We study functions $f: U \underset{\text{open}}{\subset} \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$.

Writing $z = x+iy$, these are instances of functions $\mathbb{R}^2 \to \mathbb{R}^2$, and the notion of continuity is the same, but we introduce a different (more restrictive) notion of differentiability.

<u>Def</u>: The <u>(complex) derivative</u> of f at $z \in U$ (if it exists) is

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \qquad (\text{ie. } f(z+h) = f(z) + hf'(z) + o(|h|)).$$

The catch is: this limit has to hold for $h \to 0$ <u>in $\mathbb{C}$</u> ...

<u>Def</u>: We say $f: U \to \mathbb{C}$ is <u>analytic</u> (or <u>holomorphic</u>) if $f'(z)$ exists for all $z \in U$.

<u>Ex</u>: • assume f only takes real values, $f(z) \in \mathbb{R}$ $\forall z \in \mathbb{C}$ ... then in the def$^{\underline{n}}$ the numerator is always real, so taking $h \to 0$ in $\mathbb{R}$ we get $f'(z) \in \mathbb{R}$, while taking h imaginary we get $f'(z) \in i\mathbb{R}$. So: the complex derivative of a function which takes real values either doesn't exist or is equal to 0...!

<u>Complex vs. real differentiability</u>: we can treat $f: U \to \mathbb{C}$ as a function of 2 real variables $x+iy$. If $f'(z)$ exists then, taking h real, resp. imaginary, we find:

$$\left.\begin{aligned} f'(z) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f((x+h)+iy) - f(x+iy)}{h} = \frac{\partial f}{\partial x} \\ f'(z) &= \lim_{\substack{ih\to 0\\ ih\in i\mathbb{R}}} \frac{f(x+i(y+h)) - f(x+iy)}{ih} = -i\frac{\partial f}{\partial y} \end{aligned}\right\} \Rightarrow \underbrace{\boxed{\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}}_{\text{Cauchy-Riemann eq.}}$$

Equivalently, writing $f = u+iv$ for real-valued functions $u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$, this becomes

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}, \quad \text{ie. } Df(z): \mathbb{R}^2 \to \mathbb{R}^2 \text{ is of the form } \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

This is the matrix of complex multiplication by $f'(z) = a+ib$ viewed as $\mathbb{R}$-linear transformation on $\mathbb{R} \oplus i\mathbb{R} \simeq \mathbb{C}$.

In the language of differentials, df ($= du + i\,dv$) complex valued 1-form on $U \subset \mathbb{R}^2$ can be written in terms of $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$ as:

$$\begin{aligned} df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy &= \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)}_{} (dx+i\,dy) + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}_{} (dx - i\,dy) \\ &= \underset{(\text{def.})}{\partial f/\partial z}\; dz + \partial f/\partial \bar{z}\; d\bar{z} \qquad (*) \end{aligned}$$

Then: if $f'(z)$ exists then
$$\begin{cases} \dfrac{\partial f}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) = 0 & \text{(Cauchy-Riemann eq.)} \\ \dfrac{\partial f}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right) = f'(z). \end{cases}$$

Conversely! if f is real differentiable at z then (*) gives
$$f(z+h) = f(z) + \underset{\substack{\uparrow \\ \text{linear } \mathbb{R}^2 \to \mathbb{R}^2,}}{Df(z)}\,h + o(|h|) = f(z) + \frac{\partial f}{\partial z} h + \frac{\partial f}{\partial \bar z}\bar h + o(|h|)$$
$$Df(z) = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar z}\,d\bar z$$

$\Rightarrow$ the complex derivative exists iff $\dfrac{\partial f}{\partial \bar z} = 0$.

$\rightarrow$ <u>Prop:</u> f is analytic $\Leftrightarrow$ f is differentiable and $Df \in \left\{\begin{pmatrix} a & -b \\ b & a\end{pmatrix},\ a,b \in \mathbb{R}\right\}$ $= \mathbb{R}\cdot SO(2) \subset M_{2\times 2}(\mathbb{R})$

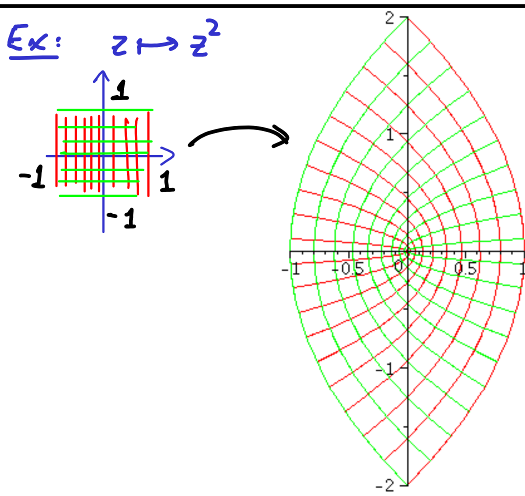
(rescale + rotate: <u>conformal</u> transformations)

$\Leftrightarrow \dfrac{\partial f}{\partial \bar z} = 0$

$\Leftrightarrow \dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y} = 0$

(Cauchy Riemann eqn)

<u>Ex:</u> $z \mapsto z^2$



<u>Remark:</u> geometrically, conformal transformations of the plane preserve angles between vectors (and orientation). So: analytic functions in 1 complex variable are conformal mappings (differentiable, in 2 real variables). If you draw a square grid in the plane and map it by f, the resulting curves meet at right angles everywhere.

* <u>The miracle:</u> even though analyticity only requires the existence of a complex derivative, it has many far reaching consequences, which we'll see and prove in next few classes.

Among these: 1) if $f: U \to \mathbb{C}$ is analytic then it has derivatives to all orders! (unlike real case where eg. $f(x) = x^{7/3}$ is only C^2, not C^∞)

2) the Taylor series expansion of f at any point $z_0 \in U$ is convergent and equal to f over a disc $B_r(z_0) \subset U$; in particular $f(z_0+h)$ can be expressed as a power series in h! (unlike: $f(x) = \exp(-\frac{1}{x^2})$ has all derivatives zero at $x=0$, so Taylor series converges to 0, not f).

3) local determination: if $f, g: U \to \mathbb{C}$ analytic, U connected, $f = g$ over any subset of U that has a limit point (eg. a small ball, or a small real interval, or...) then $f = g$ on all of U!!!

... and more! But first let's see examples and work out basic properties.

<u>Ex</u>: • <u>polynomials $\mathbb{C}[z]$</u>: $P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{i=1}^{n} (z-\alpha_i)$ are analytic,

and the complex derivative = usual derivative

(follows from usual rules of differentiation, which hold in the complex case too).

→ by contrast, a polynomial in 2 variables $P(x,y)$ can be rewritten as a polynomial in $z, \bar{z}$ (set $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$), $\mathbb{C}[x,y] \simeq \mathbb{C}[z,\bar{z}]$.

Check: $\frac{\partial}{\partial z}(z^k \bar{z}^\ell) = k z^{k-1} \bar{z}^\ell$, $\frac{\partial}{\partial \bar{z}}(z^k \bar{z}^\ell) = \ell z^k \bar{z}^{\ell-1}$, so such a polynomial is analytic iff there are no $\bar{z}$'s in the expression.

• <u>rational functions $\mathbb{C}(z)$</u>: $f(z) = \frac{P(z)}{Q(z)} = c\,\frac{\prod(z-\alpha_i)}{\prod(z-\beta_j)}$ (removing common factors we assume $\alpha_i \neq \beta_j \ \forall i,j$)

This function has <u>zeros</u> at the α_i, and <u>poles</u> at the β_j.

The <u>order</u> of a zero or pole is the multiplicity of the root α_i or β_j in P or Q.

Rational functions are analytic on their domain of definition $= \mathbb{C} - \{\text{poles}\}$.

• They are also conveniently viewed as functions on the <u>Riemann sphere</u> $S = \mathbb{C} \cup \{\infty\}$ (= 1-point compactification of $\mathbb{C}$), with values in S.

Namely $f(z) = \frac{P(z)}{Q(z)}$ has a unique extension to a continuous map $S \to S$, under which poles $\mapsto \infty$, and at $z=\infty$ we have

$\infty \mapsto \lim_{z\to\infty} \frac{P(z)}{Q(z)} \in \mathbb{C} \cup \{\infty\}$ $\begin{cases} \text{pole of order } \deg Q - \deg P & \text{if } \deg Q > \deg P. \\ \text{zero of order } \deg P - \deg Q & \text{if } \deg P > \deg Q. \end{cases}$

$\Rightarrow$ as a map $S \to S$, #poles (with multiplicities) = # zeros (with mult.)

$= \max(\deg P, \deg Q) =: \underline{\deg(f)}$.

<u>Note</u>: $\forall c \in S$, the eq^n $f(z) = c$ also has exactly $\deg(f)$ sol^s (with multiplicities). This is because for $c \in \mathbb{C}$, $\deg(f-c) = \deg(f)$. (The roots of $f-c$ are those of $P - cQ$...).

<u>Ex</u>: • $f(z) = z^2$ zero of order 2 at $z=0$, pole of order 2 at $z=\infty$ • $f(z) = \frac{z}{z^2-1}$ zeros of order 1 at $z=0$ and ∞, poles of order 1 at $z=\pm 1$

<u>Note</u>: the statement that rational functions are analytic maps $S \to S$ can be understood near $z = \infty$ by working via change of coords. $z = \frac{1}{w}$: $f(z)$ is analytic near $z=\infty$ if $f\left(\frac{1}{w}\right)$ is analytic near $w=0$. Similarly, near infinite values (poles), consider $\frac{1}{f}$.

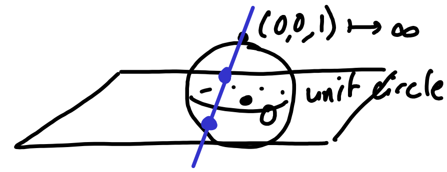
In fancier language, S is a Riemann surface, ie. has open cover by two subsets $S - \{\infty\} \simeq \underset{z}{\mathbb{C}}$ and $S - \{0\}$ also $\simeq \underset{w}{\mathbb{C}}$, and the change of coordinates $z = 1/w$ is analytic, so we can define analytic functions $S \to S$ = functions whose expressions in these coords. are analytic. But... don't need all this to study rational f^{ns}

Alternative viewpoint: (why "sphere"?)

- can identify S with the unit sphere in $\mathbb{R}^3$ by stereographic projection $S^2 \to \mathbb{C}\cup\infty$

$$(x,y,z) \longmapsto \frac{x+iy}{1-z} \quad \text{if } z<1$$
$x^2+y^2+z^2=1$

$$(0,0,1) \longmapsto \infty$$



<u>Fact:</u> This is a conformal map $S^2 \xrightarrow{\sim} \mathbb{C}\cup\infty$ (ie. preserves angles)

So... rational functions $f(z)=\frac{P(z)}{Q(z)}$ determine conformal maps $S^2\to S^2$ $(\deg(f)\text{-to-}1)$.
($\leftrightarrow$ analytic functions $S\to S$)

... and in fact <u>all</u> conformal maps $S\to S$ are given by rational functions!
(we can't prove this yet).

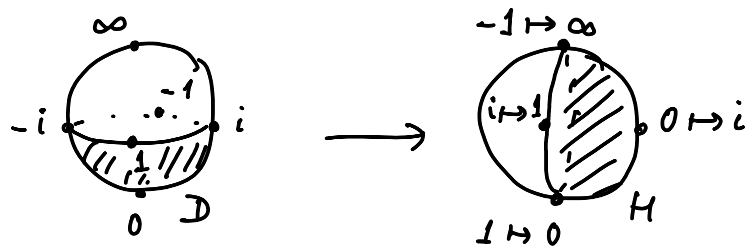
<u>Example:</u> the special case $\deg(f)=1$ is of particular interest – <u>fractional linear transformations</u>
(aka Möbius transformations)

$$f(z)=\frac{az+b}{cz+d}, \quad \det\begin{pmatrix}a&b\\c&d\end{pmatrix}\neq 0 \quad \text{(else common root)}.$$

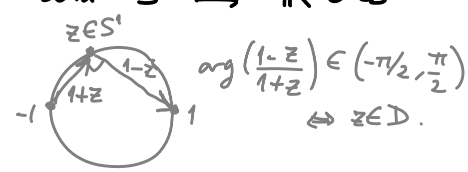
These are homeomorphisms $S\to S$ – the automorphisms of the Riemann sphere.
They form a group under composition! ($\leftrightarrow$ multiplication of matrices $\begin{pmatrix}a&b\\c&d\end{pmatrix}$!).

<u>Ex:</u> $f(z)=\frac{1}{z}$ maps $0\leftrightarrow\infty$
$S^1 \circlearrowleft$ by $e^{i\theta}\mapsto e^{-i\theta}$ (swaps hemispheres of S^2).

<u>Ex:</u> $f(z)=i\,\frac{1-z}{1+z}$ maps unit disk $D=\{|z|<1\} \xrightarrow[\text{analytic diffeomorphism}]{\sim} H=\{\operatorname{Im} z>0\}$ upper half plane



and $S^1 \to \mathbb{R}\cup\infty$



The analytic isom. $D\simeq H$ is important & useful in various areas of geometry.

- One way to understand the relation between $z\mapsto\frac{az+b}{cz+d}$ and $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ is to note that

$$\mathbb{CP}^1=\left(\mathbb{C}^2-0\right)/(z_1,z_2)\sim(\lambda z_1,\lambda z_2)\ \forall\lambda\in\mathbb{C}^* \xrightarrow{\sim} S$$

set of 1-dim $\mathbb{C}$ subspaces of $\mathbb{C}^2$

$$[z_1,z_2] \longmapsto z_1/z_2$$
$$[z,1] \longleftarrow z\in\mathbb{C}$$
$$[1,0] \longleftarrow \infty$$

and $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ maps $[z,1]\mapsto[az+b,cz+d]$.

* Since $\lambda.\mathrm{Id}$ acts by Id, we find $\mathrm{Aut}(S)\simeq PGL(2,\mathbb{C})\simeq SL(2,\mathbb{C})/\pm I$.

* $\mathrm{Aut}(S)$ acts simply transitively on triples of distinct points in S:

$\forall a_1,a_2,a_3\in S$ distinct, $\forall b_1,b_2,b_3\in S$ distinct, $\exists!\, f\in\mathrm{Aut}(S)$ st. $f(a_i)=b_i$.

* Main class of examples: <u>power series</u> $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (centred at $z=0$)

(or similarly, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ centred at z_0).

Recall the radius of convergence: $\frac{1}{R} = \limsup |a_n|^{1/n}$.
$R \in [0,\infty]$

→ for $|z|<R$, the series converges (absolutely: $\sum |a_n||z|^n$ converges) by the root test,

$$\limsup \left(|a_n z^n|^{1/n}\right) = \frac{|z|}{R} < 1 \Rightarrow \text{comparison with geometric series}$$

→ for $|z|>R$ the series diverges; for $|z|=R$ it depends...

→ convergence is uniform over smaller disc $\overline{D}_r = \{|z| \le r\}$ $\forall r<R$.

This is by the Weierstrass M-test: $\sup_{z\in\overline{D}_r} |a_n z^n| = |a_n| r^n$, $\sum |a_n| r^n$ converges $(r<R)$

$\Rightarrow \sum a_n z^n$ converges uniformly on $\overline{D}_r$.

This is because of uniform Cauchy criterion for partial sums $s_n = \sum_{k=0}^{n} a_k z^k$:

for $n>m\ge N$, $\sup_{z\in\overline{D}_r} |s_n(z)-s_m(z)| = \sup_{\overline{D}_r} \left|\sum_{m+1}^{n} a_k z^k\right| \le \sum_{m+1}^{n} |a_k| r^k \le \sum_{N+1}^{\infty} |a_k| r^k \xrightarrow[\text{as } N\to\infty]{} 0$

→ hence $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is continuous over $D_R = \{|z|<R\}$.

→ the series $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence as f;

the partial sums $s_n(z)$ are analytic, $\left.\begin{matrix} s_n \to f \\ s_n' \to g \end{matrix}\right\}$ uniformly on $\overline{D}_r$ $\forall r<R$

$\Rightarrow$ <u>Thm:</u> $\| f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on D_R and $f'(z) = g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

<u>PF.</u> We work on the smaller disk D_r $(r<R)$ where uniform convergence holds; $D_R = \bigcup_{r<R} D_r$

we've already seen that, for real f^ns of 1 real variable, $\left.\begin{matrix} s_n \to f \\ s_n' \to g \end{matrix}\right\}$ uniformly $\Rightarrow f'=g$.

Unfortunately the proof used mean value thm., which doesn't hold here. But for power series there's an easier proof using mean value inequalities, thanks to... bounds on s_n'', which also converges uniformly on $\overline{D}_r$ hence $\exists$ uniform bound $|s_n''(z)| \le M$ $\forall n\in\mathbb{N}$ $\forall z\in\overline{D}_r$.

So: for $z, z+h \in D_r$, mean value inequalities (for $s_n(z+th)$, $t\in[0,1]$)

imply $|s_n(z+h) - s_n(z) - s_n'(z)h| \le \frac{1}{2} M |h|^2$.

Taking limit as $n\to\infty$ we get $|f(z+h) - f(z) - g(z)h| \le \frac{1}{2} M |h|^2 \rightsquigarrow f'(z) = g(z)$. □