Iterated and Riemann integrals in several variables

* f continuous function on an n-cell $D = [a_1,b_1] \times \dots \times [a_n,b_n] \subset \mathbb{R}^n$

$\Rightarrow$ we can define $\int_D f = \int_D f \, dx_1 \dots dx_n = \int_D f \, |dx|$

↑ why? clearer after diff. forms

either 1) as iterated integral:

$$\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \cdots \left(\int_{a_n}^{b_n} f(x_1 \dots x_n)\, dx_n \right) \dots dx_2 \right) dx_1 \qquad \text{or any order}$$

2) as Riemann integral: split D into small cubes Q_i, and bound f between piecewise constant functions $s = s_i = \min f(Q_i)$ on $\mathrm{int}(Q_i)$, $S = S_i = \max f(Q_i)$ — " —

$\rightarrow$ estimate $\sum s_i \, \mathrm{vol}(Q_i) \le \int_D f \, |dx| \le \sum S_i \, \mathrm{vol}(Q_i)$

If f is continuous, hence uniformly continuous, then $\sup |S - s| \to 0$ as $\mathrm{diam}(Q_i) \to 0$, so this defines the integral uniquely.

Fubini's thm says: for continuous f, the iterated integrals for different orders of integration are all equal.

* if f is only piecewise continuous, integrability still holds if the regions of D where f is continuous are sufficiently regular - eg. delimited by smooth hypersurfaces.

Specifically: when decomposing D into small cubes Q_i, want $\sum_{f_{|Q_i} \text{ not } C^0} \mathrm{vol}(Q_i) \longrightarrow 0$ as one subdivides further – over such cubes, $(S_i - s_i)$ doesn't $\to 0$ as step size $\to 0$, but if vol $\to 0$ we still have $\int_D (S - s)\, |dx| = \sum (S_i - s_i)\, \mathrm{vol}(Q_i) \to 0$.

* Thus we can define integrals over regions of $\mathbb{R}^n$ delimited by hypersurfaces by either
  - extending f by 0 outside of the given region, and integrating the resulting piecewise continuous function
  - using changes of coords. (via implicit function thm) to make the region of integration an n-cell. This requires change of variables...

Thm: $\varphi : U \to V$ diffeomorphism, f continuous on V, then

$$\int_V f(y)\, |dy| = \int_U f(\varphi(x))\, |\det D\varphi(x)| \, d|x|.$$

(won't prove. The geometric input is that if Q_i is a small cube $\ni x$ then $\varphi(Q_i) \approx$ small parallelepiped $\ni \varphi(x)$, with $\mathrm{vol}(\varphi(Q_i)) \sim |\det D\varphi(x)| \cdot \mathrm{vol}(Q_i)$.)

* We also want to consider path integrals such as, given a path $\gamma \in C^1([0,1],\mathbb{R}^2)$, $\gamma(t)=(x(t),y(t))$ and a <u>differential</u> (1-form) $\omega = p(x,y)\,dx + q(x,y)\,dy$ (p,q C^0)

the path integral $\int_\gamma \omega = \int_\gamma p\,dx + q\,dy = \int_0^1 \big(p(\gamma(t))\,x'(t) + q(\gamma(t))\,y'(t)\big)\,dt$

→ this is independent of the parametrization of the path, by change of variable + chain rule.

→ if we reverse the path $(-\gamma)(t) = \gamma(1-t)$, then $\int_{-\gamma}\omega = -\int_\gamma \omega$.

→ given $f \in C^1(\mathbb{R}^2,\mathbb{R})$, define $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, then $\int_\gamma df = f(\gamma(1)) - f(\gamma(0))$

This generalizes to arbitrary dimensions, using the language of <u>differential forms</u>.

* on $\mathbb{R}^n$, the symbols $dx_1,\dots,dx_n$ can be viewed as the differentials of the coordinate functions $x_1,\dots,x_n$; they form a basis of $T^* = \mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ linear forms on the space of tangent vectors $T \simeq \mathbb{R}^n$ ($dx_i(v) = v_i$ i^{th} component).
  Differential 1-forms are therefore functions with values in T^*.
* we now consider the <u>exterior powers</u> $\Lambda^k T^* =$ vector space with basis $\{dx_{i_1}\wedge\dots\wedge dx_{i_k} \mid i_1<\dots<i_k\}$, which are parts of the exterior algebra generated by T^*, ie. quotient of tensor algebra by setting $dx_i\wedge dx_j = -dx_j \wedge dx_i$. (NB: $\Lambda^0 = \mathbb{R}$)
  ($\Rightarrow$ $\alpha\wedge\beta = -\beta\wedge\alpha$ for all 1-forms).
  $\alpha\wedge\alpha = 0$

<u>Def</u>: A <u>k-form</u> on an open subset $U\subset\mathbb{R}^n$ is a function with values in $\Lambda^k T^*$:

$$\omega(x) = \sum_{i_1<\dots<i_k} p_{i_1\dots i_k}(x)\,dx_{i_1}\wedge\dots\wedge dx_{i_k}. \quad \left(\text{also denoted } = \sum_{|I|=k} p_I\,dx_I\right)$$

→ can evaluate on k vectors: $\omega(x)(\underbrace{v_1,\dots,v_k}_{\in\mathbb{R}^n,\ v_j=(v_{ij})_{i=1..n}}) = \sum_{i_1<\dots<i_k} p_{i_1\dots i_k}(x)\det\left((v_{ij})_{\substack{i\in\{i_1\dots i_k\}\\ j\in\{1..k\}}}\right) \in \mathbb{R}$.
(an alternating multilinear form)

* The space of C^∞ k-forms on $U\subset\mathbb{R}^n$ is usually denoted $\Omega^k(U)$ ($=C^\infty(U,\Lambda^k T^*)$)
  We can multiply k-forms by functions, or take exterior products ($\wedge: \Omega^k\times\Omega^\ell \to \Omega^{k+\ell}$)

$$(f\,dx_{i_1}\wedge\dots\wedge dx_{i_k})\wedge(g\,dx_{j_1}\wedge\dots\wedge dx_{j_\ell}) = (fg)\,dx_{i_1}\wedge\dots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\dots\wedge dx_{j_\ell}$$

($=0$ if $I\cap J\neq\emptyset$, $=\pm(fg)\,dx_{I\sqcup J}$ if $I\cap J=\emptyset$)

* The <u>exterior derivative</u> $d:\Omega^k\to\Omega^{k+1}$ is $d\left(\sum_I p_I\,dx_I\right) = \sum_{I,j}\frac{\partial p_I}{\partial x_j}\,dx_j\wedge dx_I$

<u>Eg</u>: $\Omega^0\to\Omega^1$: $df = \sum \frac{\partial f}{\partial x_i}dx_i$.

$\Omega^1(\mathbb{R}^2)\to\Omega^2(\mathbb{R}^2)$: $d(p\,dx + q\,dy) = \left(-\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right)dx\wedge dy$.

Prop: $\| \ d^2 = 0$ ie. $\forall \omega \in \Omega^k$, $d(d\omega) = 0$.

(follows from: $\dfrac{\partial^2 p_I}{\partial x_j \partial x_k} = \dfrac{\partial^2 p_I}{\partial x_k \partial x_j}$, $dx_j \wedge dx_k + dx_k \wedge dx_j = 0$)

Say ω is closed if $d\omega = 0$, exact if $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}$.

The above says: exact $\Rightarrow$ closed.

Thm (Poincaré lemma): $\|$ for $U \subset \mathbb{R}^n$ convex open, $\omega \in \Omega^k$ is exact iff ω is closed.
$1 \le k \le n$

Remark: This leads to de Rham cohomology, a key invariant in diff. topology!

$H^k_{dR}(U) := \ker(d : \Omega^k(U) \to \Omega^{k+1}(U)) / \mathrm{Im}(d : \Omega^{k-1}(U) \to \Omega^k(U)) = \{\text{closed } k\text{-forms}\}/\{\text{exact}\}$.

The Poincaré lemma says $H^k_{dR}(U) = 0$ for $U \subset \mathbb{R}^n$ convex and $k \geq 1$

while $H^1_{dR}(\mathbb{R}^2 - \{0\}) \neq 0$ detects $\mathbb{R}^2 - \{0\}$ isn't simply connected.

* Pullback of differential forms: if $\varphi : U \to V$ is a smooth map ($U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$)

then we have a map $\varphi^* : \Omega^k(V) \to \Omega^k(U)$ defined by

$$(\varphi^*\omega)(\underset{\in U}{x})(\underset{\in \mathbb{R}^n}{v_1, \dots, v_k}) = \omega(\underset{\in V}{\varphi(x)})(\underset{\in \mathbb{R}^m}{D\varphi(x)v_1, \dots, D\varphi(x)v_k})$$

Basic properties: $\begin{cases} (1) \text{ for functions } (k=0), \quad \varphi^*(f) = f \circ \varphi \\ (2) \quad \varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \\ (3) \quad \varphi^*(d\alpha) = d(\varphi^*\alpha). \end{cases}$ (follows from chain rule)

Concretely, denoting by (x_i) coords. on U, (y_j) on V, $\varphi^*(dy_j) = d(y_j \circ \varphi) = \sum_i \dfrac{\partial \varphi_j}{\partial x_i} dx_i$ $(= d\varphi_j)$

and $\varphi^*\left(\sum_J p_J(y)\, dy_{j_1} \wedge \dots \wedge dy_{j_k}\right) = \sum_J p_J(\varphi(x))\, \underbrace{d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}}_{= \sum_I \det\left(\frac{\partial(\varphi_{j_1}, \dots, \varphi_{j_k})}{\partial(x_{i_1}, \dots, x_{i_k})}\right) dx_I}$

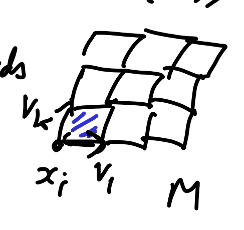
Especially: for $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and $k = n$,

$$\varphi^*(dy_1 \wedge \dots \wedge dy_n) = (\det D\varphi)\, dx_1 \wedge \dots \wedge dx_n$$

* Integration of differential forms:

given $\omega = \sum_I p_I(x)\, dx_I \in \Omega^k(U)$, we can integrate ω over a k-dimensional submanifold $M \subset U$ parametrized by a smooth map from a k-cell $D \subset \mathbb{R}^k$ to $U \subset \mathbb{R}^n$ (or any other nice enough domain for integration), $\varphi : D \hookrightarrow U$, $M = \varphi(D)$,

$\int_M \omega = \lim \sum_i \omega(x_i)(v_1, \dots, v_k)$ splitting M into small grid parallelepipeds



ie. $$\boxed{\int_M \omega = \int_D \omega(\varphi(t))\left(\frac{\partial \varphi}{\partial t_1}, \dots, \frac{\partial \varphi}{\partial t_k}\right) dt_1 \dots dt_k = \int_D \varphi^* \omega}$$

Explicitly in components, writing $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, if $\omega = \sum_{|I|=k} P_I(x)\, dx_I$:

$$\int_M \omega = \int_D \sum_I P_I(\varphi(t)) \det\left(\left(\frac{\partial \varphi_i}{\partial t_j}\right)\right)_{\substack{i \in I \\ 1 \le j \le k}} |dt|.$$

* check: for 1-forms this agrees with path integral formula $\int_\gamma P_i\, dx_i = \int P_i(\gamma(t)) \frac{dx_i}{dt}\, dt$

What this formula means is:

$$\begin{cases} \bullet \text{ for } n\text{-forms on } D \subset U \subset \mathbb{R}^n, & \int_D f\, dx_1 \wedge \dots \wedge dx_n = \int_D f\, |dx|. \\ \bullet \text{ for general } \varphi : D^k \to U \subset \mathbb{R}^n, & \int_{\varphi(D)} \omega = \int_D \varphi^* \omega \end{cases}$$

$\leftarrow$ k-form on $D \subset \mathbb{R}^k$ $\Rightarrow$ usual integral.

* Can similarly integrate k-forms over M = finite union of parametrized pieces.

* <u>pullback formula</u>: given a smooth map $\varphi : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$, $\omega \in \Omega^k(V)$, and a k-dimensional $M \subset U$: $\boxed{\int_{\varphi(M)} \omega = \int_M \varphi^* \omega}$.

This is basically equivalent to change of variables formula for usual $\int_D f\, |dx|$, and implies that $\int_M \omega$ is independent of the manner in which we parametrize M as the image of a map $\varphi : D \to U$ (or union of pieces) as long as all reparametrizations are <u>orientation-preserving</u> (ie. we compose $\varphi : D \to U$ with a diffeomorphism $g : \underset{\mathbb{R}^k}{D'} \to \underset{\mathbb{R}^k}{D}$ st. $\det(Dg) > 0$ everywhere).

<u>Ex</u>: $\omega = \frac{x\, dy - y\, dx}{x^2 + y^2}$ on $\mathbb{R}^2 - \{0\}$, C_r = circle of radius r, oriented counterclockwise: (as path $(r,0) \to (r,0)$)

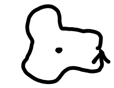
Pulling back via $\varphi : (r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, (polar coordinates),

$$\varphi^* \omega = \frac{(r\cos\theta)(r\cos\theta\, d\theta) - (r\sin\theta)(-r\sin\theta\, d\theta)}{r^2} = d\theta$$

So $\int_{C_r} \omega = \int_{\{r\} \times [0, 2\pi]} \varphi^* \omega = \int_0^{2\pi} d\theta = 2\pi$ (independent of r)

<u>Note</u>: $d\omega = 0$ (by direct calc. or using $\varphi^*(d\omega) = d(\varphi^* \omega) = d(d\theta) = 0$)

ie. ω is closed, but not exact! if $\exists f(x,y)$ on $\mathbb{R}^2 - \{0\}$ st $df = \omega$ then path integral $\int_{C_r} \omega = \int_{C_r} df = f(r,0) - f(r,0) = 0$. $H^1_{dR}(\mathbb{R}^2 - 0) \neq 0$.

But... path integral is independent of radius r, or in fact same for any 

This is a consequence of <u>Stokes' theorem</u> (next time).