<u>Last time</u>: • covering maps $p: E \to B$ ($E$ and $B$ path-connected & loc path conn$^{d}$)
induce an <u>injective</u> homomorphism $p_*: \pi_1(E,e_0) \to \pi_1(B,b_0)$, whose image
$H = \mathrm{Im}(p_*) \subset \pi_1(B,b_0)$ (= those homotopy classes of loops in $(B,b_0)$ which lift to loops in $(E,e_0)$ (rather than just paths))
determines the covering up to equivalence (= homeomorphism $\begin{array}{ccc} E & \xrightarrow[\sim]{h} & E' \\ & {\scriptstyle p}\searrow \; \swarrow{\scriptstyle p'} & \\ & B & \end{array}$).
Namely: given 2 coverings $\begin{cases} p: (E,e_0) \to (B,b_0) \\ p': (E',e_0') \to (B,b_0) \end{cases}$ & corresp. subgroups $H = \mathrm{Im}\, p_*$, $H' = \mathrm{Im}\, p'_*$:

- $\exists$ base point preserving equivalence ($h(e_0) = e_0'$) iff $H = H'$
- $\exists$ equivalence (not necess. mapping $e_0 \mapsto e_0'$) iff $H, H'$ are conjugate subgroups of $\pi_1(B,b_0)$.

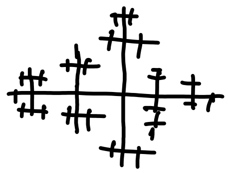
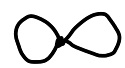
---

<u>Universal covering space</u>:

<u>Def</u>: ‖ If $p_0: E_0 \to B$ covering and $E_0$ is simply connected, say $E_0$ is a <u>universal covering</u> of $B$.

<u>Note</u>: this corresponds to the trivial subgroup $p_{0*}(\pi_1(E_0)) = \{1\} \subset \pi_1(B)$; unique up to equiv$^{ce}$ by the above.

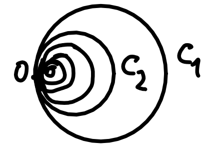
<u>Ex</u>: • $p: \mathbb{R} \to S^1$
$p \times p: \mathbb{R}^2 \to S^1 \times S^1 = $ torus

•  infinite tree $\longrightarrow$ 
(horizontal edges $\to a$, vertical $\to b$).

• <u>Thm</u>: ‖ $p_0: E_0 \to B$ universal covering, $p': E' \to B$ any path-connected covering, then
‖ $\exists$ covering map $q_0: E_0 \to E'$ s.t. $p' \circ q_0 = p_0$; and $q_0$ is univ. covering of $E'$.

$q_0$ is constructed by lifting: $\begin{array}{ccc} & & E' \\ & {\scriptstyle q_0}\nearrow & \downarrow {\scriptstyle p'} \\ E_0 & \xrightarrow[p_0]{} & B \end{array}$ ($\exists$ since $p_{0*}(\pi_1(E_0)) = \{1\} \subset p'_*(\pi_1(E'))$).
& can show it's a covering map as well.

So, in fact, if $B$ has a universal covering, all other coverings can then be obtained as quotients!

• Some spaces have no universal covering!
<u>Ex</u>: "Hawaiian earrings" $= \bigcup_{n \geq 1} C_n$ circles of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$ in $\mathbb{R}^2$ 



Any covering space must evenly over a neighborhood of the origin, which prevents it from being simply connected. (for $n$ suff$^{tly}$ large, loop around $C_n$ lifts to a loop).

• If one avoids such pathological examples - assuming $B$ is (semi) locally simply connected, can build univ. cover as space of pairs $(b, \gamma)$ where $\begin{cases} b \in B \\ \gamma = \text{homotopy class of path } b_0 \to b \end{cases}$

This has a preferred topology for which any simply conn$^{d}$ nbd $U \ni b$ is evenly covered: if $b' \in U$, adding a path $b \rightsquigarrow b'$ inside $U$ or its inverse gives a preferred bijection $\{\text{htpy classes of paths } b_0 \to b\} \longleftrightarrow \{\text{htpy classes of paths } b_0 \to b'\}$ independent of choice of path $b \rightsquigarrow b'$ inside $U$ since $U$ simply connected).

---

<u>Seifert-Van Kampen theorem</u> = given $X = U \cup V$, $U, V, U \cap V \subset X$ open & path connected this describes $\pi_1(X)$ in terms of $\pi_1(U)$ and $\pi_1(V)$. We've already seen a simpler statement: $\pi_1(X)$ is generated by the images of

$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_*"] & \pi_1(X) \\ \pi_1(V) \arrow[ur, "j_*"'] & \end{tikzcd}$$

* To formulate the thm, need to discuss the notion of <u>free product of groups</u>.

given groups $G_1, \dots, G_n$, consider <u>words</u> $(x_1 \in G_{i_1} \dots, x_m \in G_{i_m})$ $(m \geq 0)$ (finite)

Say $(x_1 \dots x_m)$ is a <u>reduced word</u> if (1) no $x_j = e_{G_{i_j}}$ (else delete it from the list)

(2) consecutive elements aren't in same $G_i$, else rewrite $(\dots, x_j, x_{j+1}, \dots)$ to $(\dots, x_j x_{j+1}, \dots)$

Define product $(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) =$ rewrite $(x_1, \dots, x_m, y_1, \dots, y_n)$ to a reduced word.

inverse $(x_1, \dots, x_m)^{-1} = (x_m^{-1}, \dots, x_1^{-1})$. identity = empty word.

<u>Def:</u> The <u>free product</u> $G = G_1 * \dots * G_n$ is the set of reduced words, with this operation.

<u>Ex:</u> $\mathbb{Z} * \mathbb{Z} \simeq \{a^k \mid k \in \mathbb{Z}\} * \{b^l \mid l \in \mathbb{Z}\} = \{\text{red. words } a^{k_1} b^{l_1} a^{k_2} b^{l_2} \dots\}$ start & end w/ either a or b.

More generally, the <u>free group</u> on generators $\{a_i\}$ is defined to be the free product of cyclic groups $G_i = \{a_i^k \mid k \in \mathbb{Z}\} (\simeq \mathbb{Z})$

<u>Observe:</u> $G = G_1 * \dots * G_n$ contains subgroups $\simeq G_1, \dots, G_n$ (words of length 1) (also denoted $G_i$) st. $G_i \cap G_j = \{e\}$, $G_1 \dots G_n$ generate $G$, "no relations among elements of these subgroups".

• <u>Alternative characterization:</u> for any group $H$ and any homomorphisms $h_j : G_j \to H$,

(★) $\exists$ unique homomorphism $h : G \to H$ s.t.
$$\begin{tikzcd} G_j \arrow[r, hook] \arrow[rr, bend right, "h_j"'] & G \arrow[r, "h"] & H \end{tikzcd}$$
commutes $\forall j$.

Indeed: expressing elements of $G$ as reduced words, we must have $h(x_1 \dots x_m) = h_{i_1}(x_1) \dots h_{i_m}(x_m)$. each $x_j \in G_{i_j}$

---

<u>Seifert-Van Kampen</u>: Let $X = U \cup V$, $U$ and $V$ open in $X$, $U \cap V$ path-connected $\ni x_0$.

The inclusions
$$\begin{tikzcd} & U \arrow[dr, hook, "j_1"] & \\ U \cap V \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X \\ & V \arrow[ur, hook, "j_2"'] & \end{tikzcd}$$
induce homomorphisms on $\pi_1$.

By univ. property of free product, $\exists$ unique homomorphism $h$ st.

$$\begin{tikzcd} \pi_1(U,x_0) \arrow[d] \arrow[rr, bend left, "j_{1*}"] & & \\ \pi_1(U,x_0) * \pi_1(V,x_0) \arrow[r, "h"] & \pi_1(X,x_0) \\ \pi_1(V,x_0) \arrow[u] \arrow[ur, bend right, "j_{2*}"'] & \end{tikzcd}$$

commutes.

(define $h$ on words in elements of $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$ using $j_{1*}$ and $j_{2*}$ on each component of the word!)

Thm (Seifert-Van Kampen):

this part is the generation result we saw last week

this is new.

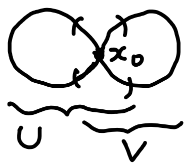
The homomorphism $h$ defined above is surjective, and its kernel $N$ is the smallest normal subgroup of $\pi_1(U,x_0) * \pi_1(V,x_0)$ which contains all elements of the form $i_{1*}(g)^{-1} i_{2*}(g)$ $\forall g \in \pi_1(U \cap V, x_0)$. I.e. $\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0) / N$.

Corollary 1: if $U \cap V$ is simply connected then $\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0)$.

Corollary 2: if $V$ is simply connected then $\pi_1(X,x_0) \cong \pi_1(U,x_0)/N$, where $N$ is the smallest normal subgroup containing the image of $i_{1*}: \pi_1(U \cap V, x_0) \to \pi_1(U,x_0)$.

---

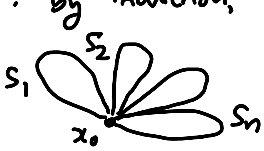
Ex. 1: figure 8:




$\Rightarrow$ $U, V$ deformation retract onto circles
$U \cap V$ contractible

Hence $\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0) \cong \mathbb{Z} * \mathbb{Z}$ free group gen^d by loops around the two circles.

Ex. 2: by induction, wedge of $n$ circles: $X = \bigcup_{i=1}^{n} S_i$, $S_i$ homeo to $S^1$ $\forall i$, $S_i \cap S_j = \{x_0\}$.



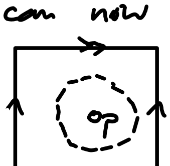

$\Rightarrow \pi_1(X,x_0) =$ free group on $n$ generators $a_i$ = loops generating $\pi_1(S_i,x_0)$.

(Similarly for a finite graph with $n$ loops).

---

Fundamental groups of surfaces can also be calculated using Van Kampen!

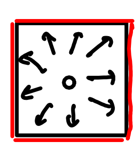
eg. can now calculate $\pi_1$ of torus in a different way (easiest is still: $\mathbb{R}^2 \xrightarrow[p \times p]{\text{univ. cover}} T$ ).




$T \simeq I \times I \big/ \begin{matrix} (x,0) \sim (x,1) & \forall x \\ (0,y) \sim (1,y) & \forall y. \end{matrix}$

let $U = T \setminus \{p\}$
$V$ = open ball of radius $< \frac{1}{2}$ around $p$.

$U$ deformation retracts onto wedge of two circles

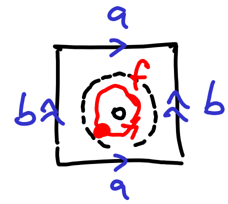


$/\sim$

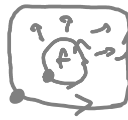
$V$ is simply connected.

$U \cap V \simeq D^2 - pt$ has homotopy type of $S^1$.

Using Corollary 2 above: $\pi_1(T) \cong \pi_1(U)/N$ where $N$ is normal generated by the image of the loop $f$ which generates $\pi_1(U \cap V)$ (and its conjugates)



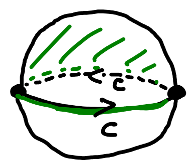
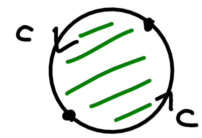

$\pi_1(U)$ is a free group on gen's. $a, b$; and then the image of $[f]$ under the inclusion $U \cap V \hookrightarrow U$ is $aba^{-1}b^{-1}$

[The "obvious" picture  $\to aba^{-1}b^{-1}$ needs to be corrected slightly: base point should be fixed $\in U \cap V$!]

So we set $aba^{-1}b^{-1} = 1$ ie. $ab = (aba^{-1}b^{-1})ba = ba$, get abelian group $\simeq \mathbb{Z}^2$

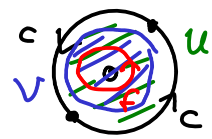
$$\pi_1(T) \cong \langle a, b \mid ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

generators ↗    ↑ relations

- Similarly for $\pi_1(\mathbb{RP}^2)$, using $\mathbb{RP}^2 \simeq S^2/_{x \sim -x}$  $\simeq B^2/\sim$ 


$x \sim -x \quad \forall x \in S^1 = \partial B^2$

Now write $\mathbb{RP}^2 = U \cup V$, $\quad U = \mathbb{RP}^2 - \{p\}$
$V =$ disc centered at $p$



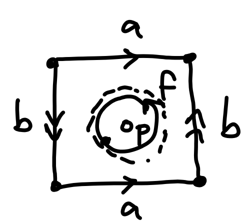

$U$ deformation retracts onto the boundary $S^1/_{x \sim -x} \xrightarrow[z \mapsto z^2]{\sim} S^1$
so $\pi_1(U) \cong \mathbb{Z}$ w/ generator $c$.

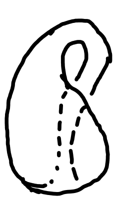
$V$ is simply connected. $\quad U \cap V \simeq D^2 - pt$ has homotopy type of $S^1$

$\pi_1(\mathbb{RP}^2) \cong \pi_1(U)/N$, $\quad N$ normal subgroup generated by image of generator $[f] \in \pi_1(U \cap V)$ under inclusion, which is $c^2$.

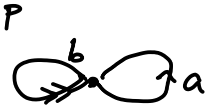
So $\quad \pi_1(\mathbb{RP}^2) = \langle c \mid c^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

- <u>Klein bottle</u>: $\quad$ recall $K = I \times I/\sim \quad (x,0) \sim (x,1)$
$(0,y) \sim (1, 1-y)$



Again write $K = U \cup V$, $\quad U = K - \{p\}$ $\quad \rightsquigarrow \pi_1(K) \cong \pi_1(U)/N$
$V =$ disc centered at $p$

$U$ retracts onto boundary $\cong$ figure 8 space  so


$\pi_1(U) \cong$ free group on generators $a, b$.

$U \cap V$ has homotopy type of $S^1$, and the generator $[f] \in \pi_1(U \cap V) \cong \mathbb{Z}$ maps under inclusion to $aba^{-1}b$

So $\pi_1(K) \cong \langle a, b \mid aba^{-1}b = 1 \rangle \quad$ not abelian: $ab = b^{-1}a$, not $ba$!

ie: $aba^{-1} = b^{-1}$ : b conjugate to its inverse!

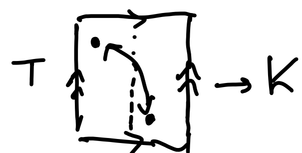
But this contains an index 2 subgroup H gen'd by $a^2$ and $b$, which commute! ( $aba^{-1} = b^{-1}$ $\Rightarrow$ taking inverses, $ab^{-1}a^{-1} = b$, so

$$a^2ba^{-2} = a(aba^{-1})a^{-1} = ab^{-1}a^{-1} = b$$

So $a^2b = ba^2$ ✓ ). ( $\Rightarrow$ subgroup $H \cong \mathbb{Z}^2$ ).

(can show, by rearranging letters via $ab = b^{-1}a$, this contains all words with even # of a's so it is an index 2 subgroup.)

This subgroup corresponds to a deg. 2 covering map by the torus, $T \to K$!




quotient by $(x,y) \sim (x+\frac{1}{2}, 1-y)$

Ie. map $(x,y) \in I \times I/\sim_T$ to $\begin{cases} (2x, y) & \text{if } x \leq 1/2 \\ (2x-1, 1-y) & \text{if } x \geq \frac{1}{2} \end{cases}$ in $I \times I/\sim_K$.

<u>Cool fact</u> that this relates to: if you coat a klein bottle in paint all over, the paint forms a torus.