Differentiation in one variable (Rudin ch 5 = McMullen §5)

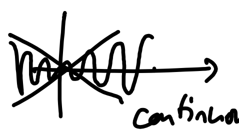
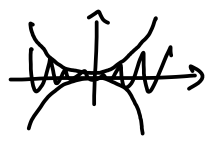
Def: $f : [a,b] \to \mathbb{R}$ is differentiable at x if $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} =: f'(x)$ exists.

(ie. $\forall \varepsilon\ \exists \delta$ st. $0<|t-x|<\delta \Rightarrow \left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$).

- **Prop:** f differentiable at $x \Rightarrow f$ continuous at x. (The converse is false, eg. $|x|$ at 0).

Pf: $f(t)-f(x) = \underbrace{\frac{f(t)-f(x)}{t-x}}_{\to f'(x) \text{ as } t\to x} \cdot \underbrace{(t-x)}_{\to 0 \text{ as } t\to x}$ + multiplication is continuous $\Rightarrow f(t)-f(x) \to f'(x)\,0 = 0$.

- Usual rules of calculation hold: derivatives of $f+g$, fg, …; $(f\circ g)'(x) = f'(g(x)).\,g'(x)$ chain rule.
(see Rudin p 104-105).

- **Ex:**
  - $\begin{cases} f(x) = x \sin\frac{1}{x} & (x\neq 0) \\ f(0) = 0 \end{cases}$  For $x\neq 0$, $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$
  continuous but not differentiable at 0 ($\nexists \lim_{x\to 0} \frac{f(x)}{x}$).
  - $\begin{cases} g(x) = x^2 \sin\frac{1}{x} \\ g(0) = 0 \end{cases} \Rightarrow$  differentiable ($g'(0)=0$) but g' not continuous at 0.
  - $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(n!\,x)$ continuous (series converges uniformly, since $\sum\frac{1}{n^2}$ conv.), nowhere differentiable!
  (See also Rudin 7.18 for a related example).

* **Mean value theorem:** $f : [a,b] \to \mathbb{R}$ differentiable $\Rightarrow \exists c\in(a,b)$ st. $f(b)-f(a) = f'(c).(b-a)$.

Follows logically from easier results:

(1) if $f : [a,b]\to\mathbb{R}$ has a local max (or min) at $x\in(a,b)$ (ie. max of $f_{|(x-\delta,\,x+\delta)}$) and f is differentiable at x, then $f'(x) = 0$.

(because $\frac{f(t)-f(x)}{t-x}$ is ≥ 0 for $t\in(x-\delta,x)$, ≤ 0 $t\in(x,x+\delta)$ $\Rightarrow$ take lim. as $t\to x$ from left and from right.)

(2) if $f : [a,b]\to\mathbb{R}$ is differentiable and $f(a) = f(b)$ then $\exists c\in(a,b)$ st. $f'(c) = 0$.
clear if f is constant; otherwise look at max or min of f over $[a,b]$ & apply (1)

(3) mean val. thm = apply (2) to $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$.

Corollary: mean value inequality: $m \leq f'(x) \leq M$ $\forall x\in(a,b) \Rightarrow m(b-a) \leq f(b)-f(a) \leq M(b-a)$.

* Generalization: **Taylor's theorem:**

$f : [a,b]\to\mathbb{R}$ n times differentiable. The deg.$(n-1)$ Taylor polynomial of f at a is:
$$P(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$ Then $\exists c\in(a,b)$ st. $f(b) = P(b) + \frac{f^{(n)}(c)}{n!}(b-a)^n$.

Pf:
- subtracting $P(x)$ from both sides, we can reduce to the case $f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0$. (and $P=0$).
- let $g(x) = f(x) - f(b)\frac{(x-a)^n}{(b-a)^n} \Rightarrow g(b) = g(a) = 0$ + still have $g'(a) = \ldots = g^{(n-1)}(a) = 0$.

- now: the mean value thm for g; $g(a)=g(b)=0 \Rightarrow \exists x_1 \in (a,b)$ st. $g'(x_1)=0$.
  ———"——— g': $g'(a)=g'(x_1)=0 \Rightarrow \exists x_2 \in (a,x_1)$ st. $g''(x_2)=0$

and so on until $\exists c = x_n \in (a, x_{n-1})$ st. $g^{(n)}(c)=0$. Ie: $f^{(n)}(c) - \frac{n!\, f(b)}{(b-a)^n} = 0$. □

Rmk: • can compare $f(x)$ to $P(x)$ by applying thm. to $[a,x]$ instead!
• as with mean value inequality: a bound $|f^{(n)}| \leq M$ gives a bound $|f(x)-P(x)| \leq \frac{M(x-a)^n}{n!}$ over $[a,b]$.

Rmk: there exist nonzero functions whose Taylor polynomials are all zero!
eg. $f(x) = \exp(-\frac{1}{x^2})$, $f(0)=0$: $f \in C^\infty$ (all derivatives exist), $f^{(k)}(0)=0$ $\forall k$
so the Taylor polynomials are all zero! The Taylor series of f at 0 converges but $\neq f$! (in other examples, it can also have $R=0$ ie. never converges for $x \neq a$).
Most C^∞ functions aren't analytic, ie. can't be written as power series.

Let $C^k([a,b],\mathbb{R}) = \{k\text{-times differentiable functions, } f^{(k)} \text{ continuous}\}$, with $\|f\|_{C^k} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty$.

Thm: $f_n \in C^1$, $f_n \to f$ pointwise, $f_n' \to g$ uniformly $\Rightarrow f \in C^1$ and $f' = g$ (& $f_n \to f$ uniformly)

Pf: * Fix $x \neq y \in [a,b]$, mean value theorem $\Rightarrow (*)$ $\frac{f_n(y)-f_n(x)}{y-x} = f_n'(c_n)$ for some $c_n \in [x,y]$ or $[y,x]$.
The left hand side $\to \frac{f(y)-f(x)}{y-x}$ as $n \to \infty$.
For the right hand side: (c_n) has a subsequence (c_{n_k}) converging to some $c \in [x,y]$.
Since f_n' is continuous, the uniform limit g is continuous; we claim $f_{n_k}'(c_{n_k}) \to g(c)$.
Indeed: fix $\varepsilon > 0$, let δ st. $|t-c| < \delta \Rightarrow |g(t)-g(c)| < \frac{\varepsilon}{2}$, and let N st.
$n \geq N \Rightarrow \sup |f_n' - g| < \frac{\varepsilon}{2}$ and $n_k \geq N \Rightarrow |c_{n_k} - c| < \delta$. Then for $n_k \geq N$,
$$|f_{n_k}'(c_{n_k}) - g(c)| \leq |f_{n_k}'(c_{n_k}) - g(c_{n_k})| + |g(c_{n_k}) - g(c)| < \varepsilon.$$
Hence: returning to $(*)$ and taking limit as $n \to \infty$: $\exists c \in [x,y]$ st. $\frac{f(y)-f(x)}{y-x} = g(c)$.
We now take the limit as $y \to x$: the rhs. $\to g(x)$ using continuity of g and the fact that $|c-x| \leq |y-x|$ (check this!). Hence f is differentiable at x and $f'(x) = g(x)$. (+ since g is continuous, $f \in C^1$).

* Finally: mean value ineq. $\Rightarrow$ $|f_n(x) - f(x)| \leq \underbrace{|f_n(a) - f(a)|}_{\to 0} + \underbrace{|x-a|}_{\leq (b-a)} \underbrace{\sup |f_n' - f'|}_{\to 0 \text{ since } f_n' \to g \text{ uniformly}}$
gives a uniform bound so $\sup |f_n - f| \to 0$. □

Corollary: $C^k([a,b],\mathbb{R})$ is a complete metric space

Pf: Using completeness of C^0 (uniform top), (f_n) Cauchy in $C^1 \Rightarrow f_n, f_n'$ Cauchy in $C^0 \Rightarrow$
$\exists$ uniform limits $f, g \in C^0 \overset{\text{thm}}{\Rightarrow} f \in C^1$ and $f' = g$. Now $\begin{Bmatrix} f_n \to f \\ f_n' \to f' \end{Bmatrix}$ uniformly $\Rightarrow f_n \to f$ in C^1.
This proves the case $k=1$. Repeat same argument for successive derivatives for $k > 1$. □

③

<u>Corollary</u>: $f(x) = \sum a_n x^n$ power series with radius of convergence $= R$
$\Rightarrow f(x)$ is C^∞ over $(-R,R)$, and $f'(x) = \sum n a_n x^{n-1}$.

<u>Pf</u>: $f = \sum a_n x^n$ and $g = \sum n a_n x^{n-1}$ have the same radius of convergence, so the partial sums for both converge uniformly over compact subsets of $(-R,R)$, hence $f \in C^1$ and $f' = g$. Repeat for successive derivatives ($g \in C^1$ so $f \in C^2$, ...) $\square$

<u>Integration</u> (Riemann $\int$, see Math 114 for Lebesgue integral and much more)

The definite integral of continuous functions is a <u>linear operator</u> $I_a^b : C^0([a,b]) \to \mathbb{R}$, $f \longmapsto I_a^b(f) = \int_a^b f\,dx$

for each $a<b \in \mathbb{R}$,

$\int_a^b (f+g)\,dx = \int_a^b f + \int_a^b g$
$\int_a^b cf\,dx = c\int_a^b f\,dx$

satisfying axioms:

1) If $f \geq 0$ then $\int_a^b f\,dx \geq 0$ $\quad$ ($\Rightarrow$ if $f \geq g$ then $\int_a^b f\,dx \geq \int_a^b g\,dx$).
2) If $a<c<b$ then $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$.
3) $\int_a^b 1\,dx = b-a$.

In fact, such a linear map is <u>unique</u>; the difference between different theories of integration is in how much more general functions we allow ourselves to integrate.

The Riemann integral starts from <u>step functions</u>: $s(x) : [a,b] \to \mathbb{R}$ such that $\exists\ a = x_0 < x_1 < \ldots < x_n = b$ st. $s(x)$ is constant over each (x_{i-1}, x_i), $s(x) = s_i$.
(the values at x_i don't matter). Then 2)+3) suggest we must have

$$I(s) = \int_a^b s(x)\,dx = \sum_{i=1}^n s_i\,(x_i - x_{i-1}).$$

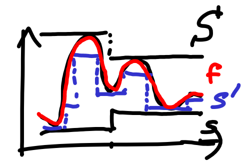
This definition of the integral for step functions satisfies the required axioms.

Next: if $s \leq f \leq S$ for s, S step functions, then $\int_a^b s\,dx \leq \int_a^b f\,dx \leq \int_a^b S\,dx$. $\quad(\star)$

In particular: $f : [a,b] \to \mathbb{R}$ bounded $\Rightarrow$ fixing $a = x_0 < x_1 < \ldots < x_n = b$, we can take $s_i = \inf f([x_{i-1}, x_i])$ and $S_i = \sup f([x_{i-1}, x_i])$, giving the lower and upper Riemann sums of f for the given partition of $[a,b]$.

Refining (ie. subdividing further) gives better bounds on f

$$\int s\,dx < \int s'\,dx < \int f\,dx < \int S'\,dx$$



Lower and upper Riemann integral:

$$I_-(f) = \sup\left\{ \int_a^b s\,dx \;\middle|\; \begin{array}{l} s \leq f \text{ on } [a,b] \\ s \text{ step function} \end{array} \right\}$$

$$I_+(f) = \inf\left\{ \int_a^b S\,dx \;\middle|\; \begin{array}{l} S \geq f \text{ on } [a,b] \\ S \text{ step function} \end{array} \right\}$$

$\forall f$ bounded $[a,b] \to \mathbb{R}$, $I_-(f) \leq I_+(f)$.

<u>Def</u>: f is <u>Riemann integrable</u>, $f \in \mathcal{R}([a,b])$, if $I_+(f) = I_-(f)$; we set $\int_a^b f\,dx = I_\pm(f)$.

Thm: Continuous functions are Riemann integrable.

Pf: The key ingredient is uniform continuity: $\forall \varepsilon>0\ \exists \delta$ st. $x,y\in[a,b]$, $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$.

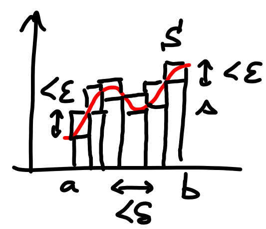
(Recall: this is proved by applying the Lebesgue number lemma to the open cover $[a,b]\subset \bigcup_{c\in\mathbb{R}} f^{-1}((c,c+\varepsilon))$: $\exists\delta>0$ st. $|x-y| = \mathrm{diam}(\{x,y\})<\delta \Rightarrow \exists c$ st. $\{x,y\}\subset f^{-1}((c,c+\varepsilon))$)

Thus: given $\varepsilon>0$, take δ as in uniform continuity, and split $a=x_0<x_1<\dots<x_n=b$ st. $x_{i+1}-x_i<\delta\ \forall i$. Then $s_i=\min f([x_i,x_{i+1}])$, $S_i=\max f([x_i,x_{i+1}])$ (attained) satisfy $S_i-s_i<\varepsilon\ \forall i$, and $s_i\le f\le S_i$ on $[x_i,x_{i+1}]$.

Let s, S = step functions taking values s_i, S_i on $[x_i,x_{i+1})$:

$s\le f\le S$ on $[a,b]$, so $I(s)\le I_-(f)$, $I(S)\ge I_+(f)$;

moreover, $S_i-s_i<\varepsilon\ \forall i$ so $I(S)-I(s)<\varepsilon(b-a)$.



Hence: $I_+(f)-I_-(f)<\varepsilon(b-a)\ \forall\varepsilon>0 \Rightarrow I_+(f)=I_-(f)$, $f\in\mathcal{R}([a,b])$. $\square$

Rmk:
- piecewise continuous functions are also integrable; and so do some stranger functions (see Rudin & see HW). However for example
$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$$
is not Riemann integrable $\left(\begin{matrix} I_-(f)=0\\ I_+(f)=b-a\end{matrix}\right)$.

  The Lebesgue integral allows more general decompositions into "measurable" subsets (rather than just sub-intervals) & allows more general functions to be integrated (including unbounded functions, which are never Riemann integrable)
  (eg for Riemann integration, $\int_0^x \frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\sqrt{x}$ only makes sense as an "improper integral" ie. $\lim_{\varepsilon\to0}\int_\varepsilon^x$, whereas Lebesgue can handle this & worse).

- In fact, Lebesgue gave a characterization of exactly which functions are Riemann integrable: $f\in\mathcal{R}([a,b])$ iff f is bounded on $[a,b]$ and the set of points where f is discontinuous has Lebesgue measure 0, which means: $\forall\varepsilon>0$
  $\exists (I_i)$ at most countable collection of open intervals st $E\subset\bigcup I_i$ and $\sum \mathrm{length}(I_i)<\varepsilon$.

- It is easy to check (do it!) that $\mathcal{R}([a,b])$ is a vector space, $I:\mathcal{R}([a,b])\to\mathbb{R}$ is linear and satisfies the above axioms.

- Fundamental thm of calculus: if f is continuous on $[a,b]$ then $F(x)=\int_a^x f(t)\,dt$ is differentiable and $F'=f$.

  Pf: $\frac{1}{h}(F(x+h)-F(h)) = \frac{1}{h}\int_x^{x+h} f(t)\,dt \xrightarrow[h\to0]{} f(x)$ using continuity of f at x to estimate the integral for $h\to0$. $\square$