Def:
- X is <u>compact</u> if every open cover $(U_i)_{i\in I}$ of X has a finite subcover.
- X is <u>limit point compact</u> if every infinite subset of X has a limit point
- X is <u>sequentially compact</u> if every sequence $\{p_n\}$ in X has a convergent subsequence.

proved last time **Thm:** X is compact $\Rightarrow$ X is limit point compact.

Thm: X sequentially compact $\Rightarrow$ X limit point compact.

Pf: Given $A \subset X$ infinite subset, pick a sequence of distinct points of A and take a convergent subsequence $\Rightarrow \exists \{a_n\}$ sequence in A, $a_n \neq a_m\ \forall n\neq m$, converging to some limit $a \in X$. Then every neighborhood of a contains a_n for all large n, hence ∞ly many points of A, including some $\neq a$. So a is a limit pt of A. □

The converse implications don't hold in general, but in metric spaces all three notions coincide! (& hence also for subspaces of metric spaces...)

Thm: For a metric space (X,d), X compact $\Leftrightarrow$ X limit pt compact $\Leftrightarrow$ X seq. compact.

Proof:
- compact $\Rightarrow$ limit point compact: already done (for all top. spaces)
- limit point compact $\Rightarrow$ sequentially compact: suppose X metric space and limit point compact, and consider a sequence $x_1, x_2, \ldots$ in X. If $\{x_1, x_2, \ldots\}$ finite, then $\exists x \in X$ st. $x_n = x$ for infinitely many n, which gives a subsequence that converges to x.

  Otherwise, $\{x_1, x_2, \ldots\}$ is infinite, so has a limit point a. So:
  $$\forall r > 0\ \exists n \text{ st. } 0 < d(a, x_n) < r.$$
  First choose $n_1 \in \mathbb{N}$ st. $x_{n_1} \in B_1(a)$, then inductively, given $n_1, \ldots, n_{k-1}$, let $\delta_k = \min\{d(x_i, a) \mid i \leq n_{k-1} \text{ and } x_i \neq a\} > 0$, and $r_k = \min(\frac{1}{k}, \delta_k)$.
  Then take n_k st. $0 < d(a, x_{n_k}) < r_k$. By construction: $n_k > n_{k-1}$, and $d(a, x_{n_k}) < \frac{1}{k}$.
  $\Rightarrow x_{n_1}, x_{n_2}, \ldots$ is a subsequence converging to a.
- seq. compact $\Rightarrow$ compact: this is the hardest part. First we show:

  <u>Lemma 1:</u> If X metric space is seq. compact, then $\forall \varepsilon > 0$ X can be covered by finitely many open balls of radius ε.

  (as we expect if X is to be compact: $X = \bigcup_{x\in X} B_\varepsilon(x)$ should have a finite subcover!)

  Proof: assume not, and choose $x_1 \in X$, then inductively choose $x_n \in X \setminus \bigcup_{i=1}^{n-1} B_\varepsilon(x_i)$

  This yields a sequence in X, which by sequential compactness must have a convergent subsequence. But this is impossible since no two terms of the sequence are within ε of each other! Contradiction. □

Lemma 2: If X metric space is sequentially compact then every open cover has a Lebesgue number ($\exists \delta > 0$ st. any subset of diameter $< \delta$ is entirely in one U_i).

(we've seen this holds for compact metric spaces, so it should hold!)

Pf: suppose $\exists$ open cover $(U_i)_{i \in I}$ with no Lebesgue number, ie. $\forall n \geq 1$ $\exists C_n \subset X$ with diameter $< \frac{1}{n}$ which isn't contained in any single U_i. Take $x_n \in C_n$.

By sequential compactness, $\exists$ subsequence (x_{n_k}) of (x_n) that converges to some $a \in X$.

Now $a \in U_{i_0}$ for some $i_0 \in I$, and so $\exists \varepsilon > 0$ st. $B_\varepsilon(a) \subset U_{i_0}$

Take k suff'ly large so that $\frac{1}{n_k} < \frac{\varepsilon}{2}$ and $d(x_{n_k}, a) < \frac{\varepsilon}{2}$.

Since C_{n_k} has diameter $< \frac{\varepsilon}{2}$, $C_{n_k} \subset B_{\frac{\varepsilon}{2}}(x_{n_k}) \subset B_\varepsilon(a) \subset U_{i_0}$, contradiction. □

(Rmk: this proof illustrates how arguments using sequential compactness are often more intuitive than those involving open covers: "if some property fails to hold uniformly, take a sequence of points where things get worse, extract a convergent subsequence, and see what goes wrong at the limit.")

Now we can prove seq. compact $\Rightarrow$ compact:

Pf: Given an open cover $X = \bigcup_{i \in I} U_i$, by Lemma 2 $\exists \delta > 0$ s.t. every subset of diameter $< \delta$ is entirely inside a single U_i. Fix $\varepsilon \in (0, \frac{\delta}{2})$: by Lemma 1, X is covered by finitely many ε-balls. Each of these has diameter $\leq 2\varepsilon < \delta$, so is contained in some U_i. This gives a finite subcover, replacing each ε-ball by one U_i containing it (and discarding the rest of the U_i's). □.

Thm: Every compact metric space (X,d) is complete, ie. every Cauchy seq. converges.

Pf: Let (x_n) Cauchy seq., by sequential compactness $\exists$ subsequence $x_{n_k} \to x \in X$.
Now $\forall \varepsilon > 0$ $\exists N$ st. $\forall m, n \geq N$, $d(x_m, x_n) < \frac{\varepsilon}{2}$. $\exists n_k \geq N$ st. $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.
Hence: $\forall n \geq N$, $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$. □.

Corollary: $\mathbb{R}$, $\mathbb{R}^n$ (with usual distances) are complete.

Pf: every Cauchy sequence is bounded, hence contained in a compact subset, hence convergent. □

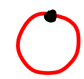
Corollary: $\mathbb{R}^X = \{$functions $X \to \mathbb{R}\}$ with uniform metric is complete.

Pf: given a Cauchy sequence $\{f_n\}$ (ie. $\forall \varepsilon > 0$ $\exists N$ st. $m, n \geq N \Rightarrow \sup |f_n - f_m| < \varepsilon$).
$\forall x \in X$, $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$ $\left(|f_n(x) - f_m(x)| \leq \sup |f_n - f_m| < \varepsilon\right)$
hence converges to some limit $f(x)$ (ie. we have a pointwise limit).
Now: given $\varepsilon > 0$, take N st. $m, n \geq N \Rightarrow \sup_x |f_n(x) - f_m(x)| < \varepsilon$.
Then $\forall n \geq N$, $\forall x \in X$, $|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \varepsilon$.
ie. $\forall n \geq N$, $\sup |f_n - f| \leq \varepsilon$, which implies $f_n \to f$ uniformly. □

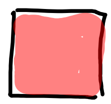
(When X is a top. space, we've seen that uniform limits of continuous functions are continuous, so we also have completeness of $C^0(X,\mathbb{R}) = \{\text{continuous } f^{n}\} \subset \mathbb{R}^X$, uniform top.
more generally: closed subsets of complete metric spaces are complete!)

Compactification

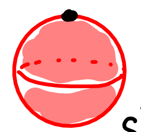
Def: A <u>compactification</u> of a (Hausdorff) top. space X is a compact (Hausdorff) space Y with an inclusion $i: X \hookrightarrow Y$ which is an embedding (ie. homeom. onto its image, ie. topology on X = subspace topology of $i(X) \subset Y$), with X open & dense in Y ($\overline{X} = Y$).

Ex: $\mathbb{R}^n \rightsquigarrow \mathbb{R}^n \cup \{\infty\}$ as in HW2; this is in fact homeo to S^n (unit sphere in $\mathbb{R}^{n+1}$)
This is not the only option: eg. $(0,1) \simeq \mathbb{R}$ compactifies to  $[0,1]$ or  S^1

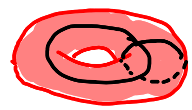
$(0,1)\times(0,1) \simeq \mathbb{R}^2$: eg.



$[0,1]\times[0,1]$



S^2



torus ($\simeq S^1 \times S^1$)

$\star$ The <u>one-point compactification</u>, if exists, is unique.
Let $Y = X \cup \{\infty\}$ (add a new point). The requirements of a compactification imply:
→ a subset $U \subset X$ is open in Y iff it is open in X (subspace top. $= \mathcal{T}_X$)
→ a subset V containing ∞ is open in Y iff $Y - V$ is closed, hence compact (we want Y compact), and a subset of X (since $\infty \in V$).

⇒ **Def:** $\mathcal{T}_Y = \{U \subset X \text{ open}\} \cup \{Y - K \,/\, K \subset X \text{ compact}\}$.

Thm: $\mathcal{T}_Y$ is a topology on $Y = X \cup \{\infty\}$, and Y is a compactification of X (in particular, Y is compact)
→ except: $\overline{X} = Y$ fails when X compact!

Pf: • axioms of a topology: case by case for U's and $(Y-K)$'s.
Arbitrary unions and finite $\cap$'s of a single type of open are still of the same type.
(note: $\bigcap (Y - K_i) = Y - (\bigcup K_i)$, a finite union of compact subsets of X is compact).
Moreover, $U \cap (Y-K) = U \cap (X - K)$ open $\subset X$
$U \cup (Y - K) = Y - (K \cap (X - U))$ closed in K hence compact ✓

• Y is compact: if $(A_i)_{i\in I}$ open cover of Y, then $\infty \in A_{i_0} = Y - K$ for some $i_0 \in I$, and now the $(A_i \cap K)$ form an open cover of K ⇒ $\exists\, i_1 \dots i_n$ st. $A_{i_1} \cup \dots \cup A_{i_n} \supset K$.
Thus $Y = A_{i_0} \cup (A_{i_1} \cup \dots A_{i_n})$ finite subcover. □

However, this Y is not always Hausdorff! One-point compactif^ns are only useful if Hausdorff.

Def: X is <u>locally compact</u> if $\forall x \in X$, $\exists K$ compact $\subset X$ which contains a neighborhood of x.

Ex: $\mathbb{R}$ is loc. compact ($x \in \mathbb{R} \Rightarrow x \in \text{int}([x-1, x+1])$), so is $\mathbb{R}^n$.
$\mathbb{R}^{\mathbb{N}}$ isn't (for any of usual topologies). Neither is $\mathbb{Q}$ with usual top ($\subset \mathbb{R}$)

Thm: The one-point compactif$^{\underline{n}}$ $Y = X \cup \{\infty\}$ is Hausdorff iff X is locally compact and Hausdorff

Pf:
- X Hausdorff $\Leftrightarrow$ we can separate points of $X \subset Y$ by open subsets (in X or in Y)
- X loc. compact $\Leftrightarrow$ $\forall x \in X$ $\exists$ opens $U \ni x$, $Y - K \ni \infty$ st. $U \subset K$ ie. $U \cap (Y-K) = \emptyset$
  $\Leftrightarrow$ we can separate points of X from ∞ by open subsets in Y. □

Countability axioms:

Def. X is first-countable if $\forall x \in X$, $\exists$ countable basis of neighborhoods at x, ie. $\exists U_1, U_2, \dots$ open $\ni x$ st. every neighborhood $V \ni x$ contains one of the U_n.

Ex: metric spaces are first-countable: at $x \in X$, take $U_n = B_{\frac{1}{n}}(x)$.

* In a first-countable space, $x \in \overline{A} \Leftrightarrow \exists$ sequence $x_n \in A$, $x_n \to x$. (else only $\Leftarrow$).

Def: X is second-countable if its topology has a countable basis.

Ex: $\mathbb{R}^n$ is second-countable, eg. basis: $\{B_r(x)\ \ x \in \mathbb{Q}^n,\ r \in \mathbb{Q}_+\}$ or $\{\prod (a_i, b_i) / a_i, b_i \in \mathbb{Q}\}$

$\mathbb{R}^\omega$ product top. is second-countable (basis = products of finite # of (a_i, b_i) $a_i < b_i \in \mathbb{Q}$ & all remaining factors are $\mathbb{R}$)

while uniform topology isn't (because $\exists$ uncountably many disjoint open subsets: balls of radius $1/2$ centered at $\{0,1\}^\omega$.)

* second-countable $\Rightarrow$ $\exists$ countable dense subset (eg: take one point in each basis open!)
  the converse holds for metric spaces (take balls of radius $\frac{1}{n}$ around points of the dense subset)
  but is false in general ($\mathbb{R}_\ell$ is first-countable, has countable dense subset, but $\nexists$ countable basis)

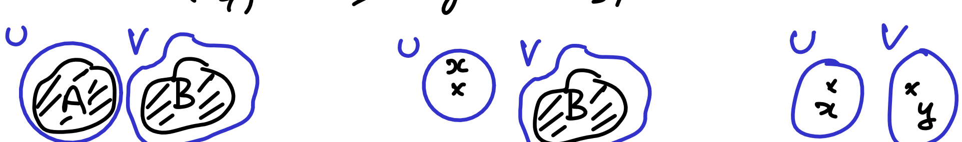
Regular and normal spaces (§31-32)

Recall: X Hausdorff := can separate points: $\forall x \neq y$, $\exists U \ni x, V \ni y$ disjoint open
(aka T_2) ($\Rightarrow T_1$: $\{x\}$ is closed $\forall x \in X$).

Stronger separation axioms:

Suppose one-point subsets $\{x\} \subset X$ are closed (T_1). Then say
- X is regular if $\forall x \in X$, $\forall B \subset X$ closed disjoint from x, $\exists$ disjoint open sets $U \ni x$, $V \supset B$.
- X is normal if $\forall A, B \subset X$ disjoint closed subsets, $\exists$ disjoint open sets $U \supset A$, $V \supset B$.

Metrizable $\Rightarrow$ Normal (T_4) $\Longrightarrow$ Regular (T_3) $\Longrightarrow$ Hausdorff (T_2) $\Longrightarrow$ T_1
(will see)



Ex: $\mathbb{R}_\ell$ is normal but not metrizable, $\mathbb{R}_\ell^2$ is regular but not normal. see Munkres §31