Math 55b Homework 9

Due Wednesday April 6, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: Integration, differential forms, Stokes' theorem (Rudin chapter 10, or McMullen's notes section 9 through DeRham cohomology); complex analytic functions (Ahlfors chapter 2, or McMullen's notes section 10 through power series).

1. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a smooth loop enclosing a region U. Use Stokes' theorem to prove that the area of U is equal to $\frac{1}{2}\int_0^1 \det(\gamma(t), \gamma'(t))\, dt$.

2. Consider the 1-form $\alpha = x\,dy + y\,dz + z\,dx$ on $\mathbb{R}^3$, and the 2-form $\beta = d\alpha$. Let $C = [-1,1]^3$ be the unit cube in $\mathbb{R}^3$, and T its top face (the square $-1 \le x \le 1$, $-1 \le y \le 1$ in the plane $z = 1$).

(a) Calculate directly the integral of α along the boundary of T oriented counterclockwise (when seen from above the cube), by integrating along the four edges. Then verify that this is equal to the integral over T of the 2-form β, as predicted by Stokes' theorem.

(b) What values do you obtain if you carry out the same calculation for every face of the cube? (don't write down all the integrals! argue using symmetry). What do the six quantities add up to, when all faces are oriented consistently around the cube?

(c) Explain the result you found in (b) in two ways: (i) by expressing the sum as a sum of path integrals; (ii) by applying Stokes' theorem to the cube C.

3. (Spherical area in rectangular coordinates)

Consider the 2-form $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ on $\mathbb{R}^3$, and its integral $\int_U \omega$ over a sufficiently nice portion of the unit sphere, $U \subset S^2$. To make things more concrete we view U as the image of a domain $V \subset \mathbb{R}^2$ under some parametrization of the sphere $f : (u,v) \mapsto f(u,v) = (x(u,v), y(u,v), z(u,v))$. (The choice is irrelevant, but for example one could use spherical angles).

(a) Express the volume of the spherical shell Σ with base U, inner radius a and outer radius b, i.e. the image of $[a,b] \times V$ under the map $F(r,u,v) = r\,f(u,v)$, in terms of a, b, and $\int_U \omega$.

Hint: compare $F^*(dx \wedge dy \wedge dz)$ and $dr \wedge f^*(\omega)$, i.e. express $(dx \wedge dy \wedge dz)(\frac{\partial F}{\partial r}, \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v})$ in terms of r and $\omega(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$ (where we evaluate the latter expression at the point $f(u,v) \in S^2$).

(b) By considering the case $a = 1$ and $b = 1 + h$ for $h \to 0$ in the above, deduce that $\int_U \omega$ is equal to the area of U. You may use without proof the fact that the volume of the thin shell Σ is approximately the thickness h times the area of U.[1]

(c) Use Stokes' theorem to verify that $\int_{S^2} \omega$ (i.e., the area of S^2) is equal to 3 times the volume of the unit ball (in agreement with the formulas you learned in high school).

[1] One way to prove this is to observe that the volume of Σ is at least its thickness h times the area of its interior face U (on the unit sphere) and at most its thickness times the area of its exterior face (on the sphere of radius $b = 1 + h$), i.e. $h\,\mathrm{area}(U) \le \mathrm{vol}(\Sigma) \le h(1+h)^2\,\mathrm{area}(U)$. Thus $\lim_{h\to 0}(\mathrm{vol}(\Sigma)/h) = \mathrm{area}(U)$.

(d) (Optional, extra credit) Explain how the definition of ω and the results in (a)-(c) above generalize to the $(n-1)$-dimensional volume element on the unit sphere S^{n-1} in $\mathbb{R}^n$.

4. Use the 2-form $\sigma = \dfrac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$ on $\mathbb{R}^3 - \{0\}$ to show that the inclusion map $i : S^2 \to \mathbb{R}^3 - \{0\}$ is not smoothly homotopic to a constant map, i.e. there does not exist a smooth map $f : S^2 \times [0,1] \to \mathbb{R}^3 - \{0\}$ such that $f_{|S^2\times\{0\}} = i$ and $f_{|S^2\times\{1\}}$ is a constant map.

Hint: apply Stokes' theorem to the pullback of σ under f (or, if the idea of a 2-form on $S^2 \times [0,1]$ is too confusing, a closely related map whose domain is a spherical shell in $\mathbb{R}^3$). Feel free to rely on results of the previous problem to find the integral of σ on the unit sphere.

Optional: formulate and prove the analogous statement for the unit sphere in $\mathbb{R}^n$.

5. (a) For any smooth function $f : U \to \mathbb{C}$, $U \subset \mathbb{C}$, define

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Show that $df = (\partial f/\partial z)\,dz + (\partial f/\partial \overline{z})\,d\overline{z}$, and that f is analytic if and only if $\partial f/\partial \overline{z} = 0$, in which case $f'(z) = \partial f/\partial z$.

(b) Show that a smooth function $f : U \to \mathbb{C}$ is analytic if and only the function $g(z) = \overline{f(\overline{z})}$ (defined on the image of U by the complex conjugation map) is analytic.

6. (a) What is the general form of a rational function f whose absolute value is 1 at every point of the unit circle $|z| = 1$? How are the zeroes and poles of f related to each other?

(Hint: consider the rational function $g(z) = 1/\overline{f(1/\overline{z})}$)

(b) What is the general form of a rational function f which defines a homeomorphism from the closed unit disc $\{|z| \leq 1\}$ to itself? Show that the set of such rational functions is a group under composition (called the group of complex automorphisms of the unit disc).

7. (a) Express $f(z) = \dfrac{1}{(1-z)^m}$ as a power series in z, for m a positive integer.

(b) Show that, for every polynomial p, the power series $\sum p(n) z^n$ represents a rational function. What is the radius of convergence of the series? What are the poles of the rational function?

8. Suppose $f(z) = \sum a_n z^n$ is analytic over the unit disc (in particular the radius of convergence of the power series is at least 1). Prove that for any $r < 1$ we have $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta = \sum |a_n|^2 r^{2n}$.

(Hint: Fourier series.)

9. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?