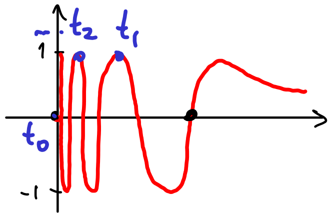
Recall: • X is connected if $\nexists\, U, V \subset X$ disjoint nonempty open subsets st. $X = U \cup V$.

• X is path-connected if $\forall x, y \in X$ $\exists$ continuous path $f: [0,1] \to X$, $f(0) = x$, $f(1) = y$.

• Thm: ‖ X is path connected $\Rightarrow$ X is connected.

Ex: the "topologist's sine curve"; let $S = \{(x,y) \mid y = \sin(\frac{1}{x}),\ x > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$.



(cf. Munkres end of §24)

S is connected (seen last time) but not path connected.

$\nexists$ continuous $f: [0,1] \to S$, $t \mapsto (x(t), y(t))$ st. $f(0) = (0,0)$, $f(1) = (\frac{1}{\pi}, 0)$

because intermediate value property for $x(t)$ gives a sequence $t_n \searrow t_0$ st. $x(t_n) \to 0$, $y(t_n) = 1$, so $\lim_{n\to\infty} f(t_n) = (0,1) \neq f(t_0) = (0,0)$.

However, for nice enough spaces the two notions are equivalent. For example:

Thm: ‖ $A \subset \mathbb{R}^n$ open $\Rightarrow$ A is connected iff A is path connected.

Pf: Assume A open in $\mathbb{R}^n$: then the path components of A (ie. equivalence classes for the equivalence relation "can be joined by a continuous path in A") are open. Indeed, if $x \in A$ then $\exists\, r > 0$ st. $B_r(x) \subset A$, and any two points of $B_r(x)$ can be connected inside A by a straight line segment. So all of $B_r(x)$ is in the same path component. Now: if A is not path connected then

$A = (\text{one path component}) \cup (\bigcup \text{all other path components})$ gives a separation.

(while we've already seen that if A is path connected then A is connected.) $\square$

This implies similar results for other classes of spaces, eg. top. manifolds and CW-complexes.

* For these kinds of spaces, path-components are also connected components, ie. they give a partition of X into disjoint connected open (& closed) subsets. Such a partition only exists if X is "locally connected" ie. the topology has a basis consisting of connected open subsets. (Counterexample: $\mathbb{Q} \subset \mathbb{R}$ isn't loc. conn.) (each point of $\mathbb{Q}$ is its own path component, but these aren't open).

Compactness (Munkres §26–...)

Compactness is a "finiteness/boundedness" property of nice topological spaces such as closed bounded intervals $[a,b] \subset \mathbb{R}$, or more generally, closed bounded subsets of $\mathbb{R}^n$.

Eg: any continuous map $f: \underset{\text{compact}}{K} \to \mathbb{R}$ achieves its maximum & minimum.

The definition isn't very intuitive.

Def. An <u>open cover</u> of a top. space X is a collection of open subsets $(U_i)_{i\in I}$ st. $\bigcup_{i\in I} U_i = X$.

Def. X is compact if <u>every</u> open cover $(U_i)_{i\in I}$ of X admits a <u>finite subcover</u>, ie. $\exists\, i_1,\dots,i_n$ st. $X = U_{i_1}\cup\dots\cup U_{i_n}$.

Showing a space is <u>not</u> compact is much easier than showing it is!

<u>Ex:</u> $\mathbb{R}$ is not compact: the open cover $\mathbb{R} = \bigcup_{n\in\mathbb{N}} (n-1, n+1)$ has no finite subcover.

neither is $(0,1]$ with subspace top.: $(0,1] = \bigcup_{n\in\mathbb{N}} (\frac{1}{n}, 1]$ has no finite subcover.

<u>Ex:</u> $X = \{0\}\cup\{\frac{1}{n},\ n\in\mathbb{Z}_+\}$ is compact: given <u>any</u> open cover $X = \bigcup_{i\in I} U_i$, let i_0 be such that $0\in U_{i_0}$, then U_{i_0} also contains $\frac{1}{n}$ for all large $n\geq N$, hence $U_{i_1},\dots,U_{i_N}$ containing $1,\frac{1}{2},\dots,\frac{1}{N}$ and U_{i_0} give a finite subcover.

Thm: If X is compact and $f: X\to Y$ is continuous, then $f(X)\subset Y$ is compact
subspace top

(Rmk: an open cover of $f(X)\subset Y$ with subspace top. $\iff$ $U_i\subset Y$ open, $\bigcup_{i\in I} U_i \supset f(X)$).

<u>Pf:</u> let $\bigcup_{i\in I} U_i$ open cover of $f(X)$. Then $\bigcup_{i\in I} f^{-1}(U_i)$ is an open cover of X, hence $\exists\, i_1 .. i_n$ st. $f^{-1}(U_{i_1})\cup\dots\cup f^{-1}(U_{i_n}) = X$. So $\forall x\in X$ $f(x)\in U_{i_1}\cup\dots\cup U_{i_n}$, ie. $f(X)\subset U_{i_1}\cup\dots\cup U_{i_n}$ finite subcover. $\square$.

* Once we know subsets of $\mathbb{R}^n$ are compact iff closed and bounded, taking $Y=\mathbb{R}$, this gives the extreme value theorem. To get started on this right away:

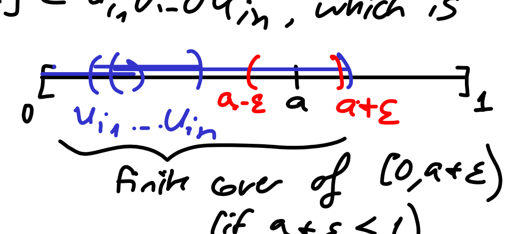
Thm: $[0,1]$ (with subspace top. $\subset\mathbb{R}$) is compact.

<u>Pf:</u> Let $\{U_i\}_{i\in I}$ open cover of $[0,1]$.

Let $A = \{x\in[0,1] \,/\, \exists \text{ finite subcover } U_{i_1}\cup\dots\cup U_{i_n}\supset[0,x]\}$.

$A\neq\emptyset$ (contains 0). We want to show $1\in A$. Let $a = \sup(A)\in[0,1]$.

- First we show $a\in A$: $\exists\, i_0$ st $a\in U_{i_0}$; since U_{i_0} is open, $\exists\, \varepsilon>0$ st. $B_\varepsilon(a)\subset U_{i_0}$. On the other hand $a=\sup A$, so $\exists\, x\in A$ st. $x > a-\varepsilon$, and a finite subcover $[0,x]\subset U_{i_1}\cup\dots\cup U_{i_n}$. Therefore $[0,a]\subset U_{i_1}\cup\dots\cup U_{i_n}\cup U_{i_0}$, and $a\in A$.
- Next, assume $a<1$: since $a\in A$, $\exists\, i_1 .. i_n$ st. $[0,a]\subset U_{i_1}\cup\dots\cup U_{i_n}$, which is open, so $\exists\,\varepsilon>0$ st. $B_\varepsilon(a)\subset U_{i_1}\cup\dots\cup U_{i_n}$, hence $U_{i_1}\cup\dots\cup U_{i_n}$ covers $[0,x]$ for some $x>a$ (eg. $x = a+\frac{\varepsilon}{2}$ if ≤ 1, else 1), contradicts $\sup(A)=a$.



- So: $a = 1\in A$, $\exists$ finite subcover. $\square$.

Thm: X compact, $F \subset X$ closed $\Rightarrow$ F is compact.

Pf: Given an open cover of F, ie. $U_i \subset X$ open, $\bigcup_{i\in I} U_i \supset F$, let $V = X \setminus F$ open: then $\{U_i, i\in I\} \cup \{V\}$ is an open cover of X, hence $\exists$ finite subcover. Discarding V, this gives a finite subcover for F. $\square$

The converse is true in Hausdorff spaces!

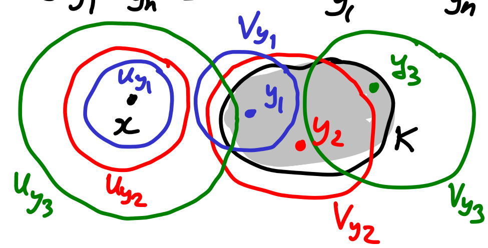
Thm: X Hausdorff, $K \subset X$ compact $\Rightarrow$ K is closed in X.

Pf: We show that $X \setminus K$ is open. Let $x \in X \setminus K$.

Since X is Hausdorff, $\forall y \in K$ $\exists$ $U_y \ni x$, $V_y \ni y$ disjoint open subsets.

Now $K \subset \bigcup_{y\in K} V_y$ is an open cover, so by compactness $\exists y_1 .. y_n$ st. $K \subset V_{y_1} \cup .. \cup V_{y_n}$.

Let $U = U_{y_1} \cap .. \cap U_{y_n} \ni x$ open.



Then $U \cap (V_{y_1} \cup .. \cup V_{y_n}) = \emptyset$, so $U \cap K = \emptyset$.

Hence: $\forall x \in X \setminus K$, $\exists\, U$ open $\ni x$ st. $U \subset X \setminus K$. $\square$

(If we tried this for an arbitrary subset of X, we'd find that $\bigcap_{y\in K} U_y$ isn't a neighborhood of x anymore. Compactness lets us reduce an infinite process to a finite one.)

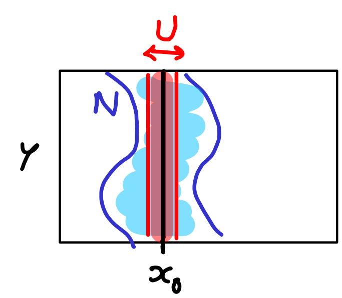
Rmk: we've actually shown more: X Hausdorff, $K \subset X$ compact, $x \in X \setminus K$ $\Rightarrow$ $\exists$ disjoint open subsets $U \ni x$, $V \supset K$, $U \cap V = \emptyset$. Ie: can separate points from compact subsets!

Ex: When X isn't Hausdorff, $K \subset X$ compact $\not\Rightarrow$ K closed in X:

eg. $X = \mathbb{R}$ with finite complement top.; <u>any</u> subset $K \subset X$ is compact.
Indeed, a nonempty open subset contains all but finitely many points, so given an open cover it is easy to find a finite subcover: take one nonempty U_i, with finite complement $\{p_1 \cdots p_k\}$, then take U_{i_j} containing p_j for $j = 1, .., k$.

Another instance of compactness allowing us to intersect infinitely many opens (or rather reduce to a finite intersection) is the <u>tube lemma</u>:

Prop: Let X top. space, Y compact top. space, $x_0 \in X$; if $N \subset X \times Y$ is open and $\{x_0\} \times Y \subset N$, then there exists a neighborhood U of x_0 in X st. $U \times Y \subset N$.



Pf: $\forall y \in Y$, $(x_0, y) \in N$ open, so $\exists$ basis open $U_y \times V_y$, U_y nbd. of x_0 in X, V_y nbd. of y in Y, st.: $(x_0, y) \in U_y \times V_y \subset N$.

Now: $\bigcup_{y\in Y} V_y = Y$ open cover. (Rmk: $\left(\bigcap_{y\in Y} U_y\right) \times Y \subset N$, but $\bigcap_{y\in Y} U_y$ not open!)

Since Y is compact, $\exists\, y_1, \dots, y_n \in Y$ st. $Y = V_{y_1} \cup \dots \cup V_{y_n}$. Let $U = U_{y_1} \cap \dots \cap U_{y_n}$.
Then U is a neighborhood of x_0 in X, and $U \times Y \subset \bigcup_{i=1}^{n} U_{y_i} \times V_{y_i} \subset N$. □

Thm: X, Y compact $\Rightarrow X \times Y$ is compact.

Pf: Let $\{A_\alpha\}$ be an open cover of $X \times Y$. For any given $x \in X$, $\{x\} \times Y$ is compact so $\exists$ finite subcollection $A_{x,1}, \dots, A_{x,n(x)}$ which suffice to cover $\{x\} \times Y$.
$A_{x,1} \cup \dots \cup A_{x,n(x)}$ is open, so by the tube lemma $\exists\, U_x \ni x$ nbd. in X such that $A_{x,1} \cup \dots \cup A_{x,n(x)} \supset U_x \times Y$. Now X is compact, and $\{U_x\}_{x \in X}$ form an open cover, so $\exists\, x_1, \dots, x_k \in X$ st. $X = U_{x_1} \cup \dots \cup U_{x_k}$.
Now $A_{x_i, j}$ $1 \le i \le k$, $1 \le j \le n(x_i)$ is a finite subcover for $X \times Y$. □

Theorem: $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded.

Pf: • if $K \subset \mathbb{R}^n$ is compact then it is closed (by above thm: $\mathbb{R}^n$ Hausdorff) and bounded:
$K \subset \bigcup_{r>0} B_r(0)$ open cover $\Rightarrow \exists$ finite subcover $\Rightarrow \exists R > 0$ st. $K \subset B_R(0)$.

• If $K \subset \mathbb{R}^n$ is closed and bounded, then it's a closed subset of $[-R, R]^n$ for some $R > 0$. $[-R, R]^n$ is a finite product of compact sets ($[-R,R] \simeq [0,1]$) hence compact; a closed subset of a compact is compact. □

Rmks: • closed and bounded are necessary conditions for compactness of a subspace of any metric space (HW!) but in "most" metric spaces, closed + bounded $\not\Rightarrow$ compact.
There are easy counterexamples (find one for HW!). More interesting: let V be <u>any</u> infinite-dimensional vector space with a norm, $d(v,v') = \|v - v'\|$. Eg. $\mathcal{F} = C^0([a,b], \mathbb{R})$ continuous f$^{\text{ns}}$ with sup norm $d(f,g) = \sup |f - g|$. (uniform topology). Then $\overline{B} = \{v \in V \,/\, \|v\| \le 1\}$ is closed & bounded but never compact. (proof uses sequential compactness). (don't use this for Munkres 26.4)

We now look at applications of compactness. We've seen

• **Thm:** If X is compact and $f: X \to Y$ is continuous, then $f(X) \subset Y$ (subspace top) is compact

Corollary: (extreme value theorem): X compact (nonempty), $f: X \to \mathbb{R}$ continuous $\Rightarrow f$ attains its max & min

Ex: (X,d) metric space, $A \subset X$ nonempty, $x \in X \Rightarrow$ define $d(x, A) = \inf \{d(x,a) \,/\, a \in A\} \ge 0$
(<u>distance</u> of x to subset A).
If A is compact then the inf is always achieved! See HW3 Problem 1 = Munkres 27.2.
Similarly, the <u>diameter</u> of a bounded subset, $\operatorname{diam}(A) = \sup \{d(x,y) \,/\, x, y \in A\}$
The sup is attained for A compact ($d: A \times A \to \mathbb{R}$ continuous, achieves its max).

Another corollary: If X is compact and Y is Hausdorff, then any continuous bijection $f: X \to Y$ is a homeomorphism.

Pf: we need to check f^{-1} is continuous as well (so $U \subset X$ open $\iff$ $f(U) \subset Y$ open)

$U \subset X$ open $\Rightarrow$ $X \setminus U$ closed hence compact $\Rightarrow$ $f(X \setminus U) = Y \setminus f(U)$ compact

Since Y is Hausdorff this implies $Y \setminus f(U)$ is closed, ie. $f(U)$ open in Y. $\square$.

(We've seen that with such assumptions a continuous bijection need not be a homeo,

eg. $[0, 2\pi) \to S^1$
$t \mapsto (\cos t, \sin t)$).