsilon(f(p)) \subset U$.
  Hence $B_\delta(p) \subset f^{-1}(U)$. So $f^{-1}(U)$ is open.
- conversely, assume U open $\Rightarrow f^{-1}(U)$ open.
  Fix $p \in X$, $\varepsilon > 0$. $B_\varepsilon(f(p))$ is open in Y, so $f^{-1}(B_\varepsilon(f(p))) \ni p$ is open in X
  Hence $\exists \delta > 0$ st. $B_\delta(p) \subset f^{-1}(B_\varepsilon(f(p)))$.
  This means $d_X(p,x) < \delta \Rightarrow x \in f^{-1}(B_\varepsilon(f(p))) \Rightarrow f(x) \in B_\varepsilon(f(p))$ ✓ $\square$

(Our first ε-δ proof, but not our last!).

We can also talk about sequences and their limits:

Def: A sequence $p_1, p_2, \dots$ in (X,d) converges to a limit $p \in X$ (write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$) if $\forall \varepsilon > 0$ $\exists N$ st. $\forall n \geq N$, $d(p_n, p) < \varepsilon$. ← p_n get closer to p!

(unique if it exists).

vs. get closer to each other

Def: A sequence $p_1, p_2, \dots$ in X is Cauchy if $\forall \varepsilon > 0$ $\exists N$ st. $\forall m,n \geq N$, $d(p_n, p_m) < \varepsilon$.

Exercise: if a sequence converges then it is Cauchy, but not necessarily vice-versa.

A metric space is complete if every Cauchy sequence converges.

Ex: $\mathbb{R}$ is complete, but $\mathbb{Q}$ (with induced metric) isn't complete.

* The notion of Cauchy seq. is specific to metric spaces, but really useful for real analysis.

Ex: $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ - if we take this to be the def^{n} of e, we can't prove directly that $x_n = \sum_{k=0}^{n} \frac{1}{k!}$ converges to e, instead use Cauchy criterion to show that the limit exists. $\square$

* <u>Interlude</u>: What is $\mathbb{R}$?

<u>Ans</u>: it's an ordered field (ie: $+ - \times /$, and $<$ compatible with usual rules) with the <u>least upper bound property</u>: every nonempty subset $E \subset \mathbb{R}$ that admits an upper bound ($\exists M \in \mathbb{R}$ st. $\forall x \in E$, $x \leq M$) has a <u>least</u> upper bound $\sup(E) \in \mathbb{R}$. (ie. $\sup(E)$ is an upper bound, and every upper bound for E is $\geq \sup(E)$).

The l.u.b. property is equivalent to completeness of $\mathbb{R}$; any ordered field with this property is isomorphic to $(\mathbb{R}, +, \times, <)$. Constructions of $\mathbb{R}$ from $\mathbb{Q}$ involve adding the missing elements (irrationals) so that l.u.b. property / completeness holds; the elements of $\mathbb{R}$ end up being either the sups of certain subsets of $\mathbb{Q}$ or the limits of Cauchy seq's in $\mathbb{Q}$

↳ see eg. Rudin ↳ see HW.

Returning to limits of sequences...

- <u>Prop</u>: if $p_n \to p$, then every open subset $U \ni p$ contains p_n for all but finitely many n.

This will be the definition of limit outside the metric case.

(<u>Pf</u>: $U \ni p$, U open $\Rightarrow \exists \varepsilon > 0$ st. $B_\varepsilon(p) \subset U$. So $\exists N$ st. $n \geq N \Rightarrow p_n \in B_\varepsilon(p) \subset U$).

- <u>Def</u>: $Z \subset X$ is <u>closed</u> if its complement $X \setminus Z$ is open.

⚠ Most subsets of X are neither open nor closed !! and... $\emptyset$ and X are <u>both</u>!

<u>Prop</u>: If $Z \subset X$ is closed, then $\forall$ sequence $\{p_n\}$ in Z which converges to a limit $p \in X$, then $p \in Z$.

(the converse is true in metric spaces and in nice enough top. spaces – "first countable")

<u>Pf</u>: Assume $\exists \{p_n\} \in Z$, $p \in X - Z$, $p_n \to p$: $\forall U \ni p$ open, U contains p_n for all but finitely many n, but $p_n \in Z$, so $U \not\subset X - Z$.

If Z is closed then $U = X - Z$ is open and we get a contradiction. □.

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* Our goal will be to reformulate / generalize all this in the context of <u>topological spaces</u>, ie. sets equipped with a <u>topology</u> which may or may not come from a metric.

<u>Def</u>: A <u>topology</u> $\mathcal{T}$ on a set X = collection of subsets of X, which we'll declare to be the <u>open sets</u> in X. Needs to satisfy axioms:

- $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$
- any union of elements of $\mathcal{T}$ is in $\mathcal{T}$
- the intersection of finitely many elements of $\mathcal{T}$ is in $\mathcal{T}$.

<u>Why bother?</u> One answer: many natural topologies do not come from a metric!

Eg, in analysis:

- on the space of (bounded) functions $f: X \to \mathbb{R}$, uniform convergence topology ($f_n \to f$ iff $\sup_x |f_n(x) - f(x)| \to 0$) comes from a metric $\left(d(f,g) = \sup_x |f(x) - g(x)|\right)$

  but pointwise convergence ($f_n \to f$ iff $\forall x \in X$ $f_n(x) \to f(x)$) doesn't. ("product topology")
- C^∞ topology on smooth functions $\mathbb{R} \to \mathbb{R}$ doesn't come from a metric either.

And on the other hand, a metric contains extraneous information for topology

<u>Eg.</u> $(\mathbb{R}^n, d)$, $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_\infty)$ have the same open sets $\Rightarrow$ same topology.