Math 55b Homework 3

Due Wednesday February 16, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Because you are also expected to complete the take-home midterm during the week of February 14-18, this assignment is on the shorter side. But... the long problem on completions of metric spaces advertised on HW1 is due with HW4, so don't hesitate to start working on it already if you have time to spare!

Material covered: Compactness in metric spaces; completeness; local compactness and compactification; separation axioms and metrizability. (Munkres §27-34; see also §43).

1. Munkres exercise 27.2.

2. Let (X, d) be a metric space. A map $f : X \to X$ is a *shrinking map* if, $\forall x, y \in X$, $x \neq y \Rightarrow d(f(x), f(y)) < d(x, y)$. There is also a slightly stronger notion: f is a *contraction* if there exists a real number $\alpha < 1$ such that, for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha\, d(x, y)$. Finally, we say a point $p \in X$ is a *fixed point* of f if $f(p) = p$.

(a) Show that shrinking maps are continuous.

(b) Show that, if (X, d) is complete, then every contraction has a unique fixed point. Show that the result is in general false for shrinking maps.

(Hint: show that given any $x_1 \in X$ the sequence defined by $x_{n+1} = f(x_n)$ is a Cauchy sequence.)

(c) Show that, if (X, d) is compact, then every shrinking map has a unique fixed point.

(Hint: Consider $A = \cap_{n\in\mathbb{N}} f^n(X)$, prove that it is non-empty, and show that $A = f(A)$ by proceeding as follows: given $x \in A$, choose x_n so that $x = f^{n+1}(x_n)$, and consider the limit of a convergent subsequence of the sequence $\{f^n(x_n)\}$. Then consider the diameter of A.)

3. (a) Let $\mathcal{B}$ be the space of bounded continuous functions from $\mathbb{R}$ to itself, equipped with the uniform metric $d(f_1, f_2) = \sup_{x\in\mathbb{R}} |f_1(x) - f_2(x)|$. Show that the composition map,

$$\begin{aligned} \mathcal{B} \times \mathcal{B} &\to \mathcal{B} \\ (f, g) &\mapsto f \circ g, \end{aligned}$$

is continuous. (Hint: use uniform continuity).

(b) Does the result remain true if do not restrict ourselves to bounded functions? Namely: denoting by $\mathcal{C}$ the space of all continuous functions from $\mathbb{R}$ to itself, with the uniform topology, does $(f, g) \mapsto f \circ g$ define a continuous map from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$?

4. Munkres exercise 29.1.

5. Munkres exercise 29.8.

6. Munkres exercise 30.4.

7. Munkres exercise 31.2.

8. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?