Lecture 20 - Friday 3/11 - Integration in one real variable.

The definite integral of continuous functions is a $\underset{\substack{\int_a^b (f+g)\,dx = \int_a^b f + \int_a^b g \\ \int_a^b cf\,dx = c\int_a^b f\,dx}}{\underline{\text{linear operator}}}$ $I_a^b : C^0([a,b]) \to \mathbb{R}$,

$f \longmapsto I_a^b(f) = \int_a^b f\,dx$

for each $a<b \in \mathbb{R}$,

satisfying axioms:

1) If $f \geq 0$ then $\int_a^b f\,dx \geq 0$ $\quad$ ($\Rightarrow$ if $f \geq g$ then $\int_a^b f\,dx \geq \int_a^b g\,dx$).
2) If $a<c<b$ then $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$.
3) $\int_a^b 1\,dx = b-a$.

In fact, such a linear map is <u>unique</u>; the difference between different theories of integration is in how much more general functions we allow ourselves to integrate.

The Riemann integral starts from <u>step functions</u>: $s(x): [a,b] \to \mathbb{R}$ such that $\exists\ a = x_0 < x_1 < \dots < x_n = b$ st. $s(x)$ is constant over each (x_{i-1}, x_i), $s(x) = s_i$.

(the values at x_i don't matter). Then 2)+3) suggest we must have

$$I(s) = \int_a^b s(x)\,dx = \sum_{i=1}^n s_i\,(x_i - x_{i-1}).$$

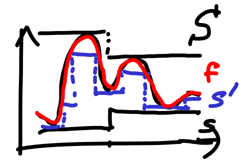
This definition of the integral for step functions satisfies the required axioms.

Next: if $s \leq f \leq S$ for s, S step functions, then $\int_a^b s\,dx \leq \int_a^b f\,dx \leq \int_a^b S\,dx$. $\quad (\ast)$

In particular: $f: [a,b] \to \mathbb{R}$ bounded $\Rightarrow$ fixing $a = x_0 < x_1 < \dots < x_n = b$, we can take $s_i = \inf f([x_{i-1}, x_i])$ and $S_i = \sup f([x_{i-1}, x_i])$, giving the lower and upper Riemann sums of f for the given partition of $[a,b]$.

Refining (i.e. subdividing further) gives better bounds on f

$$\int s\,dx < \int s'\,dx < \int f\,dx < \int S'\,dx$$



Lower and upper Riemann integral:

$$I_-(f) = \sup\left\{ \int_a^b s\,dx \;\middle|\; \begin{array}{l} s \leq f \text{ on } [a,b] \\ s \text{ step function} \end{array} \right\}$$

$$I_+(f) = \inf\left\{ \int_a^b S\,dx \;\middle|\; \begin{array}{l} S \geq f \text{ on } [a,b] \\ S \text{ step function} \end{array} \right\}$$

$\forall f$ bounded $[a,b] \to \mathbb{R}$, $I_-(f) \leq I_+(f)$.

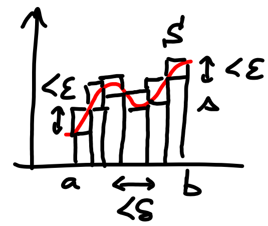
<u>Def:</u> f is <u>Riemann integrable</u>, $f \in \mathcal{R}([a,b])$, if $I_+(f) = I_-(f)$; we set $\int_a^b f\,dx = I_\pm(f)$.

<u>Thm:</u> Continuous functions are Riemann integrable.

<u>Pf:</u> The key ingredient is <u>uniform continuity</u>: $\forall \varepsilon > 0\ \exists \delta$ st. $x,y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

(Recall: this is proved by applying the Lebesgue number lemma to the open cover $[a,b] \subset \bigcup_{c \in \mathbb{R}} f^{-1}((c, c+\varepsilon))$: $\exists \delta > 0$ st. $|x-y| = \operatorname{diam}(\{x,y\}) < \delta \Rightarrow \exists c$ st. $\{x,y\} \subset f^{-1}((c,c+\varepsilon))$)

Thus: given $\varepsilon>0$, take δ as in uniform continuity, and split $a=x_0<x_1<\dots<x_n=b$.
st. $x_{i+1}-x_i<\delta\ \forall i$. Then $s_i=\min f([x_i,x_{i+1}])$, $S_i=\max f([x_i,x_{i+1}])$ (attained)
satisfy $S_i-s_i<\varepsilon\ \forall i$, and $s_i\le f\le S_i$ on $[x_i,x_{i+1}]$.



Let s, S = step functions taking values s_i, S_i on $[x_i,x_{i+1})$:
$s\le f\le S$ on $[a,b]$, so $I(s)\le I_-(f)$, $I(S)\ge I_+(f)$;
moreover, $S_i-s_i\le\varepsilon\ \forall i$ so $I(S)-I(s)\le\varepsilon(b-a)$.
Hence: $I_+(f)-I_-(f)\le\varepsilon(b-a)\ \forall\varepsilon>0 \Rightarrow I_+(f)=I_-(f)$, $f\in\mathcal{R}([a,b])$. $\square$

Rmk:
- piecewise continuous functions are also integrable; and so do some stranger functions (see Rudin & see HW). However for example
$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}\quad \text{is not Riemann integrable}\quad \begin{pmatrix} I_-(f)=0\\ I_+(f)=b-a\end{pmatrix}.$$
The Lebesgue integral allows more general decompositions into "measurable" subsets (rather than just sub-intervals) & allows more general functions to be integrated (including unbounded functions, which are never Riemann integrable)
(eg for Riemann integration, $\int_0^x \frac{1}{\sqrt{t}}\,dt=\frac12\sqrt{x}$ only makes sense as an "improper integral" ie. $\lim\limits_{\varepsilon\to0}\int_\varepsilon^x$, whereas Lebesgue can handle this & worse).
- In fact, Lebesgue gave a characterization of exactly which functions are Riemann-integrable: $f\in\mathcal{R}([a,b])$ iff f is bounded on $[a,b]$ and the set of points where f is discontinuous has Lebesgue measure 0, which means: $\forall\varepsilon>0$ $\exists(I_i)$ at most countable collection of open intervals st $E\subset\bigcup I_i$ and $\sum \text{length}(I_i)<\varepsilon$.
- It is easy to check (do it!) that $\mathcal{R}([a,b])$ is a vector space, $I:\mathcal{R}([a,b])\to\mathbb{R}$ is linear and satisfies the above axioms.
- <u>Fundamental thm</u> of calculus: if f is continuous on $[a,b]$ then $F(x)=\int_a^x f(t)\,dt$ is differentiable and $F'=f$.

<u>Pf</u>: $\frac1h(F(x+h)-F(h))=\frac1h\int_x^{x+h}f(t)\,dt \underset{h\to0}{\longrightarrow} f(x)$ using continuity of f at x to estimate the integral for $h\to0$. $\square$

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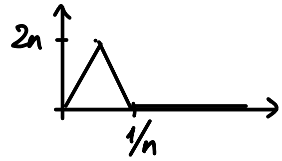
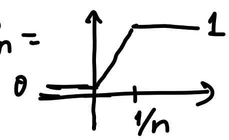
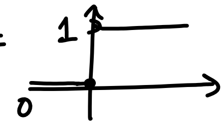
* <u>Thm</u>: $I:C^0([a,b])\to\mathbb{R}$ is continuous with respect to the uniform topology:
if $f_n\to f$ uniformly then $\int_a^b f_n\,dx\longrightarrow\int_a^b f\,dx$.
In fact, $\left|\int f\,dx-\int g\,dx\right|\le\int|f-g|\,dx\le(b-a)\sup|f-g|$.

On the other hand, pointwise convergence isn't enough: $f_n=$ 

$f_n\to0$ pointwise but $\int_0^1 f_n\,dx=1\not\to\int_0^1 0\,dx=0$.

* Besides $\|f\|_\infty = \sup |f|$, we have other norms on the vector space $C^0([a,b],\mathbb{R})$, defining <u>coarser</u> topologies (with respect to which integration is still a continuous functional) namely $\|f\|_1 = \int_a^b |f(x)|\,dx$, and also $\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}$ $\forall p \geq 1$.
(Triangle inequality follows from Hölder's inequality, cf. homework)
These are called the L^p norms; since $\|f\|_p \leq (b-a)^{1/p}\|f\|_\infty$, balls for $\|\cdot\|_p$ contain balls for $\|\cdot\|_\infty$ and the topologies defined by these metrics are <u>coarser</u> than the uniform topology (and L^p is coarser than $L^{p'}$ for $p<p'$, using Hölder ineq.).
$(C^0([a,b]), \|\cdot\|_p)$ isn't complete, its completion is the Lebesgue space $L^p([a,b])$ – Math 114!

<u>Ex:</u> $f_n =$ 

is Cauchy in L^1 norm, in fact converges in L^1 to its pointwise limit $f =$ 

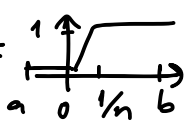
$\in \mathcal{R}$

$\left(\int_0^1 |f_n - f|\,dx = \frac{1}{2n} \to 0\right)$, but $f \notin C^0$.

* L^1 is quite natural, but so is L^2, which comes from an inner product
$\langle f, g\rangle_{L^2} = \int_a^b fg\,dx$ $\quad (\Rightarrow \|f\|_{L^2} = \sqrt{\langle f,f\rangle})$.
(Cauchy-Schwarz: $\langle f,g\rangle \leq \|f\|_{L^2}\|g\|_{L^2}$ is a special case of Hölder's ineq.)

We now return to $\|\cdot\|_\infty$ (uniform topology) and various results about $C^0([a,b])$.

* Closed & bounded subsets of $(C^0([a,b]), \|\cdot\|_\infty)$ aren't compact (In fact: the closed unit ball of an infinite-dim'l. normed vector space is <u>never</u> compact, by Riesz's theorem).

<u>Ex:</u> $f_n =$ 

$\|f_n\|_\infty = 1$ but $\nexists$ uniformly convergent subsequence

(even worse, $f_n = \sin(nx)$ doesn't even have a pointwise convergent subsequence on any interval).
So ... what kinds of subsets of $(C^0([a,b]), \|\cdot\|_\infty)$ are compact ($\Leftrightarrow$ sequentially compact).
The Ascoli-Arzela theorem gives the answer: need $\{f_n\}$ uniformly bounded + <u>equicontinuity</u>.

<u>Def:</u> A family of functions $\mathcal{F} \subset C^0(K)$, K compact metric space eg. $[a,b]$, is <u>equicontinuous</u> if $\forall \varepsilon > 0\ \exists \delta > 0$ st. $\forall f \in \mathcal{F}, \forall x,y \in K$, $d(x,y) < \delta \Rightarrow d(f(x),f(y)) < \varepsilon$.
$\uparrow$ indep't of $x \in K$ (uniform continuity) <u>and</u> of $f \in \mathcal{F}$ (equicontinuity)

<u>Prop:</u> If $f_n \to f \in C^0(K)$ uniformly, then $\{f_n\}$ is bounded in $\|\cdot\|_\infty$ ($\exists M$ st. $\forall n$, $\|f_n\|_\infty \leq M$) and equicontinuous.

Pf: given $\varepsilon>0$, $\exists N$ st. $n\geq N \Rightarrow \|f_n-f\|_\infty<\frac{\varepsilon}{3}$. f is uniformly continuous (K compact), let $\delta>0$ st. $d(x,y)<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{3}$. Then $\forall n\geq N$, $d(x,y)<\delta \Rightarrow |f_n(x)-f_n(y)|<\varepsilon$
(using triangle ineq.)
Since $f_1,\dots,f_N$ are also uniformly continuous, decreasing δ if needed we can ensure this also holds for $n<N$, thus proving equicontinuity. □

So: equicontinuity is necessary for sequential compactness of subsets of $(C^0(K),\|\cdot\|_\infty)$.

→ Thm (Arzela-Ascoli):

If a sequence $f_n\in C^0(K)$ is uniformly bounded and equicontinuous then it has a uniformly convergent subsequence. Hence: a subset of $(C^0(K),\|\cdot\|_\infty)$ is compact iff it is closed, bounded, and equicontinuous.

Proof (1st statement): • K compact metric space $\Rightarrow$ $\exists$ countable dense subset $A=\{x_1,x_2,\dots\}\subset K$.
(cover K by finitely many $\frac{1}{n}$-balls $\forall n$, take all centers).

- $\exists$ subsequence of $\{f_n\}$ st. converges pointwise at x_1 (since $\{f_n(x_1)\}$ is bounded).
$\exists$ sub-subsequence which also converges pointwise at x_2, etc...
Diagonal process: let f_{n_k} = k^{th} term of the k^{th} subsequence: then f_{n_k} converges pointwise at all points of A.
- Now we prove (f_{n_k}) is uniformly Cauchy (hence unif. convergent), using equicontinuity.
Given $\varepsilon>0$, let $\delta>0$ st. $\forall n_k, \forall x,y$, $|x-y|<\delta \Rightarrow |f_{n_k}(x)-f_{n_k}(y)|<\frac{\varepsilon}{3}$ (equicontinuity)
Let $A'\subset A$ finite subset st. $\bigcup_{x_i\in A'} B_\delta(x_i)\supset K$ (compactness of K).
Let N be st. $n_k,n_\ell\geq N \Rightarrow |f_{n_k}(x_i)-f_{n_\ell}(x_i)|<\frac{\varepsilon}{3}$ $\forall x_i\in A'$ (pointwise Cauchy + finiteness of A').
Then $\forall x\in K$ $\exists x_i\in A'$ st. $d(x_i,x)<\delta$, so $\forall n_k, n_\ell\geq N$,
$$|f_{n_k}(x)-f_{n_\ell}(x)|\leq |f_{n_k}(x)-f_{n_k}(x_i)|+|f_{n_k}(x_i)-f_{n_\ell}(x_i)|+|f_{n_\ell}(x_i)-f_{n_\ell}(x)|$$
$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$
hence: $n_k,n_\ell\geq N \Rightarrow \|f_{n_k}-f_{n_\ell}\|_\infty\leq\varepsilon$: (f_{n_k}) is Cauchy in $\|\cdot\|_\infty$, hence converges. □

Ex: $(f_n)\in C^1([a,b])$, bounded sequence in C^1-norm (ie. $\sup|f_n|\leq M$, $\sup|f_n'|\leq M$)
$\Rightarrow$ equicontinuous (using mean value ineq.) $\Rightarrow$ has subsequence that converges in C^0.

The closure of the unit ball for C^0-norm isn't compact in C^0
——"—— C^1-norm ——"—— C^1, but
The C^0-closure of the C^1-unit ball is compact in C^0!