

Urysohn metrization theorem: || If X is regular and has a countable basis, then it is metrizable.
 (Recall: regular = can separate points from closed sets. We've stated that the assumptions imply X is normal i.e. $\forall A, B \subset X$ disjoint closed subsets, $\exists$ disjoint open sets $U \supset A, V \supset B$.)

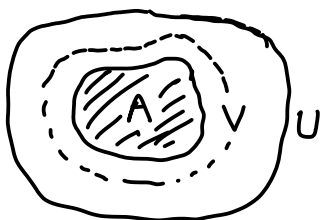
Urysohn's lemma is the key ingredient in the proof of the metrization theorem.

Thm: || X normal space, A, B disjoint closed subsets
 $\Rightarrow \exists$ continuous $f: X \rightarrow [0, 1]$ st. $f(x) = 0 \ \forall x \in A$ and $f(x) = 1 \ \forall x \in B$.

Idea: 1) construct open sets $U_q \ \forall q \in [0, 1] \cap \mathbb{Q}$ st. $A \subset U_0 \subset \dots \subset U_1 = X - B$ & moreover
 2) define $f(x) = \inf \{ q \in \mathbb{Q} / x \in U_q \}$. + show f is continuous. $p < q \Rightarrow \overline{U_p} \subset U_q$.

Step 1 uses the following reformulation of normality:

Lemma: || X is normal $\Rightarrow \forall A$ closed, $\forall U \supset A$ open, $\exists$ open V st. $A \subset V$ and $\overline{V} \subset U$.
 (in fact $\Leftrightarrow$)



Pf: A and $B = X - U$ are disjoint closed sets, so since X is normal,
 $\exists V \supset A, V' \supset B$ open such that $V \cap V' = \emptyset$.
 Moreover, $X - V'$ closed, $V \subset X - V' \Rightarrow \overline{V} \subset X - V' = U$.
 So $A \subset V \subset \overline{V} \subset X - V' \subset X - B = U$. $\square$

Proof of Urysohn's lemma:

Step 1: Given A & B disjoint closed, let $U_1 = X - B$, and let U_0 open st. $A \subset U_0 \subset \overline{U_0} \subset U_1$.

Next, we construct $U_q, q \in (0, 1) \cap \mathbb{Q}$, st. $p < q \Rightarrow \overline{U_p} \subset U_q$ by induction:

choose a labelling of $[0, 1] \cap \mathbb{Q} = \{q_0, q_1, q_2, q_3, \dots\}$ by an infinite sequence
 such that $q_0 = 0$ & $q_1 = 1$. (could eg. continue: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots$).

Assuming $U_{q_0} \dots U_{q_n}$ have already been chosen, we construct $U_{q_{n+1}}$ using the above lemma:

let $q_k = \max(\{q_0 \dots q_n\} \cap [0, q_{n+1}))$ so $q_k < q_{n+1} < q_\ell$ & none of the
 $q_\ell = \min(\{q_0 \dots q_n\} \cap (q_{n+1}, 1])$ rationals already considered lie in between.

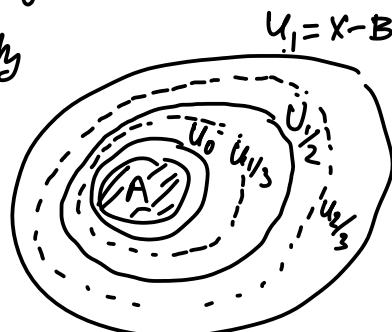
Then by induction hypothesis, $\overline{U_{q_k}} \subset U_{q_\ell}$, hence using normality

$\exists$ open V st. $\overline{U_{q_k}} \subset V \subset \overline{V} \subset U_{q_\ell}$, let $U_{q_{n+1}} = V$.

By induction, we construct in this way all the U_q 's.

and indeed $p < q \Rightarrow \overline{U_p} \subset U_q$.

also set $U_q = \emptyset$ if $q < 0$, X if $q > 1$. (still true: $p < q \Rightarrow \overline{U_p} \subset U_q$!).



Step 2: Define $f(x) = \inf Q_x$, where $Q_x = \{q \in \mathbb{Q} / x \in U_q\}$.

(2)

Since $U_{<0} = \emptyset$ and $U_{>1} = X$, $(1, \infty) \subset Q_x \subset [0, \infty)$ so $f(x) \in [0, 1] \forall x \in X$

Also, $x \in A \subset U_0 \Rightarrow f(x) = 0$, and $x \in B \Rightarrow x \notin U_1 = X - B \Rightarrow Q_x = (1, \infty)$ and $f(x) = 1$.

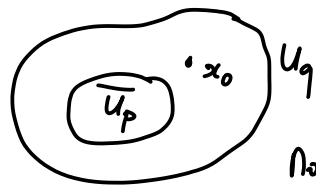
So: it only remains to show that $f: X \rightarrow [0, 1]$ is continuous! For this, observe:

• $x \in \bar{U}_q \Rightarrow f(x) \leq q$: indeed if $x \in \bar{U}_q$ then $x \in U_{q'}$ $\forall q' > q$ so $Q_x \supset \mathbb{Q} \cap (q, \infty)$.

• $x \notin U_q \Rightarrow f(x) \geq q$: indeed if $x \notin U_q$ then $Q_x \subset \mathbb{Q} \cap (q, \infty)$.

Now given an open interval (c, d) , we show $f^{-1}((c, d))$ is open in X :

Assume $x_0 \in f^{-1}((c, d))$, and let $p, q \in \mathbb{Q}$ st. $c < p < f(x_0) < q < d$.



By the above observation, $x_0 \in U_q$ and $x_0 \notin \bar{U}_p$.

$V = U_q \cap (X - \bar{U}_p)$ is open, and a neighborhood of x_0 .

Moreover, $x \in V \Rightarrow x \notin U_p$ so $f(x) \geq p$ Hence $V \subset f^{-1}([p, q]) \subset f^{-1}((c, d))$.
 $x \in \bar{U}_q$ so $f(x) \leq q$ ie. $f^{-1}((c, d)) \supset$ nbds. of its points. $\square$

Now we prove the metrization theorem, namely that if X is normal & has countable basis, then X is metrizable. We actually do this by embedding X as a subspace of a metric space, namely $[0, 1]^\omega$ with product topology or uniform topology - in fact both come from metrics.

product top: $d((x_n), (y_n)) = \sup \{ \frac{1}{n} |y_n - x_n| \}$ $\rightarrow$ then $B_\varepsilon((x_n)) = \prod_n (x_n - n\varepsilon, x_n + n\varepsilon)$

key point: for $n > \varepsilon^{-1}$ this is all of $[0, 1]$.

Step 1: $\exists$ countable collection of continuous functions $f_n: X \rightarrow [0, 1]$ st. $\forall x_0 \in X, \forall U \ni x_0$ neighborhood, $\exists n$ st. $f_n(x_0) > 0$ and $f_n \equiv 0$ on $X - U$.

Pf: This follows from Urysohn's lemma, but need to be careful so that countably many functions suffice.

Let $\mathcal{B} = \{B_n\}$ countable basis for X . If $x_0 \in U$ open then $\exists B_n \in \mathcal{B}$ st. $x_0 \in B_n \subset U$.

But then, since X is normal, $\exists V$ open st. $x_0 \in V \subset \bar{V} \subset B_n$, and $\exists B_m \in \mathcal{B}$ st. $x_0 \in B_m \subset V$, so that $x_0 \in \bar{B}_m \subset B_n \subset U$.

So: for every $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ st. $\bar{B}_m \subset B_n$, apply Urysohn's lemma to get

$g_{m,n}: X \rightarrow [0, 1]$ st. $g_{m,n} = 1$ on $\bar{B}_m$ and 0 on $X - B_n$.

This countable collection of functions has the stated property. $\square$

Step 2: $\parallel F: X \rightarrow [0, 1]^\omega$, product topology is an embedding, ie. continuous, injective, and $x \mapsto F(x) = (f_1(x), f_2(x), \dots)$ X is homeo to $F(X) \subset [0, 1]^\omega$

(so topology on X is defined by the metric $d|_{F(X)}$, QED)

Pf. • F is continuous in product topology because each component $f_1, f_2, \dots$ is continuous $X \rightarrow [0,1]$. ③

• F is injective, since $x \neq y \Rightarrow \exists U \ni x, V \ni y$ disjoint open
 $\Rightarrow \exists m, n$ st. $f_m(x) > 0, f_m = 0$ outside of U (hence at y)
 $f_n(y) > 0, f_n = 0$ outside of V (hence at x).

• finally, must show that F is a homeo $X \rightarrow Z = F(X) \subset [0,1]^{\mathbb{N}}$. since F is a continuous bijection $X \rightarrow Z$, only remains to prove: $U \subset X$ open $\Rightarrow F(U) \subset Z$ is open.

For this, let $U \subset X$ be any open set, and $x_0 \in U$. Then $\exists n$ st. $f_n(x_0) > 0$ and $f_n = 0$ outside of U . Let $V_n = \pi_n^{-1}((0, \infty)) \cap Z = \{z = (z_1, z_2, \dots) \in Z \mid z_n > 0\} \subset Z$
 $\uparrow$ open

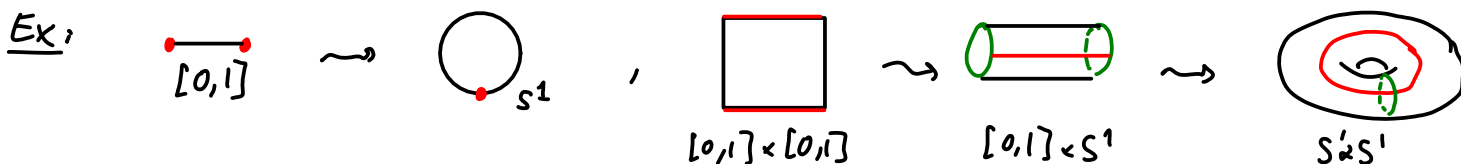
Then $x_0 \in F^{-1}(V_n) \subset U$ (since $f_n(x_0) > 0$, and $f_n(x) > 0 \Rightarrow x \in U$).

hence $F(x_0) \in V_n \subset F(U)$. This is true $\forall x_0 \in U$ ($\Leftrightarrow \forall F(x_0) \in F(U)$)
 $\uparrow$ open in Z so we conclude that $F(U)$ is open.

Hence $F: X \rightarrow Z$ is a homeomorphism, and X is homeo to a metric space! $\square$

Gluing & quotients (§22)

One good way to build interesting topological spaces is by "gluing" together simpler spaces.



The construction underlying this is the quotient topology.

Def. X top. space, A a set, $f: X \rightarrow A$ a surjective map.

The quotient topology on A is defined by:

$U \subset A$ is open $\Leftrightarrow f^{-1}(U) \subset X$ is open.

(Exercise: check this is a topology on A , in fact the finest topology for which f is continuous)

• A map $f: X \rightarrow Y$ between topological spaces is called a quotient map if f is surjective and $U \subset Y$ is open $\Leftrightarrow f^{-1}(U) \subset X$ is open.

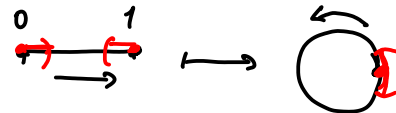
(ie. the topology on Y is the quotient topology induced by $f: X \rightarrow Y$)

• Typically, start from an equivalence relation $\sim$ on X , define A to be the set of equivalence classes $A = X/\sim$, and define $f: X \rightarrow X/\sim = A, x \mapsto [x]$.

Conversely, given any surjective map $f: X \rightarrow A$, we can define an equivalence relation on X by $x \sim x' \Leftrightarrow f(x) = f(x')$ and then $X/\sim = A$.

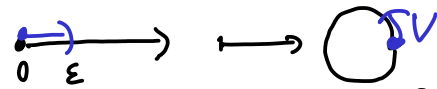
Ex: $S^1 \simeq [0,1]$ with 0 glued to 1. set $0 \sim 1$ so $\{0,1\}$ is one equiv. class. (all others are just $\{x\}$). (4)

The quotient map is $f: [0,1] \rightarrow S^1$

$$t \mapsto (\cos 2\pi t, \sin 2\pi t)$$


(Check! away from the end points f is a homeo $(0,1) \xrightarrow{\sim} S^1 - \{(1,0)\}$. so only need to check at 0 & 1. The point is: $U \ni (1,0)$ open in $S^1 \Leftrightarrow f^{-1}(U) \supset \{0,1\}$ open in $[0,1]$.)

vs. $g = f|_{[0,1)}: [0,1) \rightarrow S^1$ not a quotient map!

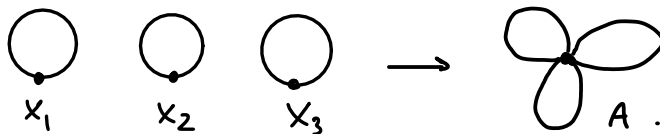


$V = g([0,\varepsilon))$ not open in S^1 vs. $g^{-1}(V) = [0,\varepsilon)$ open in $[0,1)$, wait iff!
(whereas $f^{-1}(V) = [0,\varepsilon) \cup \{1\}$ not open in $[0,1]$ ✓)

Ex: $X_1, \dots, X_n$ top spaces each $X_i \simeq S^1$, pick one point $x_i \in X_i \forall i$.

+ let $A =$ quotient space of $\coprod X_i$ by the equivalence relation $x_i \sim x_j \forall i,j$.

(glue the X_i at their base points).. This is called the wedge of the circle $X_1, \dots, X_n$.



* There is a useful characterization of continuous maps from a quotient space:

If $A = X/\sim$ and $f: X \rightarrow Y$ is a map st. $x \sim x' \Rightarrow f(x) = f(x')$,

then we can define $\bar{f}: X/\sim \rightarrow Y$ by $\bar{f}([x]) = f(x)$.

$$\begin{array}{ccc} X/\sim & \xrightarrow{\bar{f}} & Y \\ p \uparrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

Thm: If $f: X \rightarrow Y$ is a continuous map and $x \sim x' \Rightarrow f(x) = f(x')$, then equipping $X/\sim$ with the quotient topology, $\bar{f}: X/\sim \rightarrow Y$ is a continuous map.

Pf: let $p: X \rightarrow X/\sim$ the quotient map, and recall $\bar{f}([x]) = f(x)$ (indep. of $x \in [x]$)

So $\bar{f} \circ p = f$. Hence: $\forall U \subset Y$ open, $f^{-1}(U) = p^{-1}(\bar{f}^{-1}(U)) \subset X$ is open.

By definition of the quotient topology, we conclude that $\bar{f}^{-1}(U) \subset X/\sim$ is open. $\square$
($V \subset X/\sim$ open $\Leftrightarrow p^{-1}(V) \subset X$ open).

(& conversely, since $p: X \rightarrow X/\sim$ is continuous. So: $\bar{f}$ continuous $\Leftrightarrow f = \bar{f} \circ p$ continuous)

Ex: $X = \mathbb{R}^{n+1} - \{0\}$, define an equivalence relation $x \sim y$ iff x, y lie on the same line through the origin, i.e. $x = \alpha y$ for some $\alpha \in \mathbb{R}, \alpha \neq 0$. This is an equivalence relation.

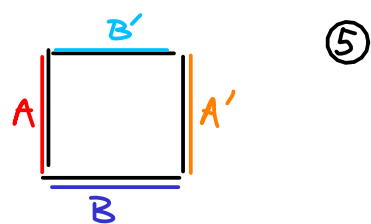
The quotient space is projective n -space, $\mathbb{RP}^n = X/\sim$ with quotient topology.

("space of lines through 0 in $\mathbb{R}^{n+1}$ ")

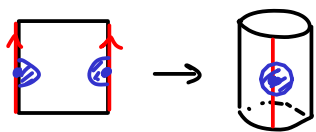
IF Y is another top space, then a continuous map $\bar{f}: \mathbb{RP}^n \rightarrow Y$ is the same thing as a continuous map $f: \mathbb{R}^{n+1} - \{0\} \rightarrow Y$ st. $f(\alpha x) = f(x) \forall \alpha \in \mathbb{R} - \{0\}, \forall x \in X$.

(more about $\mathbb{RP}^n$ on the HW.)

Ex: Various quotients of the unit square $X = [0,1]^2$: let the edges be $A = \{0\} \times [0,1]$, A' , B , B'

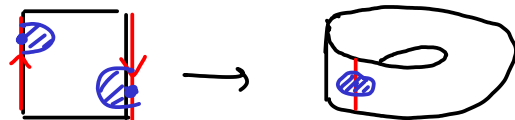


1) gluing A to A' by $(0,t) \sim (1,t)$, get a cylinder

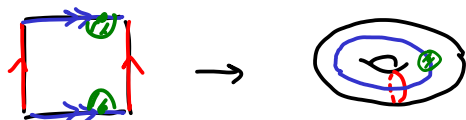


A neighborhood of a point on the gluing line corresponds to two neighborhoods of $(0,t) \in A$ and $(1,t) \in A'$ in X .

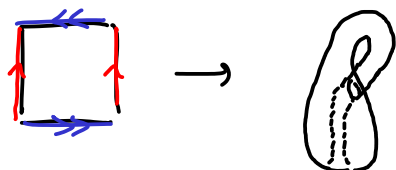
2) if instead we glue A to A' by $(0,t) \sim (1,1-t)$, we get a Möbius band!



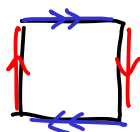
3) gluing A to A' via $(0,t) \sim (1,t)$ and B to B' by $(s,0) \sim (s,1)$ gives us the torus



4) gluing $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$, however, gives the Klein bottle, which cannot be embedded in $\mathbb{R}^3$ (can draw a picture that self-intersects).



5) gluing $(0,t) \sim (1,1-t)$ and $(s,0) \sim (1-s,1)$ is tricky to visualize, but the quotient is actually homeomorphic to $\mathbb{RP}^2$.



(Exercise: what about gluing $(0,t) \sim (t,0)$ and $(1,s) \sim (s,1)$ - what does that look like?).