556 lecture 13 - Wed. Feb 23 - The fundamental group ①

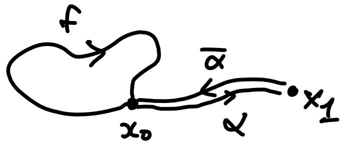
Recall: X top space, fix a base point $x_0 \in X$, consider <u>loops</u> at x_0 ie. paths from x_0 to itself, and the operation of path composition $f * g(s) = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{cases}$

Def.: The set of path homotopy classes of loops based at x_0, with operation $*$ (concatenation), is called the <u>fundamental group</u> of X, denoted $\pi_1(X, x_0)$.

Ex: $\mathbb{R}^n$ (& convex subsets of $\mathbb{R}^n$) is <u>simply connected</u>, ie. path-connected + $\pi_1 = \{1\}$

$\star$ <u>Dependence on the base point</u>: let α be a path from x_0 to $x_1 \in X$

Then for any loop f based at x_0, we get a loop at x_1 by taking $\bar{\alpha} * f * \alpha$,



and so we get a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$

$$[f] \longmapsto [\bar{\alpha} * f * \alpha] = [\bar{\alpha}] * [f] * [\alpha]$$

(recall: $*$ well-def^d on path-homotopy classes).

Prop: $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a <u>group isomorphism</u>.

Proof. • if $a, b \in \pi_1(X, x_0)$ then
$$\begin{aligned} \hat{\alpha}(a * b) &= [\bar{\alpha}]^{-1} * (a * b) * [\alpha] \\ &= [\bar{\alpha}] * a * [\alpha] * [\bar{\alpha}] * b * [\alpha] \\ &= \hat{\alpha}(a) * \hat{\alpha}(b). \end{aligned}$$
(using associativity & inverses).

So $\hat{\alpha}$ is a group homomorphism.

• let $\beta = \bar{\alpha}$ reverse path from x_1 to x_0, then $\hat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_0)$.

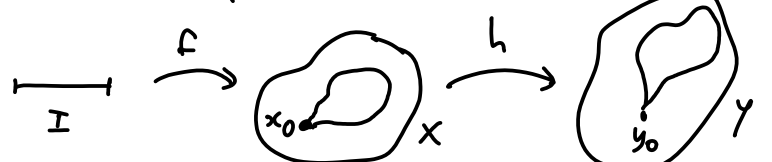
We claim $\hat{\beta}$ and $\hat{\alpha}$ are inverses of each other. Indeed: for $a \in \pi_1(X, x_0)$,
$$\begin{aligned} \hat{\beta}(\hat{\alpha}(a)) = \hat{\beta}([\bar{\alpha}] * a * [\alpha]) &= [\bar{\beta}] * [\bar{\alpha}] * a * [\alpha] * [\beta] \\ &= [\alpha] * [\bar{\alpha}] * a * [\alpha] * [\bar{\alpha}] = a. \end{aligned}$$

Hence $\hat{\beta} \circ \hat{\alpha} = id$ (and similarly $\hat{\alpha} \circ \hat{\beta} = id$ as well), so $\hat{\alpha}$ is an isomorphism. $\square$

Corollary: if X is path-connected, then $\pi_1(X, x_0)$ is independent of x_0 up to isomorphism.

Def/Prop: A continuous map $h : (X, x_0) \to (Y, y_0)$ induces a <u>group homomorphism</u> $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $h_*([f]) = [h \circ f]$.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[rr, bend right, "h \circ f"'] & X \arrow[r, "h"] & Y \end{tikzcd}$$



Check: • if $f \simeq_p f'$ via F then $h \circ f \simeq_p h \circ f'$ via $h \circ F$. So h_* is well-defined.

• $h \circ (f * g) = (h \circ f) * (h \circ g)$ (composition w/ h compatible with concatenation)

So h_* is a group homomorphism, $h_*([f] * [g]) = h_*([f]) * h_*([g])$.

⇝ π_1 as a functor: Consider the category of pointed topological spaces:

- objects = top. space + choice of base point, (X, x_0)
- morphisms = continuous maps preserving base points: $f: (X,x_0) \to (Y,y_0)$ means $f: X \to Y$ continuous & s.t. $f(x_0) = y_0$.

A functor from this category to that of groups is an assignment of a group to each pointed top space, and a group homomorphism to each map of pointed spaces, compatible with composition.

Prop: given $(X,x_0) \xrightarrow{h} (Y,y_0) \xrightarrow{k} (Z,z_0)$, $(k \circ h)_* = k_* \circ h_* : \pi_1(X,x_0) \to \pi_1(Z,z_0)$.
hence: π_1 is a functor (maps composition $k \circ h$ to composition $k_* \circ h_*$).

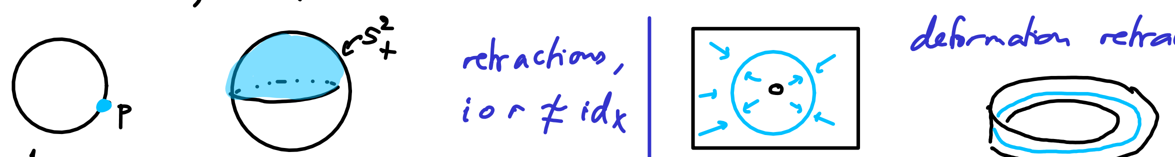
(this is just: $(k \circ h) \circ f = k \circ (h \circ f)$).

This implies: **Corollary:** if $h: (X,x_0) \to (Y,y_0)$ is a homeomorphism, then h_* is an isomorphism.

But we can do better!

Recall:
- a retraction of X onto a subset $A \overset{i}{\hookrightarrow} X$ is $r: X \to A$ s.t. $r_{|A} = id_A$, ie. $r \circ i = id_A$. Then, taking a base point $a_0 \in A$,
$$\pi_1(A, a_0) \underset{r_*}{\overset{i_*}{\rightleftarrows}} \pi_1(X, a_0) \qquad r_* \circ i_* = id \Rightarrow \mathrm{Ker}(i_*) = \{1\}, \text{ ie. } i_* \text{ injective}$$
- a deformation retraction: assume moreover that $i \circ r: X \to X$ is homotopic to id_X by a homotopy that fixes A. Then we claim i_*, r_* are inverse isom's. $\pi_1(A, a_0) \simeq \pi_1(X, a_0)$.

Ex:



retractions, $i \circ r \neq id_X$

deformation retractions

$S^1 \to P$ constant map; $S^2 \to S^2_+$, $(x,y,z) \mapsto (x,y,|z|)$; $\mathbb{R}^2 - \{0\} \to S^1$, $x \mapsto x/|x|$; Möbius band $\to S^1$

- More generally, recall a homotopy equivalence is $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$.

Thm: Homotopy equivalences induce isomorphisms $\pi_1(X, x_0) \underset{f_*}{\xrightarrow{\sim}} \pi_1(Y, f(x_0))$

This follows from the fact that homotopic maps induce the same homomorphisms on π_1:

Prop:
(1) Let $h, k: X \to Y$ homotopic via a homotopy $H: X \times I \to Y$ st. $H(x_0, t) = y_0$ $\forall t$. Then $h_* = k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

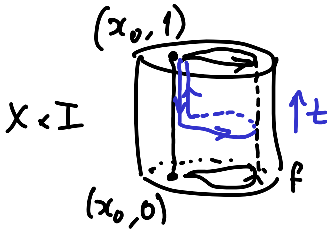
(2) If the homotopy H doesn't fix base points, let α be the path $y_0 \to y_1$ def^d by $\alpha(t) = H(x_0, t) = y_t$. Then $h_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$
$k_*: \pi_1(X,x_0) \to \pi_1(Y,y_1)$
are related by the isom. $\hat{\alpha}: \pi_1(Y,y_0) \to \pi_1(Y,y_1)$: $k_* = \hat{\alpha} \circ h_*$.

Pf: (1) given a loop $f: I \to X$ based at x_0, $I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} Y$
$(s,t) \mapsto (f(s),t) \mapsto H(f(s),t)$

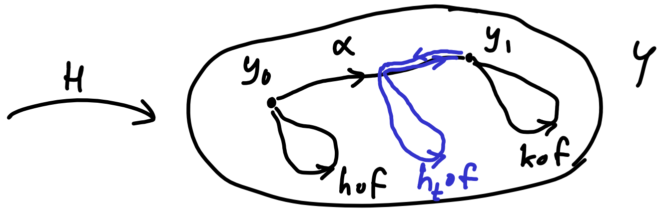
$H \circ (f \times id) : I \times I \to Y$ gives a path homotopy (based at y_0) $h \circ f \simeq_p k \circ f$, hence $h_*([f]) = k_*([f])$.

(2) now consider $I \times I \xrightarrow[F]{} X \times I$ def'd by concatenating $\begin{cases} \text{path } (x_0,1) \to (x_0,t) \\ \text{loop } f \text{ in } X \times \{t\} \\ \text{path } (x_0,t) \to (x_0,1). \end{cases}$

Then $H \circ F$ is a path homotopy in (Y, y_1) from $\alpha^{-1} * (h \circ f) * \alpha$ to $e * (k \circ f) * e$. □



F path-homotopy in $(X \times I, (x_0,1))$



→ **Pf-thm:** if $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1)$ homotopy inverses, $g \circ f \simeq id_X$

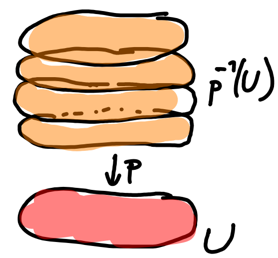
$\Rightarrow$ by the prop^n,
$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "(g \circ f)_*"'] & \pi_1(Y, y_0) \arrow[r, "g_*"] \arrow[rr, bend left, "(f \circ g)_*"] & \pi_1(X, x_1) \arrow[r, "f_*^1"] & \pi_1(Y, y_1) \end{tikzcd}$$

$(g \circ f)_* = \hat{\alpha}$ for some path $\alpha : x_0 \rightsquigarrow x_1$
$\Rightarrow$ this is an isom.

Hence f_* is injective & g_* is surjective.
Similarly, $(f \circ g)_*$ isom. $\pi_1(Y, y_0) \to \pi_1(Y, y_1) \Rightarrow g_*$ injective, f_*^1 surjective.
Hence g_* is an iso, and $f_* = (g_*)^{-1} \circ \hat{\alpha}$ is also an isom. □

At some point we'd like to show $\pi_1(S^1) \cong \mathbb{Z}$. We'll do this by introducing a key tool for the study of π_1: the notion of covering spaces.

Def: Let $p : E \to B$ be a continuous surjective map. We say p evenly covers an open subset $U \subset B$ if $p^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$ where $V_\alpha \subset E$ are disjoint open subsets, and for each $\alpha \in A$, $p_{|V_\alpha} : V_\alpha \to U$ is a homeomorphism. The V_α are called slices.

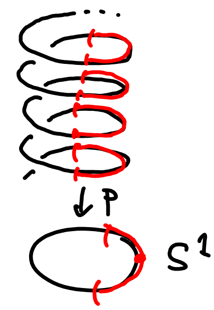


(equivalently: $\exists\ p^{-1}(U) \xrightarrow[\varphi]{\text{homeo} \ \sim} U \times A$ st. $p_{|U} = pr_1 \circ \varphi$).
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p_{|U}"'] & & U \times A \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$
A discrete top. — say diagram of maps "commutes".

Def: If every point of B has a neighborhood which is evenly covered by p, we say E is a covering space of B and p is a covering map. B is called the base of the covering.

Ex: define $p: \mathbb{R} \to S^1$
$p(t) = (\cos t, \sin t)$

This is a covering map! for instance consider $(1,0) \in S^1$ and the neighborhood $U = \{(x,y) \in S^1 \mid x > 0\}$.

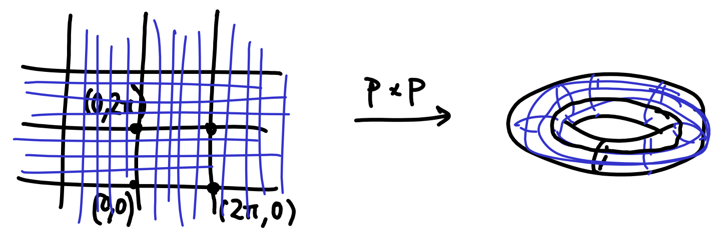


Then $p^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} \left(2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2}\right)$ and p is a homeo. on each slice.

- **Thm:** $p: E \to B$, $q: E' \to B'$ covering maps $\Rightarrow$ $p \times q: E \times E' \to B \times B'$ is a covering map.

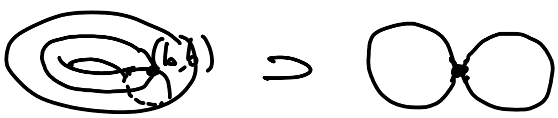
Pf: given $(b,b') \in B \times B'$, let $U \ni b$ and $U' \ni b'$ be neighborhoods st.
$p^{-1}(U) = \bigsqcup V_\alpha$, $q^{-1}(U') = \bigsqcup V'_\beta$ slices, then
$(p \times q)^{-1}(U \times U') = p^{-1}(U) \times q^{-1}(U') = \bigsqcup_{\alpha,\beta} V_\alpha \times V'_\beta$ union of open slices homeo to $U \times U'$. $\square$

Ex: consider the torus $S^1 \times S^1$:
since $\mathbb{R}$ covers S^1, $\mathbb{R}^2$ covers $S^1 \times S^1$



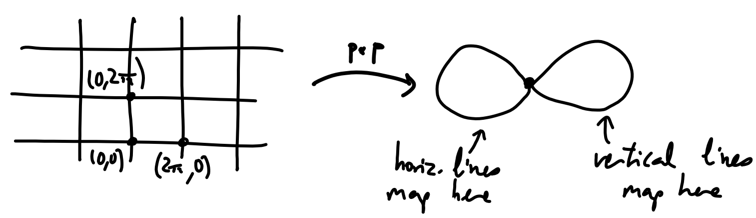
- If $p: E \to B$ is a covering, and $B_0 \subset B$ is a subspace, then by restriction we get a covering $p^{-1}(B_0) \to B_0$.

Ex: $b \in S^1$ base point on the circle, let $B_0 = (b \times S^1) \cup (S^1 \times b) \subset S^1 \times S^1$

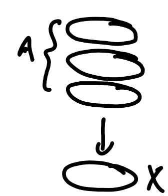


B_0 = "figure eight space" $S^1 \vee S^1$

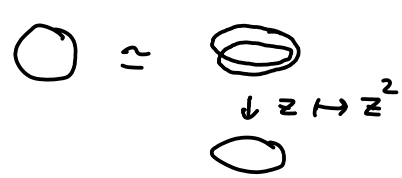
Then we have a covering $(p \times p)^{-1}(B_0) \to B_0$,
$(p \times p)^{-1}(B_0) = (\mathbb{R} \times 2\pi\mathbb{Z}) \cup (2\pi\mathbb{Z} \times \mathbb{R}) \subset \mathbb{R}^2$



- **Ex:** if X any top space, A set w/ discrete topology, then $p_1: X \times A \to X$ is a covering map.
$X \times A \cong \bigsqcup_{\alpha \in A} X \times \{\alpha\}$.



Ex: consider $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, then $p: S^1 \to S^1$, $z \mapsto z^n$ (so: $e^{i\theta} \mapsto e^{in\theta}$) is an n-fold covering.



Lifting:

Def: Given $p: E \to B$ continuous map, a lifting of a continuous map $f: X \to B$ is a map $\tilde{f}: X \to E$ st. $p \circ \tilde{f} = f$.

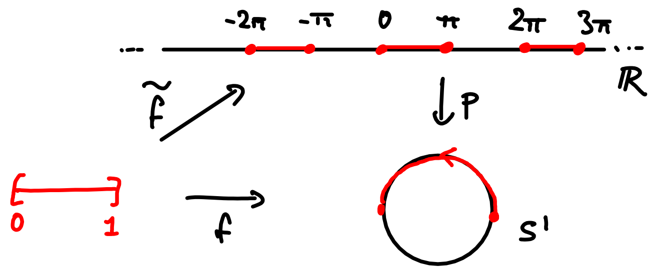
ie.
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$
commutes.

If $p: E \to B$ is a covering map, then we can locally lift, namely if $f(X) \subset U \subset B$ and U is evenly covered, then we can lift f to one of the sheets.

Key point: if $p: E \to B$ covering then paths and path homotopies in B always lift.

Ex: consider $p: \mathbb{R} \to S^1$, $p(x) = (\cos x, \sin x)$ and the path $f(s) = (\cos \pi s, \sin \pi s): I \to S^1$

This has infinitely many possible lifts to paths in $\mathbb{R}$, depending on where 0 gets lifted to.



Theorem: $p: E \to B$ covering map, $f: [0,1] \to B$ a path starting at $f(0) = b$, and $e \in p^{-1}(b)$. Then there exists a unique lift $\tilde{f}: [0,1] \to E$ st. $\tilde{f}(0) = e$.