THIS FRIDAY'S CLASS WILL BE IN SC. CENTER HALL E (basement)

Recall: • A k-form on an open subset $U \subset \mathbb{R}^n$ is a function with values in $\Lambda^k T^*$:

$$\omega = \sum_{i_1<\dots<i_k} p_{i_1\dots i_k}(x)\, dx_{i_1}\wedge\dots\wedge dx_{i_k}. \quad \left(\text{also denoted } = \sum_{|I|=k} p_I\, dx_I\right)$$

given $x \in U$, $v_1, \dots v_n \in \mathbb{R}^n \rightsquigarrow \omega(x)(v_1,\dots,v_k) = \sum_I p_I(x) \det\left((v_{ij})_{\substack{i\in I\\ 1\le j\le k}}\right)$

The space of C^∞ k-forms on $U\subset\mathbb{R}^n$: $\Omega^k(U) = C^\infty(U, \Lambda^k T^*)$.

* Exterior product $(f\, dx_{i_1}\wedge\dots\wedge dx_{i_k})\wedge(g\, dx_{j_1}\wedge\dots\wedge dx_{j_l}) = (fg)\, dx_{i_1}\wedge\dots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\dots\wedge dx_{j_l}$

$dx_i\wedge dx_j = -dx_j\wedge dx_i$ $\qquad (=0 \text{ if } I\cap J\neq\emptyset, \ = \pm(fg)\, dx_{I\cup J} \text{ if } I\cap J=\emptyset)$.

* The exterior derivative $d: \Omega^k \to \Omega^{k+1}$ is $d\left(\sum_I p_I\, dx_I\right) = \sum_{I,j} \frac{\partial p_I}{\partial x_j}\, dx_j\wedge dx_I$

Eg: $\Omega^0\to\Omega^1$: $df = \sum \frac{\partial f}{\partial x_i}\, dx_i$.

$\Omega^1(\mathbb{R}^2)\to\Omega^2(\mathbb{R}^2)$: $d(p\,dx + q\,dy) = \left(-\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) dx\wedge dy$.

* Pullback: if $\varphi: U\to V$ is a smooth map $(U\subset\mathbb{R}^n,\ V\subset\mathbb{R}^m)$

then we have a map $\varphi^*: \Omega^k(V)\to\Omega^k(U)$ defined by

$$(\varphi^*\omega)(\underset{\cap\atop U}{x})(\underset{\in\mathbb{R}^n}{v_1,\dots,v_k}) = \omega(\underset{\cap\atop V}{\varphi(x)})(\underset{\cap\atop \mathbb{R}^m}{D\varphi(x)v_1},\dots, D\varphi(x)v_k)$$

Basic properties:
(1) for functions $(k=0)$, $\varphi^*(f) = f\circ\varphi$
(2) $\varphi^*(\alpha\wedge\beta) = \varphi^*\alpha\wedge\varphi^*\beta$
(3) $\varphi^*(d\alpha) = d(\varphi^*\alpha)$. (follows from chain rule)

Concretely, denoting by (x_i) coords. on U, (y_j) on V, $\varphi^*(dy_j) = d(y_j\circ\varphi) = \sum_i \frac{\partial\varphi_j}{\partial x_i} dx_i$ $(= d\varphi_j)$

and $\varphi^*\left(\sum_J p_J(y)\, dy_{j_1}\wedge\dots\wedge dy_{j_k}\right) = \sum_J p_J(\varphi(x))\, \underbrace{d\varphi_{j_1}\wedge\dots\wedge d\varphi_{j_k}}_{=\sum_I \det\left(\frac{\partial(\varphi_{j_1},\dots,\varphi_{j_k})}{\partial(x_{i_1},\dots,x_{i_k})}\right) dx_I}$

Especially: for $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ and $k=n$,

$\varphi^*(dy_1\wedge\dots\wedge dy_n) = (\det D\varphi)\, dx_1\wedge\dots\wedge dx_n$

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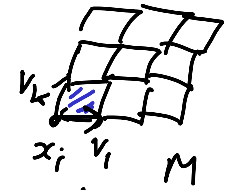
* Integration of differential forms:

given $\omega = \sum_I p_I(x)\, dx_I \in \Omega^k(U)$, we can integrate ω over a k-dimensional submanifold $M\subset U$ parametrized by a smooth map from a k-cell $D\subset\mathbb{R}^k$ to $U\subset\mathbb{R}^n$ (or any other nice enough domain for integration), $\varphi: D\hookrightarrow U$, $M=\varphi(D)$,

$\int_M \omega = \lim \sum_i \omega(x_i)(V_1 \dots V_k)$ splitting M into small grid parallelepipeds



ie. $$\boxed{\int_M \omega = \int_D \omega(\varphi(t))\left(\frac{\partial \varphi}{\partial t_1}, \dots, \frac{\partial \varphi}{\partial t_k}\right) dt_1 \dots dt_k = \int_D \varphi^* \omega}$$

Explicitly in components, writing $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, if $\omega = \sum_{|I|=k} p_I(x)\, dx_I$:

$$\int_M \omega = \int_D \sum_I p_I(\varphi(t)) \det\left(\left(\frac{\partial \varphi_i}{\partial t_j}\right)\right)_{\substack{i \in I \\ 1 \le j \le k}} |dt|$$

* <u>check</u> : • for 1-forms this agrees with path integral formula $\int_\gamma p_i\, dx_i = \int p_i(\gamma(t)) \frac{dx_i}{dt}\, dt$

• for n-forms on $D \subset U \subset \mathbb{R}^n$, $\int_D f\, dx_1 \wedge \dots \wedge dx_n = \int_D f\, |dx|$.

• for general $\varphi: D^k \to U \subset \mathbb{R}^n$, $\int_{\varphi(D)} \omega = \int_D \varphi^* \omega$ k-form on $D \subset \mathbb{R}^k$ $\Rightarrow$ usual integral.

* Can similarly integrate k-forms over $M =$ finite union of parametrized pieces.

* <u>pullback formula</u> : given a smooth map $\varphi: U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$, $\omega \in \Omega^k(V)$, and a k-dimensional $M \subset U$: $\boxed{\int_{\varphi(M)} \omega = \int_M \varphi^* \omega}$.

This is basically equivalent to change of variables formula for usual $\int_D f\, |dx|$, and implies that $\int_M \omega$ is independent of the manner in which we parametrize M as the image of a map $\varphi: D \to U$ (or union of pieces) as long as all reparametrizations are <u>orientation-preserving</u>

(ie. we compose $\varphi: D \to U$ with a diffeomorphism $g: D' \to D$ st. $\det(Dg) > 0$ everywhere). ($D' \subset \mathbb{R}^k$, $D \subset \mathbb{R}^k$)

<u>Ex</u>: $\omega = \frac{x\, dy - y\, dx}{x^2 + y^2}$ on $\mathbb{R}^2 - \{0\}$, $C_r =$ circle of radius r, oriented counterclockwise; (as path $(r,0) \to (r,0)$)

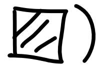
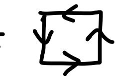
Pulling back via $\varphi: (r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, (polar coordinates),

$$\varphi^* \omega = \frac{(r\cos\theta)(r\cos\theta\, d\theta) - (r\sin\theta)(-r\sin\theta\, d\theta)}{r^2} = d\theta$$

So $\int_{C_r} \omega = \int_{\{r\} \times [0, 2\pi]} \varphi^* \omega = \int_0^{2\pi} d\theta = 2\pi$ (independent of r)

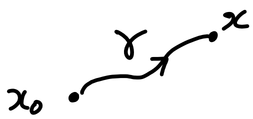
<u>Note</u>: $d\omega = 0$ (by direct calc. or using $\varphi^*(d\omega) = d(\varphi^* \omega) = d(d\theta) = 0$)

ie. ω is closed, but not exact! if $\exists f(x,y)$ on $\mathbb{R}^2 - \{0\}$ st $df = \omega$ then path integral $\int_{C_r} \omega = \int_{C_r} df = f(r,0) - f(r,0) = 0$. $H^1_{dR}(\mathbb{R}^2 - 0) \neq 0$.

Stokes' theorem: for $M \subset \mathbb{R}^n$ parametrized as $\varphi(D)$, $D \subset \mathbb{R}^k$ k-cell (or other nice domain)
define $\partial M = (k-1)$-dimensional boundary $\varphi(\partial D)$ (for $D = \prod [a_i, b_i]$, this has $2k$ pieces!)
with suitable orientation. (most relevant to us: $\partial($  $) =$  $)$.

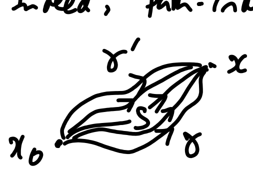
Stokes' thm: $\quad \forall \omega \in \Omega^{k-1}, \ \int_M d\omega = \int_{\partial M} \omega.$

Application: if ω is a closed 1-form on a simply connected $U \subset \mathbb{R}^n$, the path-integral



$\int_\gamma \omega$ is indept of choice of path γ from base point x_0 to x.

Indeed, path-independence comes from Stokes for the surface S traced by a path homotopy:



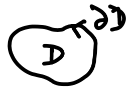
$d\omega = 0 \Rightarrow \quad 0 = \int_S d\omega = \int_{\partial S = \gamma - \gamma'} \omega = \int_\gamma \omega - \int_{\gamma'} \omega$

So we can define $f(x) = \int_\gamma \omega$ for any path $\gamma : x_0 \to x$.

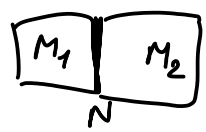
Stokes again (= fund. thm. calc.) gives $\quad \int_\gamma df = f(x) - f(x_0) = \int_\gamma \omega \quad \forall$ path γ,

and we find that $\omega = df$ is exact. ($\Rightarrow$ Poincaré lemma).

Rmk: Stokes' theorem for diff. forms in $\mathbb{R}^2$ and $\mathbb{R}^3$ specializes to all the theorems of multivariable calculus
$$\left(\begin{array}{l} k=0: \text{fund. thm. of calc. for path integrals} \\ k=1: \text{Green's theorem in } \mathbb{R}^2, \text{ curl in } \mathbb{R}^3 \\ k=2 \text{ in } \mathbb{R}^3: \text{Gauss / divergence thm.} \end{array}\right)$$

The most useful case for cx. analysis is: $D \subset \mathbb{R}^2$  $\Rightarrow \int_{\partial D} p\,dx + q\,dy = \int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx \wedge dy.$

Sketch proof:

- both sides obey pullback formula (using $\varphi^* d\omega = d(\varphi^* \omega)$, and $\partial\varphi(M) = \varphi(\partial M)$).
  so can do changes of coordinates / pullback by parametrization $D \xrightarrow{\varphi} M$.
- can decompose into pieces (either by writing ω as sum of forms with support contained in subsets that have a single parametrization, or by observing that if $M = M_1 \cup M_2$, $M_1 \cap M_2 = N \subset \partial M_i$  then ∂M_1 and ∂M_2 contain N with opposite orientations, and so
$$\int_M d\omega = \int_{M_1} d\omega + \int_{M_2} d\omega \ \ \& \ \ \int_{\partial M} \omega = \int_{\partial M_1} \omega + \int_{\partial M_2} \omega.$$
- over a k-cell, and considering each component of $\omega \in \Omega^{k-1}$ separately: eg.
  $D = \prod_{i=1}^k [a_i, b_i] = D' \times [a_k, b_k]$: $\quad \omega = f\, dx_1 \wedge \dots \wedge dx_{k-1} \Rightarrow d\omega = (-1)^{k-1} \frac{\partial f}{\partial x_k} dx_1 \wedge \dots \wedge dx_{k-1} \wedge dx_k$

So $\int_D d\omega = \int_D (-1)^{k-1}\frac{\partial f}{\partial x_k}|dx| = \int_{D'}\left(\int_{a_k}^{b_k}(-1)^{k-1}\frac{\partial f}{\partial x_k}dx_k\right)dx_1\dots dx_{k-1}$ (iterated $\int$)

$\underset{\text{fund. th. calc.}}{=} (-1)^{k-1}\int_{D'}\left(f(x_1\dots x_{k-1},b_k) - f(x_1\dots x_{k-1},a_k)\right)dx_1\dots dx_{k-1}$

$= (-1)^{k-1}\left(\int_{D'\times\{b_k\}}\omega - \int_{D'\times\{a_k\}}\omega\right) = \int_{\partial D}\omega$

using that $\int\omega$ vanishes on the other faces of D ($\perp (x_1\dots x_{k-1})$-plane) and orientation convention for ∂D (which we didn't state but is designed to make this work). $\square$

Our next topic: <u>Complex analysis</u> (in 1 complex variable)

We'll study functions $f: U \underset{\text{open}}{\subset} \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$.

Writing $z = x+iy$, these are instances of functions $\mathbb{R}^2 \to \mathbb{R}^2$, and the notion of continuity is the same, but we introduce a different (more restrictive) notion of differentiability.

<u>Def</u>: The (<u>complex</u>) <u>derivative</u> of f at $z\in U$ (if it exists) is

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \qquad (\text{ie. } f(z+h) = f(z) + hf'(z) + o(|h|))$$

The catch is: this limit has to hold for $h\to 0$ <u>in $\mathbb{C}$</u> ...

<u>Def</u>: We say $f: U\to\mathbb{C}$ is <u>analytic</u> (or <u>holomorphic</u>) if $f'(z)$ exists for all $z\in U$.

<u>Ex</u>: • assume f only takes real values, $f(z)\in\mathbb{R}$ $\forall z\in\mathbb{C}$ ... then in the def$^{\underline{n}}$ the numerator is always real, so taking $h\to 0$ in $\mathbb{R}$ we get $f'(z)\in\mathbb{R}$, while taking h imaginary we get $f'(z)\in i\mathbb{R}$. So: the complex derivative of a function which takes real values either doesn't exist or is equal to 0 ...!

<u>Complex vs. real differentiability</u>: we can treat $f: U\to\mathbb{C}$ as a function of 2 real variables $x+iy$. If $f'(z)$ exists then, taking h real, resp. imaginary, we find:

$$\left.\begin{aligned} f'(z) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}}\frac{f((x+h)+iy)-f(x+iy)}{h} = \frac{\partial f}{\partial x}\\ f'(z) &= \lim_{ih\to 0,\, ih\in i\mathbb{R}}\frac{f(x+i(y+h))-f(x+iy)}{ih} = -i\frac{\partial f}{\partial y}\end{aligned}\right\} \Rightarrow \underbrace{\boxed{\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}}_{\text{Cauchy-Riemann eq.}}$$

Equivalently, writing $f = u+iv$ for real-valued functions $u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$, this becomes $\begin{cases}\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}\\[2mm] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}\end{cases}$, ie. $Df(z): \mathbb{R}^2\to\mathbb{R}^2$ is of the form $\begin{pmatrix} a & -b\\ b & a\end{pmatrix}$

This is the matrix of complex multiplication by $f'(z) = a+ib$ viewed as $\mathbb{R}$-linear operator on $\mathbb{R}\oplus i\mathbb{R}\simeq\mathbb{C}$.