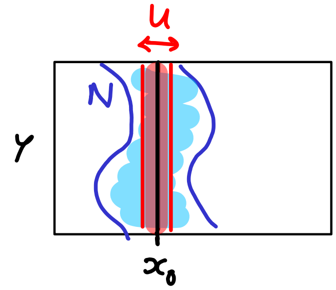
Recall:
- X is compact if every open cover $(U_i)_{i\in I}$ of X admits a finite subcover, ie. $\exists\, i_1,\dots,i_n$ st. $X = U_{i_1}\cup\dots\cup U_{i_n}$.
- $[0,1]\subset\mathbb{R}$ is compact
- X compact, $F\subset X$ closed $\Rightarrow$ F is compact;
  X Hausdorff, $K\subset X$ compact $\Rightarrow$ K is closed in X.

Tube lemma: Let X top. space, Y compact top. space, $x_0\in X$; if $N\subset X\times Y$ is open and $\{x_0\}\times Y\subset N$, then there exists a neighborhood U of x_0 in X st. $U\times Y\subset N$.



(proved last time)

$\Rightarrow$ **Thm:** X, Y compact $\Rightarrow$ $X\times Y$ is compact.

Pf: Let $\{A_\alpha\}$ be an open cover of $X\times Y$. For any given $x\in X$, $\{x\}\times Y$ is compact so $\exists$ finite subcollection $A_{x,1},\dots,A_{x,n(x)}$ which suffice to cover $\{x\}\times Y$.
$A_{x,1}\cup\dots\cup A_{x,n(x)}$ is open, so by the tube lemma $\exists\, U_x\ni x$ nbd. in X such that $A_{x,1}\cup\dots\cup A_{x,n(x)}\supset U_x\times Y$. Now X is compact, and $\{U_x\}_{x\in X}$ form an open cover, so $\exists\, x_1,\dots,x_k\in X$ st. $X = U_{x_1}\cup\dots\cup U_{x_k}$.
Now $A_{x_i,j}$ $1\le i\le k$, $1\le j\le n(x_i)$ is a finite subcover for $X\times Y$. $\square$

Theorem: $K\subset\mathbb{R}^n$ is compact iff K is closed and bounded.

Pf:
- if $K\subset\mathbb{R}^n$ is compact then it is closed (by above thm: $\mathbb{R}^n$ Hausdorff) and bounded:
  $$K\subset\bigcup_{r>0} B_r(0) \text{ open cover} \Rightarrow \exists \text{ finite subcover} \Rightarrow \exists R>0 \text{ st. } K\subset B_R(0).$$
- If $K\subset\mathbb{R}^n$ is closed and bounded, then it's a closed subset of $[-R,R]^n$ for some $R>0$. $[-R,R]^n$ is a finite product of compact sets ($[-R,R]\simeq[0,1]$) hence compact; a closed subset of a compact is compact. $\square$

Rmk: • closed and bounded are necessary conditions for compactness of a subspace of any metric space (HW!) but in "most" metric spaces, closed + bounded $\not\Rightarrow$ compact.
There are easy counterexamples (find one for HW!). More interesting: let V be any infinite-dimensional vector space with a norm, $d(v,v') = \|v-v'\|$. Eg. $\mathcal{F} = C^0([a,b],\mathbb{R})$ continuous f^{ns} with sup norm $d(f,g) = \sup|f-g|$. (uniform topology). Then $\overline{B} = \{v\in V \,/\, \|v\|\le 1\}$ is closed & bounded but never compact. (proof uses sequential compactness). (don't use this for Munkres 26.4)

We now look at applications of compactness. We've seen last time:

- **Thm:** If X is compact and $f: X \to Y$ is continuous, then $f(X) \subset Y$ is compact (subspace top).

Corollary: (extreme value theorem): X compact (nonempty), $f: X \to \mathbb{R}$ continuous $\Rightarrow$ f attains its max & min

Ex: (X,d) metric space, $A \subset X$ nonempty, $x \in X$ $\Rightarrow$ define $d(x,A) = \inf \{ d(x,a) \mid a \in A \} \geq 0$ (distance of x to subset A).

If A is compact then the inf is always achieved! See HW3 Problem 1 = Munkres 27.2.

Similarly, the diameter of a bounded subset, $\operatorname{diam}(A) = \sup \{ d(x,y) \mid x,y \in A \}$

The sup is attained for A compact ($d: A \times A \to \mathbb{R}$ continuous, achieves its max).

Another corollary: If X is compact and Y is Hausdorff, then any continuous bijection $f: X \to Y$ is a homeomorphism.

Pf: we need to check f^{-1} is continuous as well (so $U \subset X$ open $\Leftrightarrow$ $f(U) \subset Y$ open)

$U \subset X$ open $\Rightarrow$ $X \setminus U$ closed hence compact $\Rightarrow$ $f(X \setminus U) = Y \setminus f(U)$ compact

Since Y is Hausdorff this implies $Y \setminus f(U)$ is closed, ie. $f(U)$ open in Y. $\square$.

(We've seen that without such assumptions a continuous bijection need not be a homeo, eg. $[0,2\pi) \to S^1$, $t \mapsto (\cos t, \sin t)$).

In metric spaces, compactness implies uniform estimates.

- **Lebesgue number lemma:**

  (X,d) compact metric space, $(U_i)_{i \in I}$ open cover of X $\Rightarrow$ $\exists\, \delta > 0$ st. any subset of diameter $< \delta$ is entirely contained in a single open U_i.

Pf: by compactness, can assume (U_i) is a finite cover $= U_1 \cup \dots \cup U_n$.

The function $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, X \setminus U_i)$ is continuous (check: distance to a subset is a continuous function).

($X \setminus U_i$: closed set hence compact)

so achieves its min, which is therefore > 0 ($\forall x \in X$ $\exists i$ st. $x \in U_i$ and then $d(x, X - U_i) > 0$).

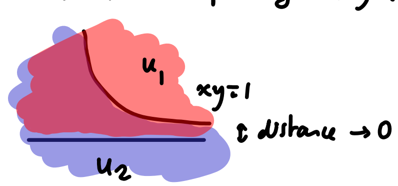
Hence $\exists\, \delta > 0$ st. $f(x) \geq \delta$ $\forall x \in X$. Thus $\forall x \in X$ $\exists\, U_i$ st. $d(x, X - U_i) \geq \delta$, ie. $B_\delta(x) \subset U_i$.

Since a subset of diameter $< \delta$ is contained in a ball of radius δ, the result follows. $\square$

This is the magic of compactness!

counterexamples: $\mathbb{R} = \bigcup$ intervals with overlaps of lengths $\to 0$ eg. $\bigcup_{n \in \mathbb{Z}} (n-1, n+1+\varepsilon_n)$, $\varepsilon_n \to 0$.

$\mathbb{R}^2 = U_1 \cup U_2$



This only makes sense for metric spaces! no notion of uniform size of neighborhood without a metric

Uniform continuity:

Def: $f: (X, d_X) \to (Y, d_Y)$ is *uniformly continuous* if $\forall \varepsilon > 0$, $\exists\, \delta > 0$ st. $\forall p,q \in X$, $d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$.

(compare with continuity: the same δ must work for every p!).

Theorem: If X and Y are metric spaces, $f: X \to Y$ continuous, and X is compact, then f is uniformly continuous.

Proof: take $\varepsilon > 0$, and consider open cover of Y by balls of radius $\frac{\varepsilon}{2}$ (so if $f(p)$, $f(q)$ land in same ball, they're less than ε apart).

$X = \bigcup_{y \in Y} f^{-1}(B_{\varepsilon/2}(y))$ open cover, so by Lebesgue number lemma $\exists\, \delta > 0$ st. if $d_X(p,q) < \delta$ then they lie in the same element of the cover, hence $d_Y(f(p), f(q)) < \varepsilon$. $\square$

Alternative notions of compactness:

Def:
- X is limit point compact if every infinite subset of X has a limit point
- X is sequentially compact if every sequence $\{p_n\}$ in X has a convergent subsequence.

Ex: in $\mathbb{R}$, $\{\frac{1}{n},\ n \geq 1\} \cup \mathbb{Z}_+$ has a limit point (0) and the sequence $1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots$ has a convergent subsequence $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$
so does $0,1,0,1,0,1,\ldots$ (eg. subsequence $0,0,\ldots$).
but $\mathbb{Z} \subset \mathbb{R}$ has no limit point & the sequence $1,2,3,\ldots$ doesn't have a convergent subsequence, so $\mathbb{R}$ is neither limit point compact nor seq. compact.

Thm: X is compact $\Rightarrow$ X is limit point compact.

Pf: Assume X is not limit point compact, ie. $\exists\, A \subset X$ infinite with no limit point.
Since A contains all of its limit points (there are none), A is closed in X, hence compact.
However, $\forall a \in A$, a isn't a limit point so $\exists\, U_a \ni a$ neighborhood of a s.t. $U_a \cap A = \{a\}$.
$(U_a)_{a \in A}$ is now an infinite open cover of A, without any finite subcover since each $a \in A$ only belongs to U_a and not to any other element of the cover. Contradiction. $\square$

Thm: X sequentially compact $\Rightarrow$ X limit point compact.

Pf: Given $A \subset X$ infinite subset, pick a sequence of distinct points of A and take a convergent subsequence $\Rightarrow \exists\, \{a_n\}$ sequence in A, $a_n \neq a_m$ $\forall n \neq m$, converging to some limit $a \in X$. Then every neighborhood of a contains a_n for all large n, hence ∞ly many points of A, including some $\neq a$. So a is a limit pt of A. $\square$

The converse implications don't hold in general, but in metric spaces all three notions coincide! (& hence also for subspaces of metric spaces...)

Thm: For a metric space (X,d), X compact $\Leftrightarrow$ X limit pt compact $\Leftrightarrow$ X seq. compact.

Proof:
- compact $\Rightarrow$ limit point compact: already done (for all top spaces)

- limit point compact $\Rightarrow$ sequentially compact: suppose X metric space and limit point compact, and consider a sequence $x_1, x_2, \dots$ in X. If $\{x_1, x_2, \dots\}$ finite, then $\exists x \in X$ st. $x_n = x$ for infinitely many n, which gives a subsequence that converges to x.
  Otherwise, $\{x_1, x_2, \dots\}$ is infinite, so has a limit point a. So:
  $\forall r > 0 \quad \exists n$ st. $\quad 0 < d(a, x_n) < r$.

  First choose $n_1 \in \mathbb{N}$ st. $x_{n_1} \in B_1(a)$, then inductively, given $n_1, \dots, n_{k-1}$, let $\delta_k = \min\{d(x_i, a) \mid i \leq n_{k-1} \text{ and } x_i \neq a\} > 0$, and $r_k = \min(\frac{1}{k}, \delta_k)$.
  Then take n_k st. $0 < d(a, x_{n_k}) < r_k$. By construction: $n_k > n_{k-1}$, and $d(a, x_{n_k}) < \frac{1}{k}$.
  $\Rightarrow x_{n_1}, x_{n_2}, \dots$ is a subsequence converging to a.

- seq. compact $\Rightarrow$ compact; this is the hardest part. First we show:

  Lemma 1: If X metric space is seq. compact, then $\forall \varepsilon > 0$ X can be covered by finitely many open balls of radius ε.

  (as we expect if X is to be compact: $X = \bigcup_{x \in X} B_\varepsilon(x)$ should have a finite subcover!)

  Proof: assume not, and choose $x_1 \in X$, then inductively choose $x_n \in X \setminus \bigcup_{i=1}^{n-1} B_\varepsilon(x_i)$ (if this isn't possible then we've covered X by finitely many balls).
  This yields a sequence in X, which by sequential compactness must have a convergent subsequence. But this is impossible since no two terms of the sequence are within ε of each other! Contradiction. $\square$

  Lemma 2: If X metric space is sequentially compact then every open cover has a Lebesgue number ($\exists \delta > 0$ st. any subset of diameter $< \delta$ is entirely in one U_i).

  (we've seen this holds for compact metric spaces, so it should hold!)

  Pf: suppose $\exists$ open cover $(U_i)_{i \in I}$ with no Lebesgue number, ie. $\forall n \geq 1 \quad \exists C_n \subset X$ with diameter $< \frac{1}{n}$ which isn't contained in any single U_i. Take $x_n \in C_n$.
  By sequential compactness, $\exists$ subsequence (x_{n_k}) of (x_n) that converges to some $a \in X$.
  Now $a \in U_{i_0}$ for some $i_0 \in I$, and so $\exists \varepsilon > 0$ st. $B_\varepsilon(a) \subset U_{i_0}$
  Take k suff'ly large so that $\frac{1}{n_k} < \frac{\varepsilon}{2}$ and $d(x_{n_k}, a) < \frac{\varepsilon}{2}$.
  Since C_{n_k} has diameter $< \frac{\varepsilon}{2}$, $C_{n_k} \subset B_{\frac{\varepsilon}{2}}(x_{n_k}) \subset B_\varepsilon(a) \subset U_{i_0}$, contradiction. $\square$

  (Rmk: This proof illustrates how arguments using sequential compactness are often more intuitive than those involving open covers: "if some property fails to hold uniformly, take a sequence of points where things get worse and worse, extract a convergent subsequence, and see what goes wrong at the limit.")

Now we can prove <u>seq. compact ⇒ compact</u>:

<u>Pf:</u> given an open cover $X = \bigcup_{i \in I} U_i$, by lemma 2 $\exists \delta > 0$ s.t. every subset of diameter $< \delta$ is entirely inside a single U_i. Fix $\varepsilon \in (0, \frac{\delta}{2})$: by lemma 1, X is covered by finitely many ε-balls. Each of these has diameter $\leq 2\varepsilon < \delta$, so is contained in some U_i. This gives a finite subcover, replacing each ε-ball by one U_i containing it (and discarding the rest of the U_i's). □.

<u>Thm:</u> ‖ Every compact metric space (X,d) is complete, ie. every Cauchy seq. converges.

<u>Pf:</u> let (x_n) Cauchy seq., by sequential compactness $\exists$ subsequence $x_{n_k} \to x \in X$.
Now $\forall \varepsilon > 0$ $\exists N$ st. $\forall m, n \geq N$, $d(x_m, x_n) < \frac{\varepsilon}{2}$. $\exists n_k \geq N$ st. $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.
Hence: $\forall n \geq N$, $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$. □.

<u>Corollary:</u> ‖ $\mathbb{R}$, $\mathbb{R}^n$ (with usual distances) are complete.

<u>Pf:</u> every Cauchy sequence is bounded, hence contained in a compact subset, hence convergent. □

<u>Corollary:</u> ‖ $\mathbb{R}^X = \{$functions $X \to \mathbb{R}\}$ with uniform metric is complete.

<u>Pf:</u> given a Cauchy sequence $\{f_n\}$ (ie. $\forall \varepsilon > 0$ $\exists N$ st. $m, n \geq N \Rightarrow \sup |f_n - f_m| < \varepsilon$).
$\forall x \in X$, $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$ $\left(|f_n(x) - f_m(x)| \leq \sup |f_n - f_m| < \varepsilon\right)$
hence converges to some limit $f(x)$ (ie. we have a pointwise limit).
Now: given $\varepsilon > 0$, take N st. $m, n \geq N \Rightarrow \sup_x |f_n(x) - f_m(x)| < \varepsilon$.
Then $\forall n \geq N$, $\forall x \in X$, $|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \varepsilon$.
ie. $\forall n \geq N$, $\sup |f_n - f| \leq \varepsilon$, which implies $f_n \to f$ uniformly. □

* When X is a top. space, we've seen that uniform limits of continuous functions are continuous, so we also have completeness of $C^0(X, \mathbb{R}) = \{$continuous $f^{n}\} \subset \mathbb{R}^X$, uniform top.
more generally: closed subsets of complete metric spaces are complete!