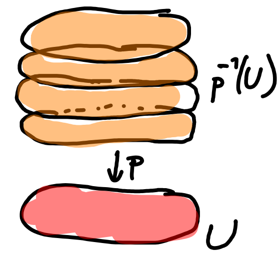
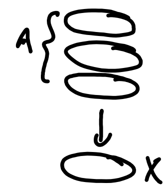
Recall: Let $p: E \to B$ be a continuous surjective map. We say p <u>evenly covers</u> an open subset $U \subset B$ if $p^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$ where $V_\alpha \subset E$ are disjoint open subsets, and for each $\alpha \in A$, $p_{|V_\alpha}: V_\alpha \to U$ is a homeomorphism. The V_α are called <u>slices</u>.



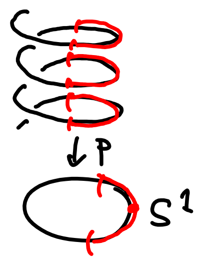
(equivalently: $\exists$
$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\text{homeo}", "\sim"', "\varphi"'] \arrow[dr, "p_{|U}"'] & U \times A \arrow[d, "pr_1"] \\ & U \end{tikzcd}$$
A discrete top. s.t. $p_{|p^{-1}(U)} = pr_1 \circ \varphi$).

Def: If every point of B has a neighborhood which is evenly covered by p, we say E is a <u>covering space</u> of B and p is a <u>covering map</u>. B is called the <u>base</u> of the covering.

- **Ex:** if X any top space, A set w/ discrete topology, then $p_1: X \times A \to X$ is a covering map.



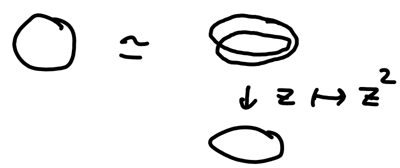
Ex: $p: \mathbb{R} \to S^1$, $p(t) = (\cos t, \sin t)$ is a covering map!



E.g. $U = \{(x,y) \in S^1 \mid x > 0\}$. (or any arc of circle)

Then $p^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} \left(2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2}\right)$ and p is a homeo. on each slice.

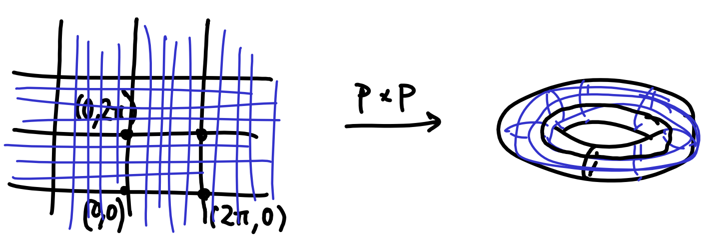
Ex: consider $S^1 = \{z \in \mathbb{C} / |z| = 1\}$, then $p: S^1 \to S^1$, $z \mapsto z^n$ (so: $e^{i\theta} \mapsto e^{in\theta}$) is an n-fold covering.



- **Thm:** $p: E \to B$, $q: E' \to B'$ covering maps $\Rightarrow$ $p \times q: E \times E' \to B \times B'$ is a covering map.

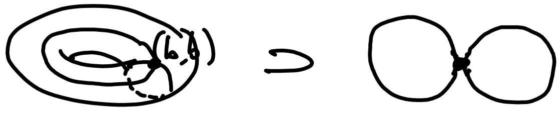
Pf: given $(b, b') \in B \times B'$, let $U \ni b$ and $U' \ni b'$ be neighborhoods st. $p^{-1}(U) = \sqcup V_\alpha$, $q^{-1}(U') = \sqcup V'_\beta$ slices, then
$(p \times q)^{-1}(U \times U') = p^{-1}(U) \times q^{-1}(U') = \bigsqcup_{\alpha, \beta} V_\alpha \times V'_\beta$ union of open slices homeo to $U \times U'$! □

Ex: consider the torus $S^1 \times S^1$: since $\mathbb{R}$ covers S^1, $\mathbb{R}^2$ covers $S^1 \times S^1$

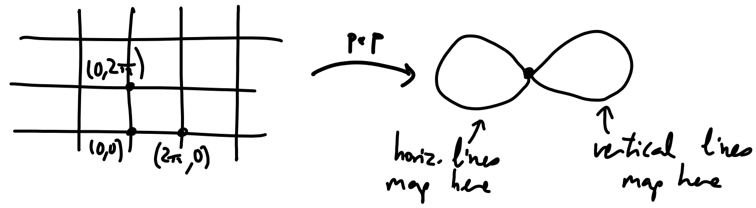


- If $p: E \to B$ is a covering, $B_0 \subset B$ subspace
  $\Rightarrow$ by restriction we get a covering $p^{-1}(B_0) \longrightarrow B_0$.

Ex: $b \in S^1$ base point on the circle, let $B_0 = (b \times S^1) \cup (S^1 \times b) \subset S^1 \times S^1$

$S^1 \times S^1$  B_0 = "figure eight space" $S^1 \vee S^1$

Then we have a covering $(p \times p)^{-1}(B_0) \to B_0$,
$(p \times p)^{-1}(B_0) = (\mathbb{R} \times 2\pi\mathbb{Z}) \cup (2\pi\mathbb{Z} \times \mathbb{R}) \subset \mathbb{R}^2$



Lifting: Def: Given $p: E \to B$ continuous map, a lifting of a continuous map $f: X \to B$ is a map $\tilde{f}: X \to E$ st. $p \circ \tilde{f} = f$.

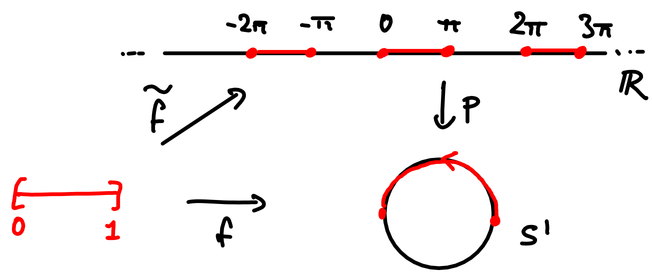
ie.
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$
commutes.

If $p: E \to B$ is a covering map, then we can locally lift, namely if $f(X) \subset U \subset B$ and U is evenly covered, then we can lift f to one of the sheets.

Key point: if $p: E \to B$ covering then paths and path homotopies in B always lift.

Ex: consider $p: \mathbb{R} \to S^1$, $p(x) = (\cos x, \sin x)$ and the path $f(s) = (\cos \pi s, \sin \pi s): I \to S^1$

This has infinitely many possible lifts to paths in $\mathbb{R}$, depending on where 0 gets lifted to.



Theorem: $p: E \to B$ covering map, $f: [0,1] \to B$ a path starting at $f(0) = b$, and $e \in p^{-1}(b)$. Then there exists a unique lift $\tilde{f}: [0,1] \to E$ st. $\tilde{f}(0) = e$.

Pf. cover B by open sets U_α which are evenly covered by p. Then the preimages $f^{-1}(U_\alpha)$ are an open cover of $[0,1]$, which is compact, so $\exists$ Lebesgue number $\delta > 0$ st. $\forall x$, $(x, x+\delta) \subset f^{-1}(U_\alpha)$ for some α. Hence we can find a finite subdivision $0 = s_0 < s_1 < \dots < s_n = 1$ st. each $f([s_i, s_{i+1}])$ lies inside one of the U_α.
Define $\tilde{f}(0) = e$. Assume we have defined $\tilde{f}(s)$ for $s \in [0, s_i]$. Then we define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ as follows. Recall $f([s_i, s_{i+1}]) \subset U$ for some U which is evenly covered by p, $p^{-1}(U) = \coprod$ slices. Let V be the slice which contains $\tilde{f}(s_i)$. The map $p|_V: V \to U$ is a homeomorphism, so has a continuous inverse

& we can define $\tilde{f}(s) = p_{|V}^{-1}(f(s))$ for $s \in [s_i, s_{i+1}]$, which extends $\tilde{f}$ continuously over $[s_i, s_{i+1}]$. Repeating the process, we obtain a continuous lift $\tilde{f}: [0,1] \to E$.

$\tilde{f}$ is unique since for each s_i there was a unique slice containing $\tilde{f}(s_i)$ and a unique way to lift $f_{|[s_i, s_{i+1}]}$ into it. $\square$

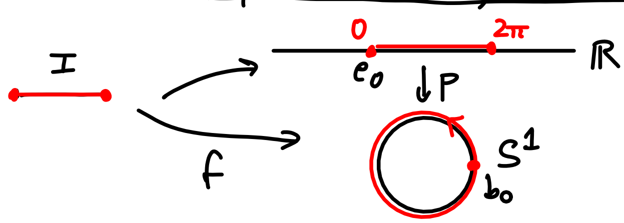
<u>Thm:</u> Let $F: I \times I \to B$ be continuous with $F(0,0) = b$, $p: E \to B$ a covering map, $e \in p^{-1}(b)$, then $\exists$ unique lift $\tilde{F}: I \times I \to E$ st. $\tilde{F}(0,0) = e$.

The proof is exactly the same, subdividing $I \times I$ into squares of side length $< \delta$ which map into open subsets of B that are evenly covered; then constructing the lift $\tilde{F}$ one square at a time.

<u>Observe:</u> if F is a path-homotopy from f to g (in B), then $\tilde{F}$ is a path homotopy (in E) from $\tilde{f}$ to $\tilde{g}$. Indeed, if $F(0,t) = b$ for all t, then $\tilde{F}(0,t) \in p^{-1}(b)$ which is a discrete subset of E (one point in each slice), so we must have $\tilde{F}(0,t) = e$ for all t (always the same point). Similarly for the other end point $\tilde{F}(1,t)$.

On the other hand, <u>loops don't always lift to loops!</u>

<u>Ex:</u>



But since path lifting is unique, given a starting point $e_0 \in p^{-1}(b_0)$, the end point is uniquely determined. This leads to a key notion:

<u>Def:</u> The <u>lifting correspondence</u> $\varphi: \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$ for a covering $\begin{matrix}(E,e_0)\\ \downarrow p \\ (B,b_0)\end{matrix}$ defined by $\varphi([f]) = \tilde{f}(1)$ where $\tilde{f}$ is the lift of f st. $\tilde{f}(0) = e_0$.

<u>Q:</u> <u>Why is φ well-defined?</u> (ie. independent of choice of f in its homotopy class?)

<u>A:</u> if F is a path homotopy $f \simeq_p g$, then its lift $\tilde{F}$ starting at e_0 is a path homotopy between $\tilde{f}$ and $\tilde{g}$, so $\tilde{f}(1) = \tilde{g}(1)$.

<u>Ex:</u> for the covering $p: \mathbb{R} \to S^1$, taking $b_0 = (1,0)$, $e_0 = 0 \in \mathbb{R}$, if f loops around the circle k times (counting ccw) then its lift $\tilde{f}$ ends at $\varphi([f]) = \tilde{f}(1) = 2\pi k$. This gives a map $\pi_1(S^1, (1,0)) \longrightarrow 2\pi\mathbb{Z}$ (surjective).

Now we know, at last, that S^1 isn't simply connected!

Prop: If E is path connected then $\varphi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective.

Pf. let $e \in p^{-1}(b_0)$, $g : I \to E$ a path from e_0 to e, then $f = p \circ g : I \to B$ is a loop at b_0 whose lift starting at e_0 is $\tilde{f} = g$. So $\varphi([f]) = e$. $\square$

Recalling **Prop:** If X is simply connected then any two paths f, g from x_0 to x_1 are path-homotopic

Pf: $f * \bar{g}$ is a loop at x_0, so $f * \bar{g} \simeq_p e_{x_0}$ (X simply connected).

Then $f \simeq_p f * (\bar{g} * g) \simeq_p (f * \bar{g}) * g \simeq_p e_{x_0} * g \simeq_p g$. $\square$.

$\Rightarrow$ **Thm:** If $p : E \to B$ is a covering and E is simply connected, then $\varphi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is a bijection.

Pf: By the above, φ is surjective. If $\varphi([f]) = \varphi([g])$ then $\tilde{f}, \tilde{g}$ are paths in E starting at e_0 and ending at the same point e_1. Since E is simply connected, $\tilde{f} \simeq_p \tilde{g}$. Hence $p \circ \tilde{f} \simeq_p p \circ \tilde{g}$, ie. $f \simeq_p g$, so $[f] = [g]$. So φ is injective. $\square$

Thm: $\pi_1(S^1) \simeq \mathbb{Z}$

Pf. consider the covering map $p : (\mathbb{R}, 0) \to (S^1, (1,0))$, $p(x) = (\cos 2\pi x, \sin 2\pi x)$.

Since $\mathbb{R}$ is simply connected, by the above thm the lifting correspondence

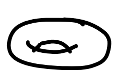
$\varphi : \pi_1(S^1, (1,0)) \to p^{-1}((1,0)) = \mathbb{Z}$ is a bijection.

We just need to show it is a group homomorphism.

Let $[f], [g] \in \pi_1(S^1)$ and let $\varphi([f]) = n$, $\varphi([g]) = m$.

Ie. the lifts $\tilde{f}$ and $\tilde{g}$ starting at 0 end at n and m.

Define a new path $h : I \to \mathbb{R}$ by $h(s) = n + \tilde{g}(s)$; this is the lift of g starting at $n = \tilde{f}(1)$. Then $\tilde{f} * h$ is a well defined path in $\mathbb{R}$, from 0 to $n+m$, and it is the lift of $f * g$ starting at 0. So $\varphi([f * g]) = n + m = \varphi([f]) + \varphi([g])$. $\square$

(Can show similarly: for , $\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$, using covering $p \times p : \mathbb{R}^2 \to S^1 \times S^1$.)

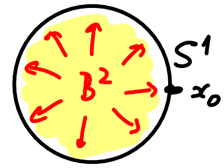
Next we study applications of $\pi_1(S^1) \simeq \mathbb{Z}$:

<u>The Brouwer fixed point theorem:</u>

Let B^n denote the <u>closed</u> ball of radius 1 in $\mathbb{R}^n$, with boundary the unit sphere S^{n-1}.

Recall that, if $A \subset X$, a <u>retraction</u> $r : X \to A$ is a continuous map st. $r(a) = a$ $\forall a \in A$.

Thm: There is no retraction of B^2 onto S^1.



Pf: if $r: B^2 \to S^1$ is a retraction, then $r \circ i = id_{S^1}$, but

$$\begin{array}{ccccc} \pi_1(S^1, x_0) & \xrightarrow[i_*]{} & \pi_1(B^2, x_0) & \xrightarrow[r_*]{} & \pi_1(S^1, x_0) \\ \wr\wr & & \wr\wr & & \\ \mathbb{Z} & & \{1\} & & \end{array}$$

$\{1\}$ (convex $\subset \mathbb{R}^2$, straight line homotopy)

$r_* \circ i_* =$ trivial hom. $\neq id: \mathbb{Z} \to \mathbb{Z}$. Contradiction. □

(More elementary way to say this: given a nontrivial loop f in S^1, $i \circ f$ is nullhomotopic in B^2, via some homotopy H from f to e_{x_0}. Then $r \circ H$ is a path-homotopy $f \rightsquigarrow e_{x_0}$ in S^1, contradiction.).

[with more alg. top., similarly $\nexists$ retraction $B^n \to S^{n-1}$ $\forall n$].

$\Rightarrow$ Brouwer fixed point theorem:

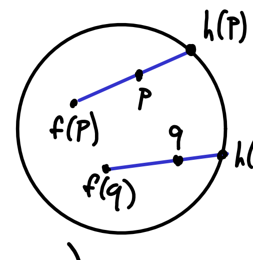
‖ If $f: B^2 \to B^2$ is continuous, then $\exists x \in B^2$ st. $f(x) = x$.

(case $n=1$ follows from intermediate value thm, cf. HW2)

[with more alg. top., the same holds for continuous maps $B^n \to B^n$ $\forall n$.]

Proof: assume $f: B^2 \to B^2$ continuous, $f(x) \neq x$ $\forall x \in B^2$.

Then define $h: B^2 \to S^1$ by mapping each $p \in B^2$ to the point where the ray from $f(p)$ to p hits $\partial B^2 = S^1$.



(formula: $h(p) = p + t(p - f(p))$ where $t > 0$ st. $\|h(p)\|^2 = 1$.
can solve by quadratic formula, so t does depend continuously on p).

This gives a continuous map $h: B^2 \to S^1$, moreover if $p \in S^1$ then $h(p) = p$, so we get a retraction $B^2 \to S^1$. Contradiction. □

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* A loop in (X, x_0) is defined as a map $I \to X$ st. $\{0,1\} \to \{x_0\}$, but since $I/_{0\sim 1}$ is homeo. to S^1, can also think of it as a map $(S^1, p_0) \xrightarrow{f} (X, x_0)$.

So $\pi_1(X, x_0)$ tells us about homotopy classes of maps $(S^1, p_0) \to (X, x_0)$... but also $S^1 \to X$.

Lemma: ‖ Let $h: S^1 \to X$ continuous, then the following are equivalent:

(1) h is nullhomotopic

(2) h extends to a continuous map $k: B^2 \to X$ $(k_{|\partial B^2 = S^1} = h)$.

(3) $h_*: \pi_1(S^1) \to \pi_1(X)$ is the trivial homomorphism.