(Note: Prof. Auroux's office hours on Mon & Tue March 21-22 are in SC 411)

Stone-Weierstrass theorem:

Thm: (Weierstrass) Polynomials are dense in $C^0([a,b])$, ie. $\forall f \in C^0([a,b])$ $\exists P_n$ polynomials st. $P_n \to f$ uniformly on $[a,b]$.

Key tool: convolution

Def: convolution: $(f*g)(x) = \int_{s+t=x} f(s)\,g(t)\,dt = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt = \int_{-\infty}^{\infty} f(s)\,g(x-s)\,ds$.

well-def'd if e.g. f and g are piecewise C^0 + one of them is compactly supported (ie. 0 outside some $[-M,M]$).
(to avoid improper integrals).

Principle: "$f*g$ inherits the best properties of f and g".

This is because (*) $(f*g)(x+h) - (f*g)(x) = \int f(x-t)\big(g(t+h)-g(t)\big)\,dt$

is bounded by $\underbrace{\left(\int |f|\,dt\right)}_{=\|f\|_{L^1}} \cdot \left(\sup_t |g(t+h)-g(t)|\right)$ (over relevant intervals, per compact support)

→ If g is C^0 then uniform continuity (over a compact interval) $\Rightarrow \lim_{h\to 0}\left(\sup|g(t+h)-g(t)|\right)=0$.
$(|g(t+h)-g(t)|<\varepsilon \;\; \forall t$ when $|h|<\delta)$. $\Rightarrow f*g$ is continuous.

→ If g is C^1 (continuously differentiable) then $f*g$ is C^1 and $(f*g)' = f*g'$.

Indeed: (*) $\Rightarrow \dfrac{(f*g)(x+h)-(f*g)(x)}{h} - f*(g')(x) = \int f(x-t)\Big(\underbrace{\dfrac{g(t+h)-g(t)}{h} - g'(t)}_{= g'(c)-g'(t) \text{ for some } c\in(t,t+h) \text{ by mean value theorem,}}\Big)\,dt \to 0$ as $h\to 0$.

hence for $h\to 0$ this $\to 0$ uniformly by uniform continuity of g' on interval of integration

→ Hence if g is C^∞ then $f*g$ is C^∞ !! (even if f isn't even continuous)

→ and... if g is a polynomial of degree d then so is $f*g$!

eg. because $g^{(d+1)}=0$ so $(f*g)^{(d+1)} = f*g^{(d+1)} = 0$, or more directly: $g(x) = \sum_{k=0}^{d} a_k x^k$

$\Rightarrow (f*g)(x) = \sum_{k=0}^{d} a_k \int f(t)\,(x-t)^k\,dt = \sum_{k=0}^{d}\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} a_k x^{k-\ell} \underbrace{\int f(t)\,t^\ell\,dt}_{\text{constant}}$

manifestly a polynomial in x.

* Approximate identities:

Def: A sequence of functions K_n approximates identity if
$\begin{cases} \cdot\; K_n \geq 0 \\ \cdot\; \int K_n\,dx = 1 \\ \cdot\; \forall \delta>0,\ \int_{|x|\geq\delta} K_n\,dx \to 0 \end{cases}$

Thm: $\left.\begin{array}{l} f \text{ compactly supported \& continuous} \\ K_n \text{ approximate identity} \end{array}\right\} \Rightarrow f * K_n \to f$ uniformly.

PF: $(f*K_n)(x) - f(x) = \int (f(x-t) - f(x))\, K_n(t)\, dt = \int_{|t|\le\delta} + \int_{|t|\ge\delta}$. Estimate each term as follows.

Given $\varepsilon > 0$, uniform continuity of f on its support $\Rightarrow \exists\, \delta$ (indep^t of x) st.

$$|t| < \delta \Rightarrow |f(x-t) - f(x)| < \varepsilon/2$$

Then $\left| \int_{-\delta}^{\delta} (f(x-t) - f(x))\, K_n(t)\, dt \right| \le \frac{\varepsilon}{2} \int_{-\delta}^{\delta} K_n(t)\, dt \le \frac{\varepsilon}{2}$. (indep^{tly} of x).

while $\left| \int_{|t|\ge\delta} (f(x-t) - f(x))\, K_n(t)\, dt \right| \le 2\|f\|_\infty \int_{|x|\ge\delta} K_n\, dt \longrightarrow 0$ as $n \to \infty$

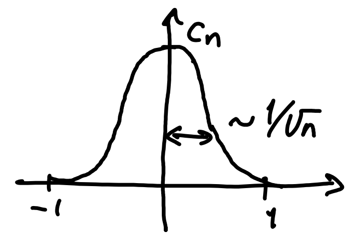
becomes $< \frac{\varepsilon}{2}$ for n suff^{tly} large.

$\rightsquigarrow \exists N$ st. $\forall x$, $|(f*K_n)(x) - f(x)| < \varepsilon \quad \forall n \ge N$. $\quad n \ge N$ (indep^t of x!)

$\square$

Ex: $K_n(x) = c_n(1-x^2)^n$ for $|x| \le 1$, 0 elsewhere

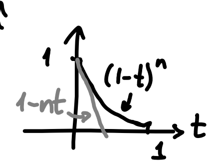
where $c_n > 0$ is chosen so that $\int_{-1}^{1} K_n\, dx = 1$.



Claim: K_n approximate identity.

Pf: • for $|x| \le \frac{1}{\sqrt{2n}}$, $(1-x^2)^n \ge 1 - nx^2 \ge \frac{1}{2}$, so $\int_{-1}^{1} (1-x^2)^n\, dx \ge \int_{-\frac{1}{\sqrt{2n}}}^{1/\sqrt{2n}} \ge \frac{1}{\sqrt{2n}}$

convexity



$\Rightarrow c_n \le \sqrt{2n}$

• for $|x| \ge \delta$, $(1-x^2)^n \le (1-\delta^2)^n$ so $\int_{|x|\ge\delta} K_n\, dx \le 2c_n(1-\delta^2)^n$

$$\le 2\sqrt{2n}\,(1-\delta^2)^n \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ **Thm.** (Weierstrass):

$\|$ $\forall f \in C^0([a,b])$ $\exists\, P_n$ polynomials st. $P_n \to f$ uniformly.

Pf: • by linear change of variables, we can assume $[a,b] = [0,1]$.

• subtracting a degree 1 polynomial from f we can assume $f(0) = f(1) = 0$.
Then extend f to $\mathbb{R}$ by $f(x) = 0$ if $x \notin [0,1]$.

• now let $K_n(x) =$ as above, and $p_n = f * K_n$.
Then K_n approx. identity, f C^0 compactly supported $\Rightarrow p_n \to f$ uniformly.

• p_n is a polynomial of degree $2n$ on $[0,1]$ because, given that $f = 0$ outside $[0,1]$, the formula $(f*K_n)(x) = \int f(x-t) K_n(t)\, dt$ for $x \in [0,1]$ doesn't involve the values of K_n outside $[-1,1]$, and $K_n|_{[-1,1]}$ is polynomial.

$\square$

- Stone's theorem generalizes this to other families of functions:

Def: $\mathcal{A} \subset C^0(K)$ is an <u>algebra</u> if $f, g \in \mathcal{A} \Rightarrow f+g \in \mathcal{A}$, $cf \in \mathcal{A}$, $fg \in \mathcal{A}$.
$\mathcal{A}$ <u>separates points</u> if $\forall a \neq b \in K$, $\exists f, g \in \mathcal{A}$ st. $f(a)=1 \quad f(b)=0$, $g(a)=0 \quad g(b)=1$

(0, 1 are arbitrary - this is equiv^t to $\mathcal{A} \to \mathbb{R}^2$, $f \mapsto (f(a), f(b))$ is surjective $\forall a \neq b$).

- For complex-valued functions, further assume $\mathcal{A}$ is conjugation-invariant, ie.
$f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$ (equivalently: $\operatorname{Re} f \in \mathcal{A}$, $\operatorname{Im} f \in \mathcal{A}$).

Thm (Stone): K compact metric space, $\mathcal{A} \subset C^0(K)$ <u>algebra</u> which separates points (+ conjugation invariant in $\mathbb{C}$ case), then $\mathcal{A}$ is dense is $(C^0(K), \|\cdot\|_\infty)$

(Weierstrass = special case $K = [a,b]$, $\mathcal{A}$ = polynomials).

Pf: • $\bar{\mathcal{A}}$ (uniform closure of $\mathcal{A}$) is an algebra $\left(f_n \to f,\ g_n \to g \Rightarrow \begin{array}{l} f+g = \lim(f_n+g_n) \\ fg = \lim(f_n g_n) \end{array}\right)$
so enough to show assumptions + $\mathcal{A}$ closed $\Rightarrow \mathcal{A} = C^0(K)$

- given $f \in \mathcal{A}$, $\mathcal{A}$ algebra & closed $\Rightarrow P(f) \in \mathcal{A}$ $\forall P$ polynomial st. $P(0)=0$
By Weierstrass, $|x|$ is a uniform limit of polynomials on $[-M, M]$, so $|f| \in \bar{\mathcal{A}} = \mathcal{A}$.
Hence: $f, g \in \mathcal{A} \Rightarrow \max(f,g) = \frac{f+g+|f-g|}{2} \in \mathcal{A}$, same for $\min(f,g)$.

- Now: given $f \in C^0(K)$, $\varepsilon > 0$, want to show $\exists h \in \mathcal{A}$ st. $\sup |h - f| \leq \varepsilon$. ($\Rightarrow f \in \bar{\mathcal{A}} = \mathcal{A}$).
given $x \in K$ $\forall y \neq x$ $\exists g_y \in \mathcal{A}$ st. $\begin{cases} g_y(x) = f(x) \\ g_y(y) = f(y). \end{cases}$ ($\mathcal{A}$ separates points).

$\exists \underset{\text{open}}{U_y} \ni y$ st. $g_y > f - \varepsilon$ on U_y; and K compact $\Rightarrow \exists y_1, \dots y_n$ st. $U_{y_1} \cup \dots \cup U_{y_n} = K$.

Then $h_x := \max(g_{y_1}, \dots, g_{y_n}) \in \mathcal{A}$ satisfies $\begin{cases} h_x > f - \varepsilon & \text{everywhere} \\ h_x(x) = f(x). \end{cases}$

By the same argument, $\exists x_1, \dots x_n$ st. $k = \min(h_{x_1}, \dots h_{x_k})$ satisfies $|k - f| < \varepsilon$ everywhere.
($\exists V_x \ni x$ open st. $h_x < f + \varepsilon$ on U_x; K compact $\Rightarrow \exists x_1, \dots x_n$ st. $V_{x_1} \cup \dots \cup V_{x_n} = K$) ◻

<u>Fourier series</u>: we consider continuous 2π-periodic functions $f: \mathbb{R} \to \mathbb{C}$ with complex values, or equivalently functions on $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, with L^2 inner product $\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \bar{f}(x) g(x)\, dx$

The complex exponentials $e_n(x) = e^{inx}$, $n \in \mathbb{Z}$ satisfy $\langle e_i, e_j \rangle = \delta_{ij}$ - orthonormality.

Def: The Fourier coefficients of f are $c_n(f) = \langle e_n, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x)\, dx$.

→ the <u>Fourier series</u> of f is $\sum_{n \in \mathbb{Z}} c_n e_n = \sum_{-\infty}^{\infty} c_n(f) e^{inx}$

<u>Q</u>: (Fourier, Dirichlet, Féjer, ...) does the Fourier series accurately represent f? (e.g. does it converge? to f?).

Def: <u>Trigonometric polynomials</u> = the vector space of <u>finite</u> linear combinations of e_n.

* Clearly this is an algebra, complex conj. invariant, and separates points of S^1, which is compact: hence by Stone-Weierstrass, trig. polynomials are <u>dense</u> in $(C^0(S^1), \|\cdot\|_\infty)$

... hence also in L^2-norm $\left(\|f\|_{L^2} = \left(\frac{1}{2\pi}\int |f|^2\,dx\right)^{1/2} \le \sup |f|\right)$.

* The n^{th} Fourier sum $f_n = s_n(f) = \sum_{-n}^{n} c_k e^{ikx} = \sum_{-n}^{n} \langle e_k, f\rangle e_k$

is the orthogonal projection of f onto $V_n = \mathrm{span}(e_{-n}, \dots, e_n)$ for $\langle\cdot,\cdot\rangle$.

Indeed: $\langle e_j, f_n\rangle = \sum_{k=-n}^{n} c_k \langle e_j, e_k\rangle = c_j = \langle e_j, f\rangle$, so $\langle e_j, f - f_n\rangle = 0 \quad \forall -n \le j \le n$.

Thus: $\forall g \in V_n, \quad \|f - f_n\|_{L^2} \le \|f - g\|_{L^2}$ – the point of V_n closest to f for $\|\cdot\|_{L^2}$

$\Big($This follows from $(f - f_n) \perp V_n$: $\quad (f-g) = \underbrace{(f - f_n)}_{\perp V_n} + \underbrace{(f_n - g)}_{\in V_n} \Rightarrow \|f-g\|^2 = \|f-f_n\|^2 + \|f_n - g\|^2 \ge \|f-f_n\|^2$. $\Big)$

$\Rightarrow$ <u>Theorem</u> (Parseval): Let $f \in C^0(S^1)$, $c_n = \langle e_n, f\rangle$ Fourier coeffts, $f_n = \sum_{-n}^{n} c_k e_k$ partial sums:

(1) $f_n \to f$ in L^2, ie. $\|f_n - f\|_{L^2}^2 = \frac{1}{2\pi}\int |f(x) - f_n(x)|^2\,dx \to 0$ as $n \to \infty$.

(2) $\sum_{n\in\mathbb{Z}} |c_n|^2 = \|f\|_{L^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx$ (in particular $\sum |c_n|^2$ converges so $c_n \to 0$ as $|n| \to \infty$)

<u>Pf</u>: (1) Since trig. polynomials $= \bigcup_n V_n$ are dense in $(C^0(S^1), \|\cdot\|_{L^2})$,

$\forall \varepsilon > 0 \ \exists N$ st. $\exists g \in V_N$ with $\|f - g\|_{L^2} < \varepsilon$.

Now for $n \ge N$, $g \in V_N \subset V_n$ and $f_n =$ closest point to f, so

$\|f - f_n\|_{L^2} \le \|f - g\|_{L^2} < \varepsilon$. This shows $f_n \to f$ in L^2.

(2) since $f_n \in V_n$ and $f - f_n \in V_n^{\perp}$, $\|f\|_{L^2}^2 = \|f_n\|_{L^2}^2 + \|f - f_n\|_{L^2}^2$

where $\|f_n\|_{L^2}^2 = \left\|\sum_{-n}^{n} c_k e_k\right\|^2 = \sum_{-n}^{n} |c_k|^2$ by orthonormality, and

$\|f - f_n\|_{L^2}^2 \to 0$ by the first part. $\square$

<u>Corollary</u>: if $f, g \in C^0(S^1)$ have same Fourier series then $\frac{1}{2\pi}\int |f-g|^2\,dx = \sum |c_n(f) - c_n(g)|^2 = 0$, hence $f = g$.

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* The fact that $f_n \to f$ in L^2 is the best approximation (in L^2 norm) of f by trig. polynomials, and that trig. polynomials are dense in $\|\cdot\|_\infty$ (so $\exists$ trig. polynomials $\to f$ uniformly) makes one hope that $f_n \to f$ uniformly or at least pointwise... alas not!

<u>Fact</u>: $\exists f \in C^0(S^1)$ st. the Fourier series of f does not converge ($s_n(f)(0)$ unbounded!) (but the example is hard to construct).

<u>Thm</u> (Dirichlet): if f is C^1 then $f_n = s_n(f) \longrightarrow f$ uniformly.

The proof uses convolution – redefine, for periodic functions, $(f*g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,g(x-t)\,dt$.

& note $c_n e_n(x) = \frac{1}{2\pi}\left(\int f(t)e^{-int}\,dt\right)e^{inx} = (f*e_n)(x)$.

So: $s_n(f) = \sum_{-n}^{n} c_k e_k = f*\left(\sum_{-n}^{n} e_k\right) = f*D_n$ where

$$D_n(x) = \sum_{-n}^{n} e^{ikx} = \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \qquad \underline{\text{Dirichlet kernel}}$$

Dirichlet's proof studies this convolution for $f\in C^1$ to prove unif. convergence.

The fact that convergence can sometimes fail makes it remarkable that $\forall f\in C^0$, f can be recovered from the partial sums $s_n(f) = f_n = \sum_{-n}^{n} c_k e^{ikx}$ ...

<u>Thm</u> (Féjer): If $f\in C^0(S^1)$ then $\frac{s_0(f)+\dots+s_{n-1}(f)}{n}$ converges uniformly to f.

The reason is that this process amounts to convolution with the Féjer kernel $F_n = \frac{D_0+\dots+D_{n-1}}{n}$, which actually approximates identity (in the sense seen above) unlike D_n.