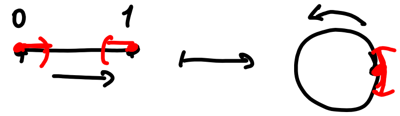
55b lecture 11 - Wed. 2/16 - Gluing & quotients (§22)

Def: given a surjective map $f: X \twoheadrightarrow A$ (e.g. $A = X/\sim$), the quotient topology on Y has $U \subset A$ open $\Leftrightarrow$ $f^{-1}(U) \subset X$ open.
Say f is a quotient map (= surjective map s.t. U open $\Leftrightarrow$ $f^{-1}(U)$ open).

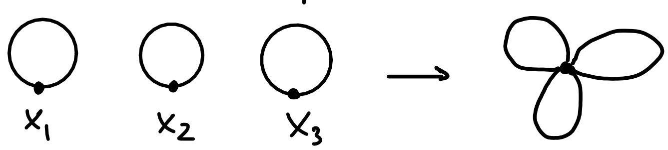
Quotient topology = the finest topology on A s.t. f is continuous.

Ex: $S^1 \simeq [0,1]$ with 0 glued to 1: set $0 \sim 1$ so $\{0,1\}$ is one equiv. class.
(all others are just $\{x\}$).
The quotient map is $f: [0,1] \longrightarrow S^1$
$t \longmapsto (\cos 2\pi t, \sin 2\pi t)$



(Check! away from the end points f is a homeo $(0,1) \xrightarrow{\sim} S^1 - \{(1,0)\}$. so only need to check at 0 & 1. The point is: $U \ni (1,0)$ open in $S^1 \Leftrightarrow f^{-1}(U) \supset \{0,1\}$ open in $[0,1]$.)

Ex: $X_1, \dots, X_n$ top. spaces each $X_i \simeq S^1$, pick one point $x_i \in X_i$ $\forall i$.
Consider the quotient space of $\coprod X_i$ by the equivalence relation $x_i \sim x_j$ $\forall i,j$.
(glue the X_i at their base points).. This is called the wedge of the circle $X_1, \dots X_n$.



★ There is a useful characterization of continuous maps from a quotient space:
if $f: X \to Y$ is a map s.t. $x \sim x' \Rightarrow f(x) = f(x')$,
then we can define $\bar{f}: X/\sim \to Y$ by $\bar{f}([x]) = f(x)$.

$$\begin{tikzcd} & X/\sim \arrow[dr, "\bar{f}"] & \\ X \arrow[ur, "p"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Thm: Equipping $X/\sim$ with the quotient topology, the above gives a bijection
$\{f: X \to Y \text{ continuous s.t. } x \sim x' \Rightarrow f(x) = f(x')\} \overset{1-1}{\longleftrightarrow} \{\bar{f}: X/\sim \to Y \text{ continuous}\}$

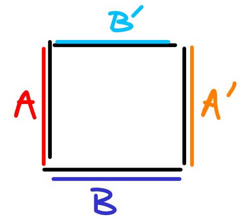
ie.: $\bar{f}: X/\sim \to Y$ is continuous $\Leftrightarrow$ $f = p \circ \bar{f}$ is continuous

Pf: $\Rightarrow$ because $p: X \to X/\sim$ is continuous. So: $\bar{f}$ continuous $\Rightarrow f = \bar{f} \circ p$ continuous.
$\Leftarrow$: let $p: X \to X/\sim$ the quotient map, and recall $\bar{f}([x]) = f(x)$
$x \mapsto [x]$ (indep. of $x \in [x]$).
So $\bar{f} \circ p = f$. Hence: $\forall U \subset Y$ open, $f^{-1}(U) = p^{-1}(\bar{f}^{-1}(U)) \subset X$ is open.
By definition of the quotient topology, we conclude that $\bar{f}^{-1}(U) \subset X/\sim$ is open. □
($V \subset X/\sim$ open $\Leftrightarrow p^{-1}(V) \subset X$ open).

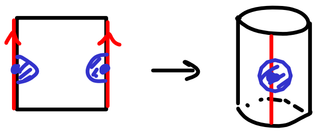
Ex: $X = \mathbb{R}^{n+1} - \{0\}$, define an equivalence relation $x \sim y$ iff x, y lie on the same line through the origin, ie. $x = \alpha y$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. This is an equivalence relation. The quotient space is projective n-space, $\mathbb{RP}^n = X/\sim$ with quotient topology. ("space of lines through 0 in $\mathbb{R}^{n+1}$")

If Y is another top space, then a continuous map $\bar{f}: \mathbb{RP}^n \to Y$ is the same thing as a continuous map $f: \mathbb{R}^{n+1}-\{0\} \to Y$ st. $f(\alpha x) = f(x)$ $\forall \alpha \in \mathbb{R}-\{0\}$, $\forall x \in X$.
(more about $\mathbb{RP}^n$ on HW4)

__Ex:__ Various quotients of the unit square $X = [0,1]^2$: let the edges be $A = \{0\} \times [0,1]$, A', B, B'

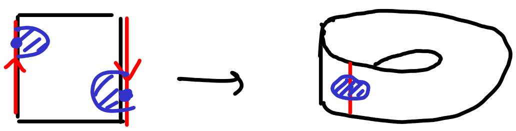


1) gluing A to A' by $(0,t) \sim (1,t)$, get a cylinder

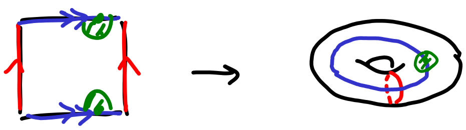


A neighborhood of a point on the gluing line corresponds to two neighborhoods of $(0,t) \in A$ and $(1,t) \in A'$ in X.

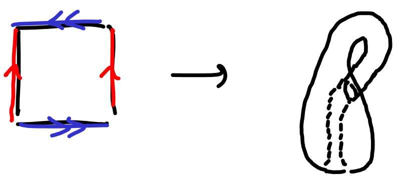
2) if instead we glue A to A' by $(0,t) \sim (1,1-t)$, we get a __Möbius band__ !



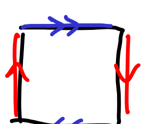
3) gluing A to A' via $(0,t) \sim (1,t)$ __and__ B to B' by $(s,0) \sim (s,1)$ gives us the torus



4) gluing $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$, however, gives the __Klein bottle__, which cannot be embedded in $\mathbb{R}^3$ (can draw a picture that self-intersects).



5) gluing $(0,t) \sim (1,1-t)$ and $(s,0) \sim (1-s,1)$ is tricky to visualize, but the quotient is actually homeomorphic to $\mathbb{RP}^2$.



__Homotopy__ = notion of continuous deformation, parametrized by $I = [0,1]$. (Munkres §51)

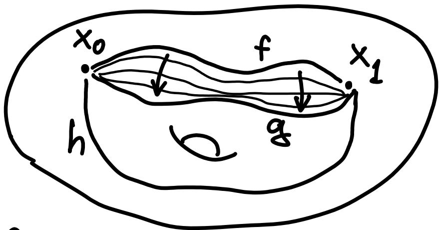
__Def:__ $f, g: X \to Y$ two continuous maps. A __homotopy__ between f and g is a continuous map $H: X \times I \to Y$ st. $H(x,0) = f(x)$, $H(x,1) = g(x)$ $\forall x \in X$.
(the "time" variable in the homotopy)
If this exists, then say f and g are __homotopic__ and write $f \simeq g$.
If f is homotopic to a constant map, we say it is __nullhomotopic__.

We'll want to study __paths__ in top spaces, ie $f: [0,1] \to X$ continuous, $f(0) = x_0$, $f(1) = x_1$.

The above notion is not useful for paths if we don't fix the end points x_0 & x_1 (see HW4).

Better notion: <u>homotopy of paths</u> only considers homotopies which keep the end points in place.

(General notion: pairs (X,A) $A \subset X$ subspace, maps of pairs $(X,A) \xrightarrow{f} (Y,B)$: $f(A) \subset B$)



$f \simeq_p g$ homotopic paths

h not homotopic to f & g.

<u>Def:</u> Two paths $f,g: I \to X$ from x_0 to x_1 are <u>(path) homotopic</u> if $\exists$ continuous $H: I \times I \longrightarrow X$

st. $H(s,0) = f(s)$, $H(s,1) = g(s)$ (homotopy)

and $H(0,t) = x_0$, $H(1,t) = x_1$ (fix end points; so $\forall t \in [0,1]$, $f_t = H_{|I \times \{t\}}$ is a path from x_0 to x_1)

Such H is a <u>path homotopy</u>, and we write $f \simeq_p g$.

<u>Lemma:</u> $\simeq$ and $\simeq_p$ are equivalence relations.

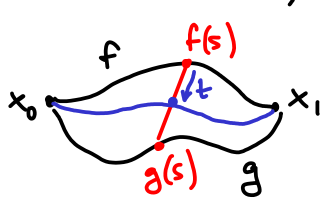
<u>Pf:</u>

- clearly $f \simeq f$ (constant homotopy $H(x,t) = f(x)$).
- if $f \simeq g$ with homotopy $F(x,t)$, then the <u>reverse</u> homotopy $G(x,t) = F(x,1-t)$ gives $g \simeq f$.
- Assume $f \simeq g$ with homotopy $F(x,t)$
  $g \simeq h$ —— " —— $G(x,t)$
  then the <u>concatenation</u> of these is

  $$H: X \times [0,1] \longrightarrow Y \text{ defined by } H(x,t) = \begin{cases} F(x,2t) & \text{if } t \in [0,\frac{1}{2}] \\ G(x,2t-1) & \text{if } t \in [\frac{1}{2},1] \end{cases}$$

  These two formulas agree at $t = \frac{1}{2}$ $(F(x,1) = g(x) = G(x,0))$ so H is well-defined and continuous (cf "pasting lemma" Thm 18.3), and gives a homotopy $f \simeq h$.
- In the case of path homotopies, can check the above constructions preserve the requirements $F(0,t) = x_0$ & $F(1,t) = x_1$, so yield path homotopies. $\square$

<u>Ex:</u> 1) If f,g are paths in $\mathbb{R}^n$ (or any convex subset of $\mathbb{R}^n$) from x_0 to x_1, we can define the <u>straight-line homotopy</u> $F(s,t) = (1-t)f(s) + t g(s)$



For each s, this connects $f(s)$ to $g(s)$ by a straight line segment. We conclude: $f \simeq_p g$ always!

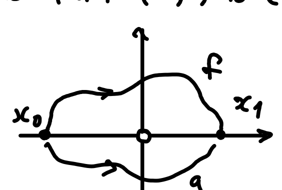
2) in the punctured plane $X = \mathbb{R}^2 - \{(0,0)\}$, let f,g be paths from $(-1,0)$ to $(1,0)$

st. f stays in the upper half plane $\{(x,y) \mid y \geq 0\}$

g —— " —— lower —— " —— $y \leq 0$

Then there is no homotopy between f & g in X.

(we'll prove this rigorously later).



Def: Spaces X, Y are <u>homotopy equivalent</u> if $\exists\ f: X \to Y$, $g: Y \to X$ st. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

(Check: this is an equiv^{ce} relation)

homotopic (vs. exact inverses would be homeo).

Def: X is <u>contractible</u> if X is homotopy eq. to $\{point\}$.

<u>Ex</u>: • $\mathbb{R}^n$ (or a convex subset of $\mathbb{R}^n$) is contractible: ie. $\{0\} \underset{r: x \mapsto 0}{\overset{i}{\rightleftarrows}} \mathbb{R}^n$

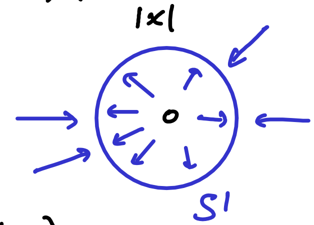
check $i \circ r$ = zero map is homotopic to $id_{\mathbb{R}^n}$ by $H(x,t) = tx$.

<u>Ex</u>: • $\mathbb{R}^2 - \{0\}$ is not contractible, but homotopy eq. to S^1, via $S^1 \underset{r: x \mapsto \frac{x}{|x|}}{\overset{i}{\rightleftarrows}} \mathbb{R}^2 - \{0\}$.

r is a <u>deformation retraction</u> of $X = \mathbb{R}^2 - \{0\}$ onto its subset $A = S^1 \subset X$, ie.
$$\begin{cases} \bullet\ r: X \to A \\ \bullet\ r_{|A} = id_A \quad (\text{ie. } r \circ i = id_A) \\ \bullet\ i \circ r: X \to A \subset X \text{ is } \simeq id_X. \end{cases}$$

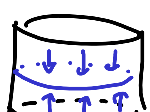
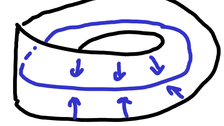
(this is called a retraction; less useful!)

(the "deformation")



(in this case, $i \circ r(x) = \frac{x}{|x|}$ homotopic to id by straight line homotopy)

Deformation retraction is a useful special case of homotopy equiv^{ce}

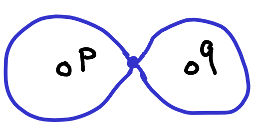
By the same argument the cylinder $S^1 \times I$  & Möbius band 

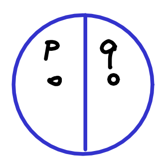
deformation retract onto "middle" S^1 by sliding points of $[0,1]$ to midpoint.

(<u>check</u>: this is consistent with the twisted gluing of $I \times I$, $(0,y) \sim (1, 1-y)$).

hence they are homotopy equivalent to S^1 (and to each other and to $\mathbb{R}^2 - \{0\}$).

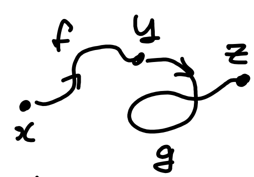
<u>Ex</u>: $\mathbb{R}^2 - \{p,q\}$ deformation retracts onto wedge of two S^1's ("figure 8" space)



(or also on "theta graph" (htpy eq. to ∞, not homeo!) )

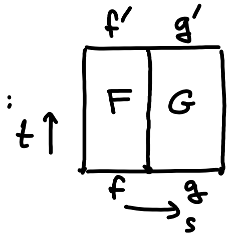
• We now focus on paths and path-homotopy as a way to define an algebraic inv^t of top. spaces (up to homotopy equiv^{ce}): the <u>fundamental group</u>. A group needs a multiplication?

Def. if f is a path from x to y and g is a path from y to z, define a path $f * g$ from x to z by running through first f then g (twice as fast):
$$(f * g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, \tfrac{1}{2}] \\ g(2s-1) & \text{if } s \in [\tfrac{1}{2}, 1] \end{cases}$$



This product is well-defined on path-homotopy classes, as long as $f(1) = g(0)$:
if $f \simeq_p f'$ and $g \simeq_p g'$ then $f * g \simeq_p f' * g'$

using homotopy $(F*G)(s,t) = \begin{cases} F(2s,t) & s \le \frac{1}{2} \\ G(2s-1,t) & s \ge \frac{1}{2} \end{cases}$. Picture:



So we define $[f]*[g] = [f*g]$

Claim: this operation is associative, and has identity & inverses.

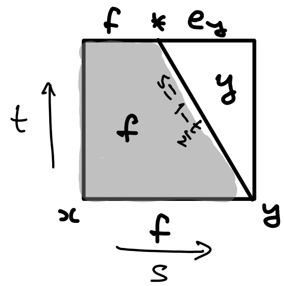
→ the "fundamental groupoid" of X: category with objects = points of X and $\mathrm{Mor}(x,y) = \{$path homotopy classes of paths $x \to y\}$.

$\begin{cases} \text{category: composition is associative} + \exists \text{ identity morphisms } x \to x \\ \text{groupoid: all morphisms have inverses.} \end{cases}$

★ **Identity:** given $x \in X$, consider the constant path $e_x : I \to X$, $e_x(s) = x\ \forall s$, & let $id_x = [e_x]$.

We claim that if f is any path from x to y, then $[f]*id_y = id_x*[f] = [f]$.

Indeed, there are explicit homotopies $f \simeq_p (f*e_y)$ & similarly, $(e_x*f) \simeq_p f$.



$$F(s,t) = \begin{cases} f\left(\frac{s}{1-t/2}\right) & s \in [0, 1-\frac{t}{2}] \\ y & s \in [1-\frac{t}{2}, 1] \end{cases}$$

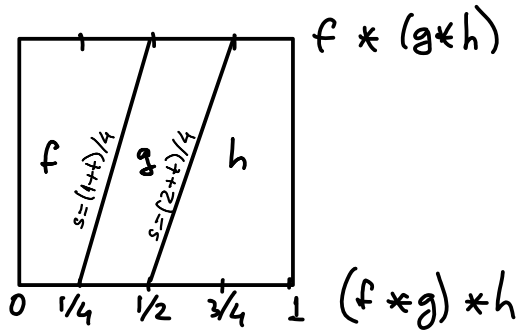
★ **Inverse:** given a path f from x to y, define the reverse path $\bar{f}(s) = f(1-s)$ from y to x. $[\bar{f}]$ is inverse to $[f]$, namely $e_x \simeq_p f*\bar{f}$ and $e_y \simeq_p \bar{f}*f$. Indeed:

$$F(s,t) = \begin{cases} f(2ts) & s \in [0,\frac{1}{2}] \\ f(2t(1-s)) & s \in [\frac{1}{2},1] \end{cases}$$

For given t this runs forward along f from $f(0)=x$ to $f(t)$ at $s=\frac{1}{2}$ then backwards to $f(0)=x$ at $s=1$. For $t=0$ get e_x, $t=1$ get $f*\bar{f}$.

(Similarly for $e_y \simeq_p \bar{f}*f$).

★ **Associativity:** given paths f, g, h with $f(1)=g(0)$ and $g(1)=h(0)$, claim $(f*g)*h \simeq_p f*(g*h)$. Both run along f then g then h, but with different parametrizations. The homotopy comes from adjusting for this:



$$\text{Let } F(s,t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & s \in [0, \frac{1+t}{4}] \\ g(4s-(1+t)) & s \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h\left(\frac{4s-(2+t)}{2-t}\right) & s \in [\frac{2+t}{4}, 1]. \end{cases}$$