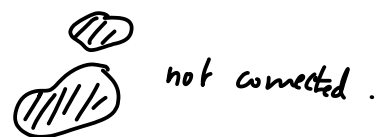


Def: A topological space X is connected if it cannot be written as $X = U \cup V$ where U, V are disjoint nonempty open sets. (such a decomposition is called a separation of X).



Prop: $[0,1] \subset \mathbb{R}$ (standard top.) is connected.

Pf: assume $[0,1] = U \cup V$ separation. Without loss of generality, $0 \in U$.

Let $a = \sup \{x \in [0,1] \text{ st. } [0,x) \subset U\}$.

- $0 \in U$, U open $\Rightarrow [0,\varepsilon) \subset U$ for some $\varepsilon > 0$, so $a > 0$.
- Can't have $a \in V$: since V is open this would imply $(a-\varepsilon, a] \subset V$ for some $\varepsilon > 0$, hence $[0,x) \not\subset U$ for $x > a-\varepsilon$, hence $\sup \{x \text{ st. } [0,x) \subset U\} \leq a-\varepsilon$, contradiction. So $a \in U$.
- but if $a < 1$, U open, $U \ni a \Rightarrow \exists \varepsilon > 0$ st. $(a-\varepsilon, a+\varepsilon) \subset U$, and by def. of a , $\exists x > a+\varepsilon$ st. $[0,x) \subset U$. Hence $[0, a+\varepsilon) \subset U$, contradicting def. of a .
- hence $a = 1$, and since U is open, $\exists \varepsilon > 0$ st. $(1-\varepsilon, 1] \subset U$, & by def. of a , $\exists x > 1-\varepsilon$ st. $[0,x) \subset U$, hence $U = [0,1]$, and $V = \emptyset$. Contradiction. $\square$

Ex: $[0,1) \cup (1,2]$ is not connected, since $[0,1)$ and $(1,2]$ are open in subspace topology & provide a separation. More generally, $x < y < z \in \mathbb{R}$, $x, z \in A, y \notin A \Rightarrow A$ d's connected.

Thm: $f: X \rightarrow Y$ continuous, X connected $\Rightarrow f(X) \subset Y$ is connected.

Pf: If $U \cup V$ is a separation of $f(X)$, then $f^{-1}(U) \cup f^{-1}(V)$ is a separation of X , contradiction!
(subspace top.: $U = f(X) \cap U' \neq \emptyset$, U' open in $Y \Rightarrow f^{-1}(U) = f^{-1}(U') \neq \emptyset$ open in X ; $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$).

Corollary: intermediate value theorem

Theorem: X connected top space, $f: X \rightarrow \mathbb{R}$ continuous.

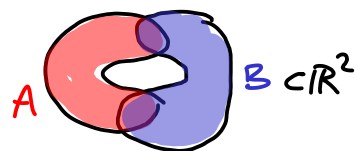
If $a, b \in X$ and r lies between $f(a)$ and $f(b)$, then $\exists c \in X$ st. $f(c) = r$.

Pf: since X is connected, so is $f(X)$. If $r \notin f(X)$ then

$U = (-\infty, r) \cap f(X)$ and $V = (r, \infty) \cap f(X)$ gives a separation of $f(X)$

(one contains $f(a)$ and the other contains $f(b)$) - contradiction. So $r \in f(X)$. $\square$

Fact: $A, B \subset X$ connected (for subspace top.) $\nRightarrow A \cap B$ connected. Ex:



But things are better for unions of connected sets, provided they overlap.

Thm: $A_i \subset X$ connected subspaces, all containing some point $p \in X$ (i.e. $\bigcap A_i \neq \emptyset$)
Then $Y = \bigcup A_i$ is connected.

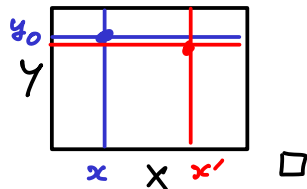
PF: assume $Y = U \cup V$ disjoint, open in Y . Without loss of generality, $p \in U$. (2)

Then $U \cap A_i$ and $V \cap A_i$ are disjoint, open in A_i . Since A_i is connected and $p \in U \cap A_i$, must have $A_i \subset U \quad \forall i$. Hence $Y = \bigcup A_i \subset U$ (and $V = \emptyset$). So Y is connected. $\square$

Corollary: $\parallel$ $\mathbb{R}$ is connected; so are open, half-open, and closed intervals in $\mathbb{R}$.

Thm: $\parallel$ X, Y connected $\Rightarrow X \times Y$ is connected.

PF: Fix $(x_0, y_0) \in X \times Y$. Then $\forall x \in X$, $A_x = (X \times \{y_0\}) \cup (\{x\} \times Y)$ is connected by previous thm (both pieces contain (x, y_0)) and now $X \times Y = \bigcup_{x \in X} A_x$ (all containing (x_0, y_0)) $\Rightarrow X \times Y$ connected.

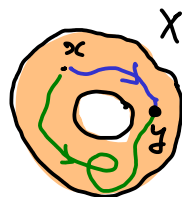


In fact, more is true: $\parallel$ $X_i, i \in I$ connected $\Rightarrow \prod_{i \in I} X_i$ with product top is connected. (won't prove).

(This is false for uniform and box topologies: eg $\mathbb{R}^I = \{\text{functions } I \rightarrow \mathbb{R}\}$ for infinite I . Say $f: I \rightarrow \mathbb{R}$ is bounded if $f(I) \subset \mathbb{R}$ bounded subset. Then $\{\text{bounded}\} \cup \{\text{unbounded}\}$ is a separation of $\mathbb{R}^I$ in uniform topology.)

Path-connectedness:

Def: $\parallel$ X top. space, $x, y \in X$, a path from x to y is a continuous map $f: [a, b] \rightarrow X$ st. $f(a) = x$ and $f(b) = y$.
 $\uparrow$ subspace top. of standard $\mathbb{R}$



two paths $x \rightarrow y$.

Def: $\parallel$ X is path-connected if every pair of points in X can be joined by a path.

Note: The relation $x \sim y \Leftrightarrow x$ and y can be connected by a path is an equivalence relation, ie. (1) $x \sim x$ (constant path $f(t) = x$)

(2) $x \sim y \Leftrightarrow y \sim x$ (backwards path $f(-t)$)

(3) $x \sim y$ and $y \sim z \Rightarrow x \sim z$
 (concatenate paths: $f = \begin{cases} f_1(t) & t \in [a, c] \\ f_2(t) & t \in [c, b] \end{cases}$)

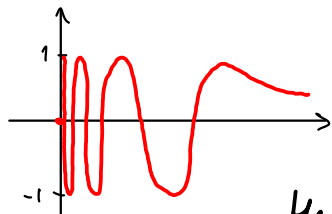
The equivalence classes are called the path components of X . (will return to these in alg. topology!)

Thm: $\parallel$ if X is path connected then X is connected.

PF: Assume not, ie $X = U \sqcup V$ disjoint open, $x \in U, y \in V$. Pick a path $f: [a, b] \rightarrow X$ connecting x to y . Then $[a, b] = \underbrace{f^{-1}(U)}_a \sqcup \underbrace{f^{-1}(V)}_b$ open subsets. This contradicts the connectedness of $[a, b]$. $\square$

The converse is false in general, but true for nice enough spaces eg. CW-complexes.

Ex: the "topologist's sine curve": let $S = \{(x, y) \mid y = \sin(\frac{1}{x}), x > 0\} \cup \{(0, 0)\} \subset \mathbb{R}^2$. (3)
 $\text{def} = S_0 \rightarrow$



the 'main' part S_0 is connected, since it's the image of $\mathbb{R}_+$ (connected) under the continuous map $x \mapsto (x, \sin \frac{1}{x})$.

Hence S is connected: if $S = U \cup V$ disjoint open, then

$S_0 = (U \cap S_0) \cup (V \cap S_0)$ disjoint & open $\Rightarrow$ one of them (eg. $V \cap S_0$) is empty.

$V \subset S - S_0 = \{(0, 0)\}$. But $\{(0, 0)\}$ not open in S , so in fact $V = \emptyset$.

On the other hand, S is not path connected: there's no path connecting $(\frac{1}{\pi}, 0)$ to $(0, 0)$.

(PF. later using compactness: the image of such a path is a closed subset of $\mathbb{R}^2$, but S isn't: $(0, 1)$ is a limit point of S not in S).

However, for nice enough spaces the two notions are equivalent.

Thm: $\parallel A \subset \mathbb{R}^n$ open $\Rightarrow A$ is connected iff A is path connected.

PF: already seen: path connected $\Rightarrow$ connected. We show: not path connected $\Rightarrow$ not connected.

Assume A open in $\mathbb{R}^n$: then the path components of A are open.

Indeed, if $x \in A$ then $\exists r > 0$ st. $B_r(x) \subset A$, and any two points of $B_r(x)$ can be connected inside A by a straight line segment. So all of $B_r(x)$ is in the same path component. Now: if A is not path connected then

$A = (\text{one path component}) \cup (\cup \text{all other path components})$ gives a separation. $\square$

This implies similar results for other classes of spaces, eg. top. manifolds and CW-complexes.

* For these kinds of space, path-components are also connected components, ie.

they give a partition of X into disjoint connected open (& closed) subsets.

Such a partition only exists if X is "locally connected" ie. the topology has a basis consisting of connected open subsets. (Counterexample: $\mathbb{Q} \subset \mathbb{R}$ isn't loc. conn.)

(each point of $\mathbb{Q}$ is its own path component, but these aren't open).

Compactness (Numbers §26-...)

Compactness is a "finiteness/boundedness" property of nice topological spaces such as closed bounded intervals $[a, b] \subset \mathbb{R}$, or more generally, closed bounded subsets of $\mathbb{R}^n$.

Eg: any continuous map $f: K \rightarrow \mathbb{R}$ achieves its maximum & minimum.
 $\uparrow$ compact

The definition isn't very intuitive.

④

Def. An open cover of a top. space X is a collection of open subsets $(U_i)_{i \in I}$ st. $\bigcup_{i \in I} U_i = X$.

Def. X is compact if every open cover $(U_i)_{i \in I}$ of X admits a finite subcover,
ie. $\exists i_1, \dots, i_n$ st. $X = U_{i_1} \cup \dots \cup U_{i_n}$.

Showing a space is not compact is much easier than showing it is!

Ex: $\mathbb{R}$ is not compact: the open cover $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (n-1, n+1)$ has no finite subcover.

neither is $(0, 1]$ with subspace top.: $(0, 1] = \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 1]$ has no finite subcover.

Ex: $X = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{Z}_+\}$ is compact: given any open cover $X = \bigcup_{i \in I} U_i$,
let i_0 be such that $0 \in U_{i_0}$, then U_{i_0} also contains $\frac{1}{n}$ for all large $n \geq N$,
hence $U_{i_1}, \dots, U_{i_N}$ containing $1, \frac{1}{2}, \dots, \frac{1}{N}$ and U_{i_0} give a finite subcover.

Thm: If X is compact and $f: X \rightarrow Y$ is continuous, then $f(X) \subset Y$ is compact
 $\uparrow$ subspace top
(Remark: an open cover of $f(X) \subset Y$ with subspace top. $\iff U_i \subset Y$ open, $\bigcup_{i \in I} U_i \supset f(X)$).

Pf: let $\bigcup_{i \in I} U_i$ open cover of $f(X)$. Then $\bigcup_{i \in I} f^{-1}(U_i)$ is an open cover of X ,
hence $\exists i_1, \dots, i_n$ st. $f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_n}) = X$. So $\forall x \in X$ $f(x) \in U_{i_1} \cup \dots \cup U_{i_n}$,
ie. $f(X) \subset U_{i_1} \cup \dots \cup U_{i_n}$ finite subcover. $\square$.

Once we know subsets of $\mathbb{R}^n$ are compact iff closed and bounded, taking $Y = \mathbb{R}$,
this gives the extreme value theorem. To get started on this right away:

Thm: $[0, 1]$ (with subspace top. $\subset \mathbb{R}$) is compact.

Pf: let $\{U_i\}_{i \in I}$ open cover of $[0, 1]$.

Let $A = \{x \in [0, 1] \mid \exists \text{ finite subcover } U_{i_1} \cup \dots \cup U_{i_n} \supset [0, x]\}$.

$A \neq \emptyset$ (contains 0). We want to show $1 \in A$. Let $a = \sup(A) \in [0, 1]$.

• First we show $a \in A$: $\exists i_0$ st. $a \in U_{i_0}$; since U_{i_0} is open, $\exists \varepsilon > 0$ st. $B_\varepsilon(a) \subset U_{i_0}$.

On the other hand $a = \sup A$, so $\exists x \in A$ st. $x > a - \varepsilon$, and a finite subcover $[0, x] \subset U_{i_1} \cup \dots \cup U_{i_n}$. Therefore $[0, a] \subset U_{i_1} \cup \dots \cup U_{i_n} \cup U_{i_0}$, and $a \in A$.

• Next, assume $a < 1$: since $a \in A$, $\exists i_1, \dots, i_n$ st. $[0, a] \subset U_{i_1} \cup \dots \cup U_{i_n}$, which is open, so $\exists \varepsilon > 0$ st. $B_\varepsilon(a) \subset U_{i_1} \cup \dots \cup U_{i_n}$, hence

$U_{i_1} \cup \dots \cup U_{i_n}$ covers $[0, x]$ for some $x > a$

(eg. $x = a + \frac{\varepsilon}{2}$ if ≤ 1 , else 1), contradicts $\sup(A) = a$.

• So: $a = 1 \in A$, $\exists$ finite subcover. $\square$.

