Def: Spaces X, Y are <u>homotopy equivalent</u> if $\exists\, f: X \to Y$, $g: Y \to X$ st. $f \circ g \simeq id_Y : Y \to Y$ and $g \circ f \simeq id_X : X \to X$

(Check: this is an equiv^ce^ relation)

homotopic (ie. $\exists\, H: X \times I \to X$ continuous, $H(x,0) = g\circ f(x)$, $H(x,1) = x$)

- Many of the simplest examples arise from <u>deformation retractions</u>:

Def:
- given $A \subset X$ a <u>retraction</u> is a map $r: X \to A$ st. $r_{|A} = id_A$ or equivalently, denoting by $i: A \hookrightarrow X$ the inclusion, $r \circ i = id_A$.
- Say $r: X \to A$ is a <u>deformation retraction</u> if additionally, $i \circ r : X \to A \hookrightarrow X$ is homotopic to id_X. (sometimes impose further: by a homotopy that fixes A.)

It then follows that $A \underset{r}{\overset{i}{\rightleftarrows}} X$ is a homotopy equivalence.

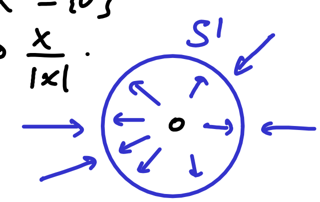
Ex: • $\mathbb{R}^n$ (or a convex subset of $\mathbb{R}^n$) deformation retracts onto $\{0\}$: $\{0\} \underset{r: x \mapsto 0}{\overset{i}{\rightleftarrows}} \mathbb{R}^n$

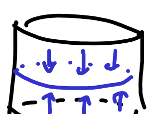
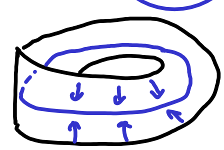
check $i \circ r =$ zero map is homotopic to $id_{\mathbb{R}^n}$ by $H(x,t) = tx$.

(say $\mathbb{R}^n$ is <u>contractible</u>)

Ex: • $\mathbb{R}^2 - \{0\}$ is not contractible, but deform. retracts to S^1 by $S^1 \underset{r: x \mapsto \frac{x}{|x|}}{\overset{i}{\rightleftarrows}} \mathbb{R}^2 - \{0\}$.

(in this case, $i \circ r(x) = \frac{x}{|x|} \simeq id$ by straight line homotopy)



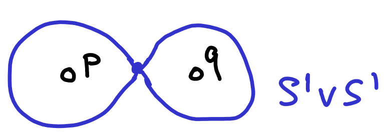
By the same argument the cylinder $S^1 \times I$  & Möbius band 

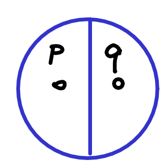
deformation retract onto "middle" S^1 by sliding points of $[0,1]$ to midpoint.

(<u>check</u>: this is consistent with the twisted gluing of $I \times I$, $(0,y) \sim (1, 1-y)$).

hence they are homotopy equivalent to S^1 (and to each other and to $\mathbb{R}^2 - \{0\}$).

Ex. $\mathbb{R}^2 - \{p,q\}$ deformation retracts onto wedge of two S^1's ("figure 8" space)



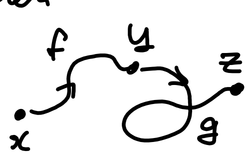
(or also on "theta graph" (htpy eq. to ∞, not homeo!) 

)

- We now focus on paths and path homotopy as a way to define an algebraic inv^t^ of top. spaces (up to homotopy equiv^ce^): the <u>fundamental group</u>. A group needs a multiplication?

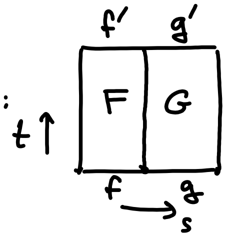
Def. if f is a path from x to y and g is a path from y to z, define a path $f * g$ from x to z by running through first f then g (twice as fast):

$$(f * g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$



This product is well-defined on path-homotopy classes, as long as $f(1)=g(0)$:
if $f \simeq_p f'$ and $g \simeq_p g'$ then $f * g \simeq_p f' * g'$
using homotopy $(F*G)(s,t) = \begin{cases} F(2s,t) & s \le \frac{1}{2} \\ G(2s-1,t) & s \ge \frac{1}{2} \end{cases}$.

Picture:



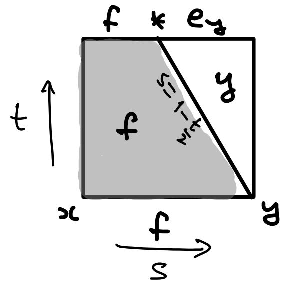
So we define $[f] * [g] = [f * g]$

<u>Claim:</u> this operation is <u>associative</u>, and has <u>identity</u> & <u>inverses</u>.
$\rightsquigarrow$ the <u>"fundamental groupoid"</u> of X: category with objects = points of X
$\mathrm{Mor}(x,y) = \{$path-homotopy classes of paths $x \to y\}$.

$\begin{cases} \text{category: composition is associative} + \exists \text{ identity morphisms } x \to x \\ \text{groupoid: all morphisms have inverses.} \end{cases}$

★ <u>Identity:</u> given $x \in X$, consider the constant path $e_x : I \to X$, $e_x(s) = x \ \forall s$, & let $id_x = [e_x]$.
We claim that if f is any path from x to y, then $[f] * id_y = id_x * [f] = [f]$.

Indeed, there are explicit homotopies
$f \simeq_p (f * e_y)$
& similarly, $(e_x * f) \simeq_p f$.



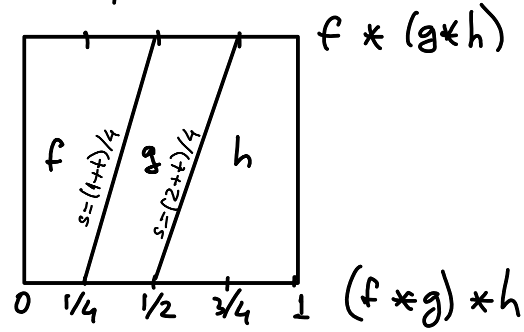
$$F(s,t) = \begin{cases} f\left(\frac{s}{1-t/2}\right) & s \in [0, 1-\frac{t}{2}] \\ y & s \in [1-\frac{t}{2}, 1] \end{cases}$$

★ <u>Inverse:</u> given a path f from x to y, define the reverse path $\bar{f}(s) = f(1-s)$ from y to x.
$[\bar{f}]$ is inverse to $[f]$, namely $e_x \simeq_p f * \bar{f}$ and $e_y \simeq_p \bar{f} * f$. Indeed:

$$F(s,t) = \begin{cases} f(2ts) & s \in [0, \frac{1}{2}] \\ f(2t(1-s)) & s \in [\frac{1}{2}, 1] \end{cases}$$

For given t this runs forward along f from $f(0)=x$ to $f(t)$ at $s=\frac{1}{2}$
then backwards to $f(0)=x$ at $s=1$. For $t=0$ get e_x
$t=1$ $\quad f * \bar{f}$.
(Similarly for $e_y \simeq_p \bar{f} * f$).

★ <u>Associativity:</u> given paths f, g, h with $f(1)=g(0)$ and $g(1)=h(0)$, claim
$(f*g)*h \simeq_p f*(g*h)$. Both run along f then g then h, but with different parametrizations. The homotopy comes from adjusting for this:



Let $$F(s,t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & s \in [0, \frac{1+t}{4}] \\ g(4s-(1+t)) & s \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h\left(\frac{4s-(2+t)}{2-t}\right) & s \in [\frac{2+t}{4}, 1]. \end{cases}$$

<u>Fundamental group</u>: Groups are much easier to study than groupoids! want to be able to multiply always, not worrying whether end points match. Thus we fix a <u>base point</u> $x_0 \in X$ and only consider paths from x_0 to itself – ie. <u>loops</u> (based at x_0).

<u>Def</u>: The set of path homotopy classes of loops based at x_0, with operation $*$ (concatenation), is called the <u>fundamental group</u> of X, denoted $\pi_1(X, x_0)$.

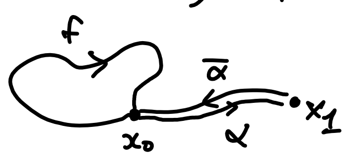
<u>Ex</u>: in $\mathbb{R}^n$ (or a convex domain in $\mathbb{R}^n$), every loop at x_0 is path-homotopic to the identity (ie. the constant path at x_0) by the straight-line homotopy
$$F(t,s) = (1-t)f(s) + t\,x_0$$
So $\pi_1(\mathbb{R}^n, x_0) = \{id\}$.

<u>Def</u>: X is <u>simply-connected</u> if $X \neq \phi$ is path-connected, and for $x_0 \in X$, $\pi_1(X, x_0) = \{1\}$.

<u>Ex</u>: we'll see at some point: $\pi_1(S^1, x_0) \simeq \mathbb{Z}$ ("#turns of a loop around the circle")

$\star$ <u>Dependence on the base point</u>:

If x_0, x_1 are in the same path-component of X, let α be a path from x_0 to x_1. Then for any loop f based at x_0, we get a loop at x_1 by taking $\bar{\alpha} * f * \alpha$,



and so we get a map $\hat{\alpha}: \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$
$$[f] \longmapsto [\bar{\alpha} * f * \alpha] = [\bar{\alpha}] * [f] * [\alpha]$$
(recall: $*$ well-def^d on path-homotopy classes).

<u>Prop</u>: $\hat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a <u>group isomorphism</u>.

<u>Proof</u>.
- if $a, b \in \pi_1(X, x_0)$ then
$$\begin{aligned}\hat{\alpha}(a * b) &= [\bar{\alpha}]^{-1} * (a * b) * [\alpha] \\ &= [\bar{\alpha}] * a * [\alpha] * [\bar{\alpha}] * b * [\alpha] \\ &= \hat{\alpha}(a) * \hat{\alpha}(b).\end{aligned}$$
(using associativity & inverses).

  So $\hat{\alpha}$ is a group homomorphism.
- let $\beta = \bar{\alpha}$ reverse path from x_1 to x_0, then $\hat{\beta}: \pi_1(X, x_1) \to \pi_1(X, x_0)$. We claim $\hat{\beta}$ and $\hat{\alpha}$ are inverses of each other. Indeed: for $a \in \pi_1(X, x_0)$,
$$\begin{aligned}\hat{\beta}(\hat{\alpha}(a)) = \hat{\beta}([\bar{\alpha}] * a * [\alpha]) &= [\bar{\beta}] * [\bar{\alpha}] * a * [\alpha] * [\beta] \\ &= [\alpha] * [\bar{\alpha}] * a * [\alpha] * [\bar{\alpha}] = a.\end{aligned}$$

  Hence $\hat{\beta} \circ \hat{\alpha} = id$ (and similarly $\hat{\alpha} \circ \hat{\beta} = id$ as well), so $\hat{\alpha}$ is an isomorphism. $\square$

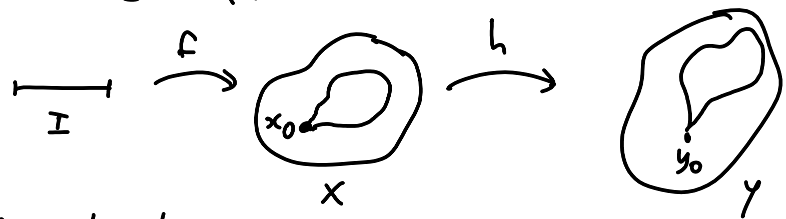
<u>Corollary</u>: if X is path-connected, then $\pi_1(X, x_0)$ is independent of x_0 up to isomorphism.

<u>Rmk</u>: when α is a loop at x_0, we get an <u>automorphism</u> $\hat{\alpha}$ of $\pi_1(X, x_0)$. This is in fact an inner automorphism = conjugation by $[\alpha]$: $a \mapsto [\alpha]^{-1} * a * [\alpha]$.

* **π_1 as a functor:** Consider the category of pointed topological spaces:
  - objects = top. space + choice of base point, (X, x_0)
  - morphisms = continuous maps preserving base points: $f : (X, x_0) \to (Y, y_0)$ means $f : X \to Y$ continuous & s.t. $f(x_0) = y_0$.

Def/Prop: A continuous map $h : (X, x_0) \to (Y, y_0)$ induces a group homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $h_*([f]) = [h \circ f]$.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[rr, bend right, "h \circ f"'] & X \arrow[r, "h"] & Y \end{tikzcd}$$



Check:
- if $f \simeq_p f'$ via F then $h \circ f \simeq_p h \circ f'$ via $h \circ F$. So h_* is well-defined.
- $h \circ (f * g) = (h \circ f) * (h \circ g)$ (composition w/ h compatible with concatenation)

  So h_* is a group homomorphism, $h_*([f] * [g]) = h_*([f]) * h_*([g])$.

Prop: given $(X, x_0) \xrightarrow{h} (Y, y_0) \xrightarrow{k} (Z, z_0)$, $(k \circ h)_* = k_* \circ h_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$.
hence: π_1 is a functor (maps composition $k \circ h$ to composition $k_* \circ h_*$).

(this is just: $(k \circ h) \circ f = k \circ (h \circ f)$).

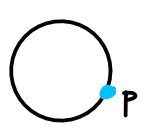
This implies: **Corollary:** if $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism.

But we can do better!

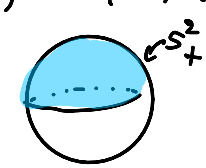
Recall:
- a retraction of X onto a subset $A \overset{i}{\hookrightarrow} X$ is $r : X \to A$ st. $r_{|A} = id_A$, ie. $r \circ i = id_A$. Then, taking a base point $a_0 \in A$,

  $\pi_1(A, a_0) \underset{r_*}{\overset{i_*}{\rightleftarrows}} \pi_1(X, a_0)$ $\quad r_* \circ i_* = id \Rightarrow Ker(i_*) = \{1\}$, ie. i_* injective
- a deformation retraction: assume moreover that $i \circ r : X \to X$ is homotopic to id_X by a homotopy that fixes A. Then we claim i_*, r_* are inverse isom's. $\pi_1(A, a_0) \simeq \pi_1(X, a_0)$.

Ex:



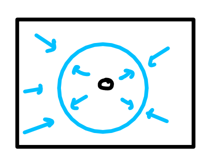
$S^1 \to P$
constant map



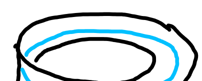
$S^2 \to S^2_+$
$(x, y, z) \mapsto (x, y, |z|)$

retractions, $i \circ r \neq id_X$

deformation retractions



$\mathbb{R}^2 - \{0\} \to S^1$
$x \mapsto x/|x|$



Möbius band $\to S^1$

- More generally, recall a homotopy equivalence is $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ st. $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$.

Thm: Homotopy equivalences induce isomorphisms $\pi_1(X, x_0) \underset{f_*}{\overset{\sim}{\longrightarrow}} \pi_1(Y, f(x_0))$

This follows from the fact that homotopic maps induce the same homomorphisms on π_1:

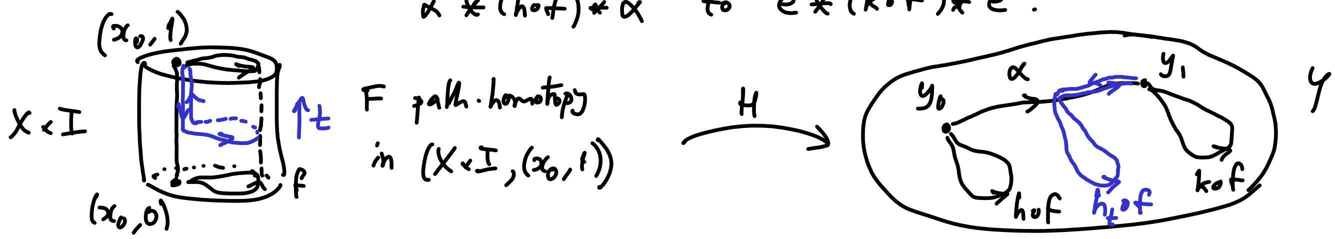
Prop: (1) Let $h, k : X \to Y$ homotopic via a homotopy $H : X \times I \to Y$ s.t. $H(x_0, t) = y_0 \ \forall t$. Then $h_* = k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

(2) If the homotopy H doesn't fix base points, let α be the path $y_0 \to y_1$ def^{d} by $\alpha(t) = H(x_0, t) = y_t$. Then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $k_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$ are related by the isom. $\hat{\alpha} : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$: $k_* = \hat{\alpha} \circ h_*$.

Pf: (1) given a loop $f : I \to X$ based at x_0,
$$I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} Y$$
$$(s,t) \longmapsto (f(s), t) \longmapsto H(f(s), t)$$
$H \circ (f \times id) : I \times I \to Y$ gives a path homotopy (based at y_0) $h \circ f \simeq_p k \circ f$, hence $h_*([f]) = k_*([f])$.

(2) now consider $I \times I \xrightarrow[F]{} X \times I$ def^{d} by concatenating $\begin{cases} \text{path } (x_0, 1) \to (x_0, t) \\ \text{loop } f \text{ in } X \times \{t\} \\ \text{path } (x_0, t) \to (x_0, 1). \end{cases}$

Then $H \circ F$ is a path homotopy in (Y, y_1) from $\alpha^{-1} * (h \circ f) * \alpha$ to $e * (k \circ f) * e$. □



→ **Pf-thm:** if $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1)$ homotopy inverses, $g \circ f \simeq id_X$

$\Rightarrow$ by the prop^{n},
$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "(g \circ f)_*"'] & \pi_1(Y, y_0) \arrow[r, "g_*"] \arrow[rr, bend left, "(f \circ g)_*"] & \pi_1(X, x_1) \arrow[r, "f_*^1"] & \pi_1(Y, y_1) \end{tikzcd}$$
$(g \circ f)_* = \hat{\alpha}$ for some path $\alpha : x_0 \rightsquigarrow x_1$
$\Rightarrow$ this is an isom.

Hence f_* is injective & g_* is surjective.

Similarly, $(f \circ g)_*$ isom. $\pi_1(Y, y_0) \to \pi_1(Y, y_1) \Rightarrow g_*$ injective, f_*^1 surjective.

Hence g_* is an iso, and $f_* = (g_*)^{-1} \circ \hat{\alpha}$ is also an isom. □