Math 55b Homework 6

Due Wednesday March 9, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: fundamental group calculations; statement of Seifert-Van Kampen; classification of covering spaces (Munkres §59-60, §79 [74 in some international editions]).

1. Munkres exercise 59.4.

2. Munkres exercise 60.5.

3. Munkres exercise 79.1 [int'l edition: 74.1]

4. Munkres exercise 79.4 [int'l edition: 74.4]

5. Show that there do not exist any covering maps:
(a) from $\mathbb{RP}^2$ to the torus,
(b) from the torus to $\mathbb{RP}^2$,
(c) from $\mathbb{R}^2$ to $\mathbb{RP}^2$.

6. Let $p : E \to B$ and $p' : E' \to B$ be maps of topological spaces. The *fiber product* of E and E' over B, denoted $E \times_B E'$, is the set

$$E \times_B E' \;=\; \{(e, e') \in E \times E' \mid p(e) = p'(e')\},$$

with the subspace topology from the inclusion in $E \times E'$. Note that $E \times_B E'$ also has a natural map q to B, sending (e, e') to $p(e) = p'(e')$.
(a) Show that if $p : E \to B$ and $p' : E' \to B$ are covering maps, then $q : E \times_B E' \to B$ is a covering map as well.
(b) Suppose $B = S^1$, $E = S^1$ and $E' = S^1$, with the maps

$$p = p' : z \;\mapsto\; z^2$$

(here we think of S^1 as $\{z \in \mathbb{C} \,|\, |z| = 1\}$). Describe the covering space $q : E \times_B E' \to B$ in this case.
(c) Same as the preceding question, but now take

$$p : z \;\mapsto\; z^2 \;\text{ and }\; p' : z \;\mapsto\; z^3.$$

7. Let X be the figure eight space, with base point x_0 where the two circles are glued. Let a and b be the two free generators of $\pi_1(X, x_0)$ corresponding to the loops around each circle.

(a) What are the connected degree 2 coverings of X up to equivalence? Draw them! Make sure to label and orient the edges (as in Figure 60.3).

(b) For each covering you drew in (a), give a list of generators for the corresponding subgroup of $\pi_1(X, x_0)$.

(Remark: what does this say about index 2 subgroups of the free group on two generators?)

8. What is the fundamental group of a torus with one point removed? With two points removed? Can you come up with an example of a degree 2 covering map between these two spaces?

9. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?