Math 55b Homework 8

Due Wednesday March 30, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: Equicontinuity, Stone-Weierstrass, Fourier series; differentiation in several variables (Rudin chapters 7-8-9, or McMullen's notes sections 7-8).

0. Sometime over the weekend of March 26-27, please complete the week 8 feedback survey (in Canvas).

1. Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\int_0^1 x^n\, f(x)\, dx = 0$ for all $n \in \mathbb{N}$, then $f(x) = 0$ on $[0, 1]$. (Hint: use the Weierstrass theorem).

2. Let c_n be the Fourier coefficients of a continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, and $k \in \mathbb{N}$ an integer.

(a) Show that if $\sum |c_n|$ is convergent then the Fourier series $\sum c_n e^{inx}$ converges uniformly to f.

(b) Show that if $\sum |n|^k |c_n|$ is convergent then $f \in C^k$.

(c) Conversely, show that if $f \in C^k$ then $\sum n^{2k} |c_n|^2$ converges, and in particular $n^k |c_n| \to 0$.

(Note: this shows that the amount of differentiability of a function can be read off from the rate of decay of its Fourier coefficients!)

(d) Deduce Dirichlet's theorem: if $f \in C^1$ then $\sum c_n e^{inx}$ converges uniformly to f.

(Hints for (b) and (c): how are the Fourier coefficients of f' related to those of f? for (d): observe that $2|c_n| \leq \frac{1}{n^2} + n^2 |c_n|^2$)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the unique 2π-periodic function such that $f(x) = (x - \pi)^2$ for $x \in [0, 2\pi]$.

(a) Show that the Fourier coefficients of f are $c_n = \dfrac{2}{n^2}$ for $n \neq 0$, and $c_0 = \dfrac{\pi^2}{3}$.

(b) Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

(Hint: use evaluation at $x = 0$ and Parseval's theorem. Convergence can be justified using the criterion in part (a) of the previous problem.)

4. Recall that the Dirichlet kernel is $D_n(x) = \displaystyle\sum_{k=-n}^{n} e^{ikx} = \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}}$. The Féjer kernel is defined to be $K_N(x) = \frac{1}{N} \displaystyle\sum_{n=0}^{N-1} D_n(x)$.

(a) Show that $K_N(x) = \dfrac{(e^{iNx/2} - e^{-iNx/2})^2}{N(e^{ix/2} - e^{-ix/2})^2} = \dfrac{\sin^2(Nx/2)}{N \sin^2(x/2)}$.

(b) Show that K_N approximates identity, in the sense that: (i) $K_N \geq 0$, (ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N\,dx = 1$, and (iii) $\forall \delta > 0$, $\int_{\delta \leq |x| \leq \pi} K_N\,dx \to 0$ (in fact, K_N converges uniformly to 0 on $[-\pi, -\delta] \cup [\delta, \pi]$).

(c) Let f be a continuous 2π-periodic function, and denote by $s_n = \sum_{k=-n}^{n} c_k e^{ikx}$ the partial sums of the Fourier series of f. We consider the arithmetic mean $\sigma_N = \frac{1}{N}(s_0 + \cdots + s_{N-1})$.

(The process of replacing a series by the arithmetic means of its partial sums is called *Cesaro summation*; for convergent series it gives the same limit, but this sometimes turns a divergent series into a convergent sequence).

Show that σ_N is the convolution of f with K_N, in the sense that

$$\sigma_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,K_N(t)\,dt.$$

(d) Deduce Féjer's theorem: for any continuous 2π-periodic function f, the sequence σ_N converges uniformly to f.

(This is a remarkable result, considering that the Fourier series of a continuous function need not even converge pointwise – even though the counterexamples are fairly pathological. The point is that the Féjer kernel approximates identity (in the sense of part (b)) whereas the Dirichlet kernel does not).

5. Let f be a real-valued function on $\mathbb{R}^2$, and suppose the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist for every (x, y). Prove or disprove each of the following assertions:

(a) f is continuous.

(b) if the partial derivatives of f are bounded (i.e. $\exists M > 0$ such that $|\partial f/\partial x| \leq M$ and $|\partial f/\partial y| \leq M$ everywhere), then f is continuous.

(c) if the partial derivatives of f are bounded, then f is differentiable.

(Hint: consider functions of the form $x^k y^\ell/(x^2+y^2)$.)

6. (a) Give an example showing that a differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\det Df(x) \neq 0$ for all x does not need to be injective.

(b) Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable and $\sup_{x\in\mathbb{R}^2} |Df(x) - I| \leq \alpha$ for some constant $\alpha < 1$, then f is a bijection.

(c) Consider the statement: if $U \subset \mathbb{R}^2$ is a connected open subset, and $f : U \to \mathbb{R}^2$ is a differentiable map which satisfies $\sup_{x\in U} |Df(x) - I| \leq \alpha < 1$, then f is injective. Show that this statement is true if U is an open ball, but false for some other connected open subsets of $\mathbb{R}^2$ (give an example).

7. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?