Topologies on infinite products: given topological spaces X_i, $i \in I$ index set:

What is the natural topology on $X = \prod_{i\in I} X_i = \{(p_i)_{i\in I} \mid p_i \in X_i \ \forall i \in I\}$?

First idea: **Def:** the box topology on $\prod_{i\in I} X_i$ has basis $\{\prod_{i\in I} U_i \mid U_i \subset X_i \text{ open } \forall i\}$

(this is a basis: box $\cap$ box $=$ box, since $(\prod U_i) \cap (\prod V_i) = \prod (U_i \cap V_i)$)

This is actually too fine for most purposes.

Example: consider the diagonal map $\Delta: \mathbb{R} \to \mathbb{R}^\omega = \mathbb{R}^{\mathbb{N}}$ $(= \mathbb{R}\times\mathbb{R}\times\ldots)$, $\Delta(x) = (x,x,x,\ldots)$

giving $\mathbb{R}^{\mathbb{N}}$ the box topology, Δ is not continuous!! (unlike case of finite products)

Indeed, let $U = (-1,1) \times (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{3},\frac{1}{3}) \times \ldots$ open in box topology.

$\Delta^{-1}(U) = \bigcap_{n\geq 1} (-\frac{1}{n},\frac{1}{n}) = \{0\}$ not open in $\mathbb{R}$.

Better: **Def:** the product topology on $X = \prod X_i$ has basis
$\{\prod_{i\in I} U_i \mid U_i \subset X_i \text{ open, and } U_i = X_i \text{ for all but finitely many } i\}$

(This is the same as the box topology if I is finite; for infinite I this is coarser)

Unless otherwise specified, the product topology is the one we'll use on $\prod X_i$.

Theorem: $f: Z \to X = \prod X_i$ (product top) is continuous $\iff$ each component $f_i: Z \to X_i$ is continuous.
$z \mapsto (f_i(z))_{i\in I}$

Ex: this now implies the diagonal map $\Delta: \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is continuous, since each $\Delta_i =$ identity.

Pf:
- the projection $p_i: X \to X_i$ to the i^{th} factor is continuous ($\forall U \subset X_i$ open, $p_i^{-1}(U)$ is open in product top.). Hence, if f is continuous, so is $f_i = p_i \circ f$.
- conversely, assume all f_i are continuous, and consider basis element $\prod U_i \subset X$ where $U_i = X_i$ for all but finitely many i,
  then $f^{-1}(\prod U_i) = \{z \in Z \mid (f_i(z))_{i\in I} \in \prod U_i\} = \bigcap_{i\in I} f_i^{-1}(U_i)$
  Each $f_i^{-1}(U_i) \subset Z$ is open, and all but finitely many are $= f_i^{-1}(X_i) = Z$, so can be omitted from the intersection. So $f^{-1}(\prod U_i)$ is the intersection of finitely many open sets in Z, hence open. □

Similarly: (exercise) a sequence $x_n = (x_{n,i})_{i\in I} \in \prod X_i$, $n = 1,2,3\ldots$ converges to $y = (y_i)_{i\in I} \in \prod X_i$
$\iff \forall i \in I$, the sequence $x_{n,i} \in X_i$ converges to y_i.

Ex: given a set X & top. space Y, let $\mathcal{F} = \{\text{functions } X \to Y\} = Y^X$ with product top.
Then a sequence $f_n \in \mathcal{F}$ converges to $f \in \mathcal{F}$ iff $\forall x \in X$, $f_n(x) \to f(x)$ in Y.
So: the product topology is the topology of pointwise convergence.

On products of metric spaces, there is another natural topology, finer than product but coarser than box topology – the <u>uniform topology</u>

This works similarly to the construction of $d_\infty(x,y) = \sup(|y_i - x_i|)$ on $\mathbb{R}^n$, but for an infinite product the sup might be infinite. So:

- <u>First step</u>: can replace the metric on (X,d) by $\hat{d}(x,y) = \min(d(x,y),1)$, this is still a metric (check!) and induces the same topology as d (same balls of radius ≤ 1!)
- Now, given metric spaces $(X_i, d_i)_{i\in I}$, replace each d_i by bounded metric $\hat{d}_i$, and define a metric $d_\infty(x,y) = \sup\{\hat{d}_i(x_i,y_i) \mid i \in I\}$ on $\prod X_i$
  $(= \sup\{d_i(x_i,y_i)\}$ if it's ≤ 1, else $1)$

This is called the <u>uniform metric</u> and induces the <u>uniform topology</u>.

<u>Ex</u>: on $\mathbb{R}^X = \{\text{functions } X \to \mathbb{R}\}$, (with usual distance on $\mathbb{R}$), this is
$$d_\infty(f,g) = \sup_{x\in X} |f(x) - g(x)| \text{ if } \leq 1, \text{ else } 1.$$
so $f_n \to f \iff d_\infty(f_n, f) \to 0 \iff \sup_{x\in X} |f_n(x) - f(x)| \to 0$ uniform convergence!

<u>Rmk</u>: The ball of radius $r \leq 1$ around $x = (x_i)_{i\in I}$ is <u>contained</u> in $P_r(x) = \prod_{i\in I} B_r(x_i)$, but not equal to it (unless I is finite)!
Indeed, $d(x_i, y_i) < r \ \forall i \in I$ only implies $d_\infty(x,y) = \sup_{i\in I}\{d(x_i,y_i)\} \leq r$!
The ball $B_r(x)$ only contains those y for which the sup is $< r$.
In fact: $B_r(x) = \bigcup_{0<r'<r} P_{r'}(x) \subset P_r(x)$... and $P_r(x)$ is not open for d_∞!

<u>Theorem</u>: The uniform topology on $\prod (X_i, d_i)$ is finer than the product topology, and coarser than the box topology (strictly if I is infinite).

<u>Pf</u>: 1) Let $x = (x_i) \in \prod X_i$, and $\prod U_i \ni x$ a basis element in the product top., then $\forall i \ \exists \varepsilon_i > 0$ st. $B_{\varepsilon_i}(x_i) \subset U_i$. Without loss of generality we can assume $\varepsilon_i \leq 1 \ \forall i$, and $\varepsilon_i = 1$ for all but finitely many i (whenever $U_i = X_i$).
So $\varepsilon = \inf(\varepsilon_i) > 0$, and $B_\varepsilon^{d_\infty}(x) \subset P_\varepsilon(x) \subset \prod B_{\varepsilon_i}(x_i) \subset \prod U_i$.
So $\prod U_i$ is open in uniform top: $\mathcal{T}_{product} \subset \mathcal{T}_{uniform}$.

2) $B_r^{d_\infty}(x) = \underbrace{\bigcup_{0<r'<r} P_{r'}(x)}_{\text{open in box}}$ $\Rightarrow$ balls of uniform top. are open in box topology, so $\mathcal{T}_{uniform} \subset \mathcal{T}_{box}$. $\square$

- <u>Rmk</u>: on $\mathbb{R}^{\mathbb{N}}$ the product topology is actually metrizable, using a clever modification of d_∞ (see Munkres Thm. 20.5), while box isn't metrizable (Munkres end of §21).
  On uncountable products, neither box nor product are metrizable (——"——).

Ex: in $\mathcal{F} = \{f : [0,1] \to \mathbb{R}\}$, $f_n(x) = x^n$ converges pointwise (i.e. in product top.) to
$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x<1 \\ 1 & x=1 \end{cases}$, but not uniformly $\left(\sup_x |f_n(x) - f(x)| = 1 \not\to 0\right)$.

$g_n(x) = \frac{1}{n}$ (constant functions) $\to g(x) = 0$ uniformly (& pointwise) but not in box topology (g_n never $\in U = \{f \,/\, |f(x)| < x \ \forall x>0\}$ box nbd. of $g=0$)
in fact it's almost impossible for a nonconst. seq. of functions to converge in box top.

- The notion of uniform convergence is important in real analysis because it is well behaved wrt continuity and differentiability. For example:

Thm: given a top space X, metric space Y, and a sequence of functions $f_n : X \to Y$, if f_n is continuous $\forall n$ and $f_n \to f$ uniformly then f is continuous.

Pf. let $V \subset Y$ open, $p \in f^{-1}(V)$. $\exists \varepsilon > 0$ st. $B_\varepsilon(f(p)) \subset V$. Let N be s.t. $\sup_{q \in X} d(f_N(q), f(q)) < \frac{\varepsilon}{3}$.
Let $U \ni p$ open st. $q \in U \Rightarrow d(f_N(p), f_N(q)) < \frac{\varepsilon}{3}$ (continuity of f_N). Then using triangle ineq.: $\forall q \in U$,
$d(f(p), f(q)) \leq d(f(p), f_N(p)) + d(f_N(p), f_N(q)) + d(f_N(q), f(q)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. So $U \subset f^{-1}(B_\varepsilon(f(p))) \subset f^{-1}(V)$. □

Corollary: $\mathcal{C}(X,Y) = \{\text{continuous } f : X \to Y\}$ is a closed subspace of $(\mathcal{F}(X,Y) = Y^X$, uniform top.).

Connected spaces (Munkres §23-24)

Def: A topological space X is <u>connected</u> if it cannot be written as $X = U \cup V$ where U, V are disjoint nonempty open sets.
(such a decomposition is called a <u>separation</u> of X).

not connected.

Prop: $[0,1] \subset \mathbb{R}$ (standard top.) is connected.

Pf: assume $[0,1] = U \cup V$ separation. Without loss of generality, $0 \in U$.
Let $a = \sup\{x \in [0,1] \text{ st. } [0,x) \subset U\}$.

- $0 \in U$, U open $\Rightarrow [0,\varepsilon) \subset U$ for some $\varepsilon > 0$, so $a > 0$.
- Can't have $a \in V$: since V is open this would imply $(a-\varepsilon, a] \subset V$ for some $\varepsilon > 0$, hence $[0,x) \not\subset U$ for $x > a-\varepsilon$, hence $\sup\{x \text{ st.} \dots\} \leq a - \varepsilon$, contradiction. So $a \in U$.
- but if $a < 1$, U open, $U \ni a \Rightarrow \exists \varepsilon > 0$ st. $(a-\varepsilon, a+\varepsilon) \subset U$, and by def. of a, $\exists x > a-\varepsilon$ st. $[0,x) \subset U$. Hence $[0, a+\varepsilon) \subset U$, contradicting def of a.
- hence $a = 1$, and since U is open, $\exists \varepsilon > 0$ st. $(1-\varepsilon, 1] \subset U$, & by def. of a, $\exists x > 1-\varepsilon$ st. $[0,x) \subset U$, hence $U = [0,1]$, and $V = \phi$. Contradiction. □

Ex. $[0,1) \cup (1,2]$ is not connected, since $[0,1)$ and $(1,2]$ are open in subspace topology & provide a separation. More generally, $x < y < z \in \mathbb{R}$, $x, z \in A$, $y \notin A \Rightarrow A$ disconnected.

Thm: $f: X \to Y$ continuous, X connected $\Rightarrow$ $f(X) \subset Y$ is connected.

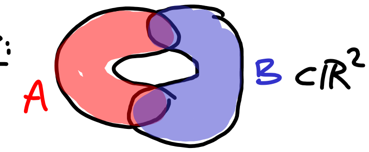
Pf: If $U \cup V$ is a separation of $f(X)$, then $f^{-1}(U) \cup f^{-1}(V)$ is a separation of X, contradiction!
(subspace top.: $U = f(X) \cap U' \neq \phi$, U' open in Y $\Rightarrow$ $f^{-1}(U) = f^{-1}(U') \neq \phi$ open in X; $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \phi$).

Corollary: intermediate value theorem

Theorem: X connected top space, $f: X \to \mathbb{R}$ continuous.
If $a, b \in X$ and r lies between $f(a)$ and $f(b)$, then $\exists c \in X$ st. $f(c) = r$.

Pf. since X is connected, so is $f(X)$. If $r \notin f(X)$ then
$U = (-\infty, r) \cap f(X)$ and $V = (r, \infty) \cap f(X)$ gives a separation of $f(X)$
(one contains $f(a)$ and the other contains $f(b)$) – contradiction. So $r \in f(X)$. $\square$

Fact: $A, B \subset X$ connected (for subspace top.) $\not\Rightarrow$ $A \cap B$ connected. **Ex:**



But things are better for unions of connected sets, provided they overlap.

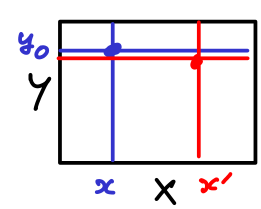
Thm: $A_i \subset X$ connected subspaces, all containing some point $p \in X$ (ie. $\cap A_i \neq \phi$)
Then $Y = \cup A_i$ is connected.

Pf: assume $Y = U \cup V$ disjoint, open in Y. Without loss of generality, $p \in U$.
Then $U \cap A_i$ and $V \cap A_i$ are disjoint, open in A_i. Since A_i is connected and $p \in U \cap A_i$,
must have $A_i \subset U$ $\forall i$. Hence $Y = \cup A_i \subset U$ (and $V = \phi$). So Y is connected. $\square$

Corollary: $\mathbb{R}$ is connected; so are open, half-open, and closed intervals in $\mathbb{R}$.

Thm: X, Y connected $\Rightarrow$ $X \times Y$ is connected.

Pf: Fix $(x_0, y_0) \in X \times Y$. Then $\forall x \in X$, $A_x = (X \times \{y_0\}) \cup (\{x\} \times Y)$
is connected by previous thm (both pieces contain (x, y_0))
and now $X \times Y = \bigcup_{x \in X} A_x$ (all containing (x_0, y_0)) $\Rightarrow$ $X \times Y$ connected. $\square$



In fact, more is true: X_i, $i \in I$ connected $\Rightarrow$ $\prod_{i \in I} X_i$ with product top is connected.
(won't prove).
(This is false for uniform and box topologies: eg. $\mathbb{R}^I = \{\text{functions } I \to \mathbb{R}\}$ for infinite I
Say $f: I \to \mathbb{R}$ is bounded if $f(I) \subset \mathbb{R}$ bounded subset. Then $\{\text{bounded}\} \cup \{\text{unbounded}\}$ is a separation of $\mathbb{R}^I$ in uniform topology.).