# Math 55b Homework 4

Due Wednesday February 23, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

**Material covered:** Urysohn metrization theorem (§33-34); quotient topology (§22); paths and homotopy (§51).

**1.*** (postponed from HW1...) In this problem, we'll see how to construct the *completion* of a metric space; that is, given a metric space $X$, we'll construct a complete metric space $X^*$ (i.e., every Cauchy sequence in $X^*$ has a limit) in which $X$ sits as a dense subset.

To start, let $(X, d)$ be any metric space, and let $\mathcal{C}(X)$ denote the set of all Cauchy sequences $\{p_n\} = p_1, p_2, p_3 \dots$ in $X$. We define an equivalence relation $\sim$ on $\mathcal{C}(X)$ by

$$\{p_n\} \sim \{q_n\} \quad \text{iff} \quad d(p_n, q_n) \to 0, \quad \text{i.e.:} \quad \forall \epsilon > 0,\ \exists N : \forall n \geq N,\ d(p_n, q_n) < \epsilon.$$

We then define the set $X^*$ to be the quotient $\mathcal{C}(X)/{\sim}$, that is, a point $P \in X^*$ is an equivalence class of Cauchy sequences in $X$. Finally, we define a distance function $D$ on $X^*$ by

$$D(\{p_n\}, \{q_n\}) \;=\; \lim_{n\to\infty} d(p_n, q_n)$$

We will take for granted the fact that $\mathbb{R}$ (with its usual distance) is complete.

(a) Show that $\sim$ is indeed an equivalence relation on $\mathcal{C}(X)$.

(b) Show that $D$ is well defined and gives a metric on $X^*$.

(Hint: you need to check three things: (1) given two Cauchy sequences in $X$, the limit in the definition of $D(\{p_n\}, \{q_n\})$ exists; (2) this quantity does not depend on the choice of $\{p_n\}$ in its equivalence class; (3) $D$ satisfies the axioms of a metric).

(c) Show that the metric space $(X^*, D)$ is complete.

(Hint: given a Cauchy sequence $P_1, P_2, \dots$ in $(X^*, D)$, the first step in showing that it converges to some limit $Q \in X^*$ is to construct $Q$. First choose a Cauchy sequence $p_{n,1}, p_{n,2}, \dots$ in $X$ to represent each $P_n$, then construct a new sequence $q_1, q_2, \dots$ in $X$ by choosing each $q_n = p_{n,k_n}$ for $k_n$ sufficiently large, so all subsequent terms of the sequence $\{p_{n,k}\}$ are within distance $1/n$ of $q_n$. Use the triangle inequality to show that $\{q_n\}$ is a Cauchy sequence, defining a point $Q \in X^*$, and finally show that $P_n$ converges to $Q$ in $(X^*, D)$. [Why can't we just choose $q_n = p_{n,n}$?])

(d) Show that the map $\iota : X \to X^*$ defined by $p \mapsto \{p, p, p, \dots\}$ is injective, and that for any $p, q \in X$ we have $D(\iota(p), \iota(q)) = d(p, q)$ (that is, $\iota$ is an *isometry*).

(e) Finally, show that the image $\iota(X) \subset X^*$ is dense.

Note that applying this construction to the metric space $\mathbb{Q}$ (with the usual distance) gives $\mathbb{R}$.

In fact, this is one of the ways in which $\mathbb{R}$ can be constructed! The real number field $\mathbb{R}$ is *characterized* by its properties (namely, $\mathbb{R}$ is an ordered field with the least upper bound property), but to prove the existence of such a field, one needs to actually construct it. The two standard approaches are via Dedekind cuts, see e.g. Rudin pp. 17-21, or by completion of $\mathbb{Q}$, see e.g. `http://www.math.ucsd.edu/~tkemp/140A/Construction.of.R.pdf`

Also note: there is another, slightly more efficient (but less explicit!) way of constructing the completion of a metric space, by embedding $X$ into the space of bounded functions from $X$ to $\mathbb{R}$ with the uniform distance. (See e.g. Munkres Theorem 43.7).

**2.** Munkres exercise 22.2.

**3.** Let $X = \mathbb{R} \times \{1, 2\}$, where $\{1, 2\}$ is equipped with the discrete topology, and consider the equivalence relation given by $(x, 1) \sim (x, 2)$ for all $x \neq 0$ (but $(0, 1) \not\sim (0, 2)$). Show that the quotient topology on $X/\sim$ is not Hausdorff.

**4.*** Let $X = \mathbb{R}^{n+1} - \{0\}$. We define an equivalence relation on $X$ by $x \sim y \Leftrightarrow x = \alpha y$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The quotient $X/\sim$ is called the $n$-dimensional real projective space, $\mathbb{RP}^n$.

(a) By considering the image under the quotient map of the open subset $X_0 = \{(x_1, \dots, x_{n+1}) \in X \mid x_{n+1} \neq 0\}$, show that $\mathbb{RP}^n$ contains an open subset $U_0$ which is homeomorphic to $\mathbb{R}^n$ and whose complement is homeomorphic to $\mathbb{RP}^{n-1}$.

(b) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient space $S^n/\sim$, where $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$ and $a \sim b \Leftrightarrow a = \pm b$ (i.e., we identify *antipodal* points on the sphere).

(c) Show that the quotient map $p : S^n \to \mathbb{RP}^n$ is a two-to-one *covering map*, i.e. that every point of $\mathbb{RP}^n$ has a neighborhood $U$ such that $p^{-1}(U)$ is the disjoint union of two open subsets $U_1, U_2 \subset S^n$, such that the restriction of $p$ to $U_i \to U$ is a homeomorphism for each $i = 1, 2$.

(d) Show that $\mathbb{RP}^1$ is homeomorphic to $S^1$. (Note: the analogue for $n \geq 2$ is false).

**5.** Let $X$ be a topological space, and consider the equivalence relation on $X$ defined by $x \sim y$ if there exists a path in $X$ from $x$ to $y$. The equivalence classes are called *path components* of $X$. Define $\pi_0(X)$ to be the set of path components of $X$.

(a) If $f : A \to Y$ is continuous and $A$ is path-connected, show that $f(A)$ is path-connected and thus contained in a single path component of $Y$.

(b) Show that if $f : X \to Y$ is a continuous function, there is an induced map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$.

(c) Show that $\pi_0$ is a functor from the category of topological spaces (with continuous functions) to the category of sets; i.e. verify that (i) for the identity map $id_X : X \to X$, $\pi_0(id_X) = id_{\pi_0(X)}$, and (ii) given $f : X \to Y$ and $g : Y \to Z$, $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.

**6.** Munkres exercise 51.1.

**7.** Munkres exercise 51.2.

**8.** Munkres exercise 51.3.

**9.** (optional, extra credit)[1]

(a) Show that the collection $\mathcal{B} = \{[a,b) \,|\, a < b,\ a,b \in \mathbb{Q}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$ which is strictly finer than the standard topology and strictly coarser than the lower limit topology.

(b) Show that $\mathcal{T}$ is regular (T3) and second-countable, hence metrizable by Urysohn's theorem.

(c)* Show that $(\mathbb{R}, \mathcal{T})$ is homeomorphic to a subspace of $\mathbb{R}$ with its usual topology (this shows more directly that $\mathcal{T}$ is metrizable).

(Hint: given a strictly increasing function $F : \mathbb{R} \to \mathbb{R}$, what kind of continuity properties should $F$ have at the various points of $\mathbb{R}$ in order for $F$ to give a homeomorphism from $(\mathbb{R}, \mathcal{T})$ to $F(\mathbb{R}) \subset \mathbb{R}$ with the usual metric topology? and how would you construct such an $F$?)

**10.** How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?

[1]The idea for this problem arose from a question asked by AJ La Motta, though this topology also appears in Munkres exercise 13.8(b).