

NOTE: • midterm will be posted on Canvas Monday, due Friday Feb 18.
 topics = everything seen up to now (ending with compactifications)
 No collaboration/no materials other than lecture notes + Munkres.

Def: A compactification of a (Hausdorff) top space X is a compact (Hausdorff) space Y with an inclusion $i: X \hookrightarrow Y$ which is an embedding (ie. homeo. onto its image, ie. topology on $X \equiv$ subspace topology of $i(X) \subset Y$), with X open & dense in Y ($\bar{X} = Y$).

Ex: $\mathbb{R}^n \rightsquigarrow \mathbb{R}^n \cup \{\infty\}$ as in HW2; this is in fact homeo to S^n (unit sphere in $\mathbb{R}^{n+1}$)
 This is not the only option: eg. $(0,1) \cong \mathbb{R}$ compactifies to $\bullet \text{---} [0,1] \bullet$ or $\bigcirc S^1$

★ The one-point compactification, if exists, is unique.

Let $Y = X \cup \{\infty\}$ (add a new point). The requirements of a compactification imply:
 $\rightarrow$ a subset $U \subset X$ is open in Y iff it is open in X (subspace top. $\equiv \tau_X$)
 $\rightarrow$ a subset V containing ∞ is open in Y iff $Y - V$ is closed, hence compact (we want Y compact), and a subset of X (since $\infty \in V$).

$\Rightarrow$ Def: $\tau_Y = \{U \subset X \text{ open}\} \cup \{Y - K \mid K \subset X \text{ compact}\}$.

Thm: Assuming X Hausdorff, τ_Y is a topology on $Y = X \cup \{\infty\}$, and Y is a (not necessarily Hausdorff) compactification of X (in particular, Y is compact) except: $\bar{X} = Y$ fails when X compact!

Pf: • axioms of a topology: case by case for U 's and $(Y - K)$'s.

Arbitrary unions and finite $\cap$'s of a single type of open are still of the same type.
 (note: $\cap (Y - K_i) = Y - (U K_i)$, a finite union of compact subsets of X is compact).

Moreover, $U \cap (Y - K) = U \cap (X - K)$ open $\subset X$ (note: K compact $\Rightarrow K$ closed).

$U \cup (Y - K) = Y - (\underbrace{K \cap (X - U)}_{\text{closed in } K \text{ hence compact}})$ ✓

• Y is compact: if $(A_i)_{i \in I}$ open cover of Y , then $\infty \in A_{i_0} = Y - K$ for some $i_0 \in I$, and now the $(A_i \cap K)$ form an open cover of $K \Rightarrow \exists i_1 \dots i_n$ st. $A_{i_1} \cup \dots \cup A_{i_n} \supset K$.
 Thus $Y = A_{i_0} \cup (A_{i_1} \cup \dots \cup A_{i_n})$ finite subcover.

• X is open in Y , subspace topology $\equiv \tau_X$, and $\bar{X} = Y$ unless $\{\infty\} \in \tau_Y$ ie. X compact. $\square$

However, this Y is not always Hausdorff! One-point compactifications are only useful if Hausdorff.

Def: X is locally compact if $\forall x \in X, \exists K$ compact $\subset X$ which contains a neighborhood of x .

Ex: $\mathbb{R}$ is loc. compact ($x \in \mathbb{R} \Rightarrow x \in \text{int}([x-1, x+1])$), so is $\mathbb{R}^n$.

$\mathbb{R}^n$ isn't (for any of usual topologies). Neither is $\mathbb{Q}$ with usual top ($\subset \mathbb{R}$)

Thm. The one-point compactification $Y = X \cup \{\infty\}$ is Hausdorff iff X is locally compact and Hausdorff

Pf. • X Hausdorff $\Leftrightarrow$ we can separate points of $X \subset Y$ by open subsets (in X or in Y)
 • X loc. compact $\Leftrightarrow \forall x \in X \exists$ opens $U \ni x, Y - K \ni \infty$ st. $U \cap K = \emptyset$ i.e. $U \cap (Y - K) = \emptyset$
 $\Leftrightarrow$ we can separate points of X from ∞ by open subsets in Y . $\square$

Countability axioms:

Def. X is first-countable if $\forall x \in X, \exists$ countable basis of neighborhoods at x ,
 i.e. $\exists U_1, U_2, \dots$ open $\ni x$ st. every neighborhood $V \ni x$ contains one of the U_n .

Ex: metric spaces are first-countable: at $x \in X$, take $U_n = B_{\frac{1}{n}}(x)$.

* In a first-countable space, $x \in \bar{A} \Leftrightarrow \exists$ sequence $x_n \in A, x_n \rightarrow x$. (else only $\Leftarrow$)

Def. X is second-countable if its topology has a countable basis.

Ex: $\mathbb{R}^n$ is second-countable, eg. basis: $\{B_r(x) \mid x \in \mathbb{R}^n, r \in \mathbb{Q}_+\}$ or $\{\prod (a_i, b_i) \mid a_i, b_i \in \mathbb{Q}\}$
 $\mathbb{R}^\omega$ product top. is second-countable (basis = products of finite # of $(a_i, b_i) \mid a_i, b_i \in \mathbb{Q}$ & all remaining factors are $\mathbb{R}$)

while uniform topology isn't (because $\exists$ uncountably many disjoint open subsets: balls of rad'us $1/2$ centered at $\{0, 1\}^\omega$).

* second-countable $\Rightarrow \exists$ countable dense subset (eg: take one point in each basis open!)
 the converse holds for metric spaces (take balls of radius $\frac{1}{n}$ around points of the dense subset)
 but is false in general ($\mathbb{R}_\ell$ is first-countable, has countable dense subset, but $\nexists$ countable basis)

Regular and normal spaces (§31-32)

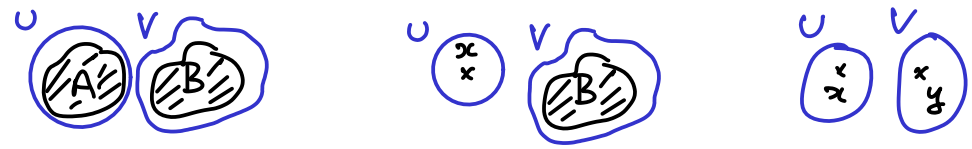
Recall: X Hausdorff := can separate points: $\forall x \neq y, \exists U \ni x, V \ni y$ disjoint open
 (aka T_2) $(\Rightarrow T_1: \{x\} \text{ is closed } \forall x \in X)$

Stronger separation axioms:

Suppose one-point subsets $\{x\} \subset X$ are closed (T_1). Then say
 • X is regular if $\forall x \in X, \forall B \subset X$ closed disjoint from $x, \exists$ disjoint open sets $U \ni x, V \supset B$.
 • X is normal if $\forall A, B \subset X$ disjoint closed subsets, $\exists$ disjoint open sets $U \supset A, V \supset B$.

Metrizable $\Rightarrow$ Normal (T_4) $\Rightarrow$ Regular (T_3) $\Rightarrow$ Hausdorff (T_2) $\Rightarrow T_1$

will see $\nearrow$



Ex: $\mathbb{R}_\ell$ is normal but not metrizable
 $\mathbb{R}_\ell^2$ is regular but not normal. } see Munkres §31 for these and more.

(3)

Theorem: || • regular + second-countable $\Rightarrow$ normal
 • Hausdorff + compact $\Rightarrow$ normal

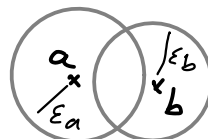
(won't prove, cf. Munkres §32. We did see, when proving compact subsets of Hausdorff spaces are closed, that compact + Hausdorff $\Rightarrow$ regular; normal was an exercise on HW2. (Munkres 26.5))

Theorem: || Metric spaces are normal.

Pf: Let A, B disjoint closed subsets $\subset (X, d)$.

$\forall a \in A, \exists \varepsilon_a > 0$ st. $B_{\varepsilon_a}(a) \subset X - B$.

$\forall b \in B, \exists \varepsilon_b > 0$ st. $B_{\varepsilon_b}(b) \subset X - A$.



Let $U = \bigcup_{a \in A} B_{\varepsilon_a/2}(a) \supset A$, $V = \bigcup_{b \in B} B_{\varepsilon_b/2}(b) \supset B$ (clearly open: $\cup$ balls)

We claim $U \cap V = \emptyset$. Indeed if $z \in U \cap V$ then $\exists a \in A, b \in B$ st.

$z \in B_{\varepsilon_a/2}(a) \cap B_{\varepsilon_b/2}(b)$. So $d(a, b) \leq d(a, z) + d(z, b) < \frac{\varepsilon_a}{2} + \frac{\varepsilon_b}{2} \leq \max(\varepsilon_a, \varepsilon_b)$.

But this is a contradiction (eg if $d(a, b) < \varepsilon_a$ then $B_{\varepsilon_a}(a) \not\subset X - B$!). $\square$

* We can now ask which topological spaces are metrizable.

We've seen: Metrizable $\Rightarrow$ first-countable and normal. ($\nLeftarrow$ counterexample: $\mathbb{R}_\ell$)

Urysohn metrization theorem: || If X is regular and has a countable basis, then it is metrizable.

(The first condition is necessary, the second one is stronger than needed. The

Nagata-Smirnov thm gives a sharper criterion but is more technical to state & prove).

Urysohn's lemma is the key ingredient in the proof of the metrization theorem.

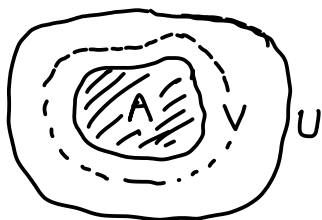
Thm: || X normal space, A, B disjoint closed subsets
 $\Rightarrow \exists$ continuous $f: X \rightarrow [0, 1]$ st. $f(x) = 0 \ \forall x \in A$ and $f(x) = 1 \ \forall x \in B$.

Idea: 1) construct open sets $U_q \ \forall q \in [0, 1] \cap \mathbb{Q}$ st. $A \subset U_0 \subset \dots \subset U_1 = X - B$
 and moreover $p < q \Rightarrow \overline{U_p} \subset U_q$. & also set $U_q = X$ for $q > 1$.

2) define $f(x) = \inf \{ q \in \mathbb{Q} / x \in U_q \}$. + show f is continuous.

Step 1 uses the following reformulation of normality:

Lemma: || X is normal $\Rightarrow \forall A$ closed, $\forall U \supset A$ open, $\exists$ open V st.
 (in fact $\Leftrightarrow$) $A \subset V$ and $\overline{V} \subset U$.



Pf: A and $B = X - U$ are disjoint closed sets, so since X is normal, $\exists V \supset A, V' \supset B$ open such that $V \cap V' = \emptyset$.
Moreover, $X - V'$ closed, $V \subset X - V' \Rightarrow \bar{V} \subset X - V'$.
So $A \subset V \subset \bar{V} \subset X - V' \subset X - B = U$. $\square$

Proof of Urysohn's lemma:

Step 1: Given A & B disjoint closed, let $U_1 = X - B$, and let U_0 open st. $A \subset U_0 \subset \bar{U}_0 \subset U_1$.

Next, we construct $U_q, q \in (0,1) \cap \mathbb{Q}$, st. $p < q \Rightarrow \bar{U}_p \subset U_q$ by induction:

choose a labelling of $[0,1] \cap \mathbb{Q} = \{q_0, q_1, q_2, q_3, \dots\}$ by an infinite sequence such that $q_0 = 0$ & $q_1 = 1$. (could eg. continue: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots$).

Assuming $U_{q_0} \dots U_{q_n}$ have already been chosen, we construct $U_{q_{n+1}}$ using the above lemma:

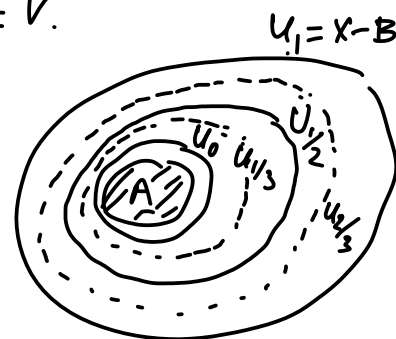
let $q_k = \max(\{q_0 \dots q_n\} \cap [0, q_{n+1}))$ so $q_k < q_{n+1} < q_\ell$ & none of the
 $q_\ell = \min(\{q_0 \dots q_n\} \cap (q_{n+1}, 1])$ rationals already considered lie in between.

Then by induction hypothesis, $\bar{U}_{q_k} \subset U_{q_\ell}$, hence using normality

$\exists$ open V st. $\bar{U}_{q_k} \subset V \subset \bar{V} \subset U_{q_\ell}$, let $U_{q_{n+1}} = V$.

By induction, we construct in this way all the U_q 's.
and indeed $p < q \Rightarrow \bar{U}_p \subset U_q$.

We also set $U_q = \emptyset$ if $q < 0$, X if $q > 1$.
(still true: $p < q \Rightarrow \bar{U}_p \subset U_q$!).



Step 2: Define $f(x) = \inf Q_x$, where $Q_x = \{q \in \mathbb{Q} / x \in U_q\}$.

Since $U_{<0} = \emptyset$ and $U_{>1} = X$, $(1, \infty) \subset Q_x \subset [0, \infty)$ so $f(x) \in [0,1] \forall x \in X$

Also, $x \in A \subset U_0 \Rightarrow f(x) = 0$, and $x \in B \Rightarrow x \notin U_1 = X - B \Rightarrow Q_x = (1, \infty)$ and $f(x) = 1$.

So: it only remains to show that $f: X \rightarrow [0,1]$ is continuous! For this, observe:

- $x \in \bar{U}_q \Rightarrow f(x) \leq q$: indeed if $x \in \bar{U}_q$ then $x \in U_{q'} \forall q' > q$ so $Q_x \supset \mathbb{Q} \cap (q, \infty)$.
- $x \notin U_q \Rightarrow f(x) \geq q$: indeed if $x \notin U_q$ then $Q_x \subset \mathbb{Q} \cap (q, \infty)$.

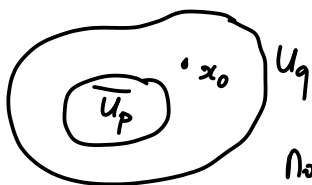
Now given an open interval (c,d) , we show $f^{-1}((c,d))$ is open in X :

Assume $x_0 \in f^{-1}((c,d))$, and let $p, q \in \mathbb{Q}$ st. $c < p < f(x_0) < q < d$.

By the above observation, $x_0 \in U_q$ and $x_0 \notin \bar{U}_p$.

$V = U_q \cap (X - \bar{U}_p)$ is open, and a neighborhood of x_0 .

Moreover, $x \in V \Rightarrow x \notin \bar{U}_p$ so $f(x) \geq p$ Hence $V \subset f^{-1}([p, q]) \subset f^{-1}((c,d))$.
 $x \in U_q$ so $f(x) \leq q$ ie. $f^{-1}((c,d)) \supset$ nbds. of its points. $\square$



Now we prove the metrization theorem, namely that if X is normal & has countable basis, then X is metrizable. We actually do this by embedding X as a subspace of a metric space, namely $[0,1]^{\omega}$ with product topology or uniform topology - in fact both come from metrics. (5)

product top: $d((x_n), (y_n)) = \sup \left\{ \frac{1}{n} |y_n - x_n| \right\} \rightarrow$ then $B_{\varepsilon}((x_n)) = \prod_n (x_n - n\varepsilon, x_n + n\varepsilon)$

key point: for $n > \varepsilon^{-1}$ this is all of $[0,1]$.

Step 1: $\left\{ \begin{array}{l} \exists \text{ countable collection of continuous functions } f_n: X \rightarrow [0,1] \text{ st. } \forall x_0 \in X, \forall U \ni x_0 \text{ neighborhood,} \\ \exists n \text{ st. } f_n(x_0) > 0 \text{ and } f_n \equiv 0 \text{ on } X - U. \end{array} \right.$

Pf: This follows from Urysohn's lemma, but need to be careful so that countably many functions suffice.

Let $\mathcal{B} = \{B_n\}$ countable basis for X . If $x_0 \in U$ open then $\exists B_n \in \mathcal{B}$ st. $x_0 \in B_n \subset U$.

But then, since X is normal, $\exists V$ open st. $x_0 \in V \subset \bar{V} \subset B_n$, and $\exists B_m \in \mathcal{B}$ st. $x_0 \in B_m \subset V$, so that $x_0 \in \bar{B}_m \subset B_n \subset U$.

So: for every $(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ st. $\bar{B}_m \subset B_n$, apply Urysohn's lemma to get

$g_{m,n}: X \rightarrow [0,1]$ st. $g_{m,n} = 1$ on $\bar{B}_m$ and 0 on $X - B_n$.

This countable collection of functions has the stated property. $\square$

Step 2: $\left\{ \begin{array}{l} F: X \rightarrow [0,1]^{\omega}, \text{ product topology} \text{ is an embedding, ie. continuous, injective, and} \\ x \mapsto F(x) = (f_1(x), f_2(x), \dots) \text{ } X \text{ is homeo to } F(X) \subset [0,1]^{\omega} \end{array} \right.$
(so topology on X is defined by the metric $d|_{F(X)}$, QED)

Pf: • F is continuous in product topology because each component $f_1, f_2, \dots$ is continuous $X \rightarrow [0,1]$.

• F is injective, since $x \neq y \Rightarrow \exists U \ni x, V \ni y$ disjoint open
 $\Rightarrow \exists m, n$ st. $f_n(x) > 0, f_n = 0$ outside of U (hence at y)
 $f_m(y) > 0, f_m = 0$ outside of V (hence at x).

• finally, must show that F is a homeo $X \rightarrow \mathbb{Z} = F(X) \subset [0,1]^{\omega}$. since F is a continuous bijection $X \rightarrow \mathbb{Z}$, only remains to prove: $U \subset X$ open $\Rightarrow F(U) \subset \mathbb{Z}$ is open.

For this, let $U \subset X$ be any open set, and $x_0 \in U$. Then $\exists n$ st. $f_n(x_0) > 0$ and $f_n = 0$ outside of U . Let $V_n = \pi_n^{-1}((0, \infty)) \cap \mathbb{Z} = \{z = (z_1, z_2, \dots) \in \mathbb{Z} / z_n > 0\} \subset \mathbb{Z}$
 open

Then $x_0 \in F^{-1}(V_n) \subset U$ (since $f_n(x_0) > 0$, and $f_n(x) > 0 \Rightarrow x \in U$).

hence $F(x_0) \in V_n \subset F(U)$. This is true $\forall x_0 \in U$ ($\Leftrightarrow \forall F(x_0) \in F(U)$)
 $\uparrow$ open in $\mathbb{Z}$ so we conclude that $F(U)$ is open.

Hence $F: X \rightarrow \mathbb{Z}$ is a homeomorphism, and X is homeo to a metric space! $\square$