Math 556 Lecture 27 – Monday 4/4 (This week's office hrs Mon/Tue 8:30–9:30 in 530, Mon 1-2 & 3-4 in math lounge)

Recall: $f : U \subset \mathbb{C} \to \mathbb{C}$ is analytic if the complex derivative $f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists at every point of U ($\Leftrightarrow$ real differentiable, and solves Cauchy-Riemann eqⁿ $\dfrac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\dfrac{\partial f}{\partial x} + i \dfrac{\partial f}{\partial y}\right) = 0$).

* Ex: polynomials, rational functions $\dfrac{P(z)}{Q(z)}$
* Main class of examples: power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (centered at $z = 0$)

(or similarly, $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ centered at z_0).

Recall the series converges for $|z| < R = 1/\limsup |a_n|^{1/n}$, uniformly on $\overline{D}_r = \{|z| \le r\}$ $\forall r < R$, hence f is continuous over $D_R = \{|z| < R\}$.

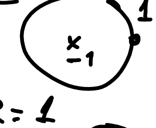
The series $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence as f;
the partial sums $s_n(z)$ are analytic, $\left.\begin{array}{l} s_n \to f \\ s_n' \to g \end{array}\right\}$ uniformly on $\overline{D}_r$ $\forall r < R$

$\Rightarrow$ Thm: $\Big\|$ $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on D_R and $f'(z) = g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

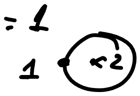
Pf: We work on the smaller disk D_r ($r < R$) where uniform convergence holds; $D_R = \bigcup_{r<R} D_r$
we've already seen that, for real fⁿˢ of 1 real variable, $\left.\begin{array}{l} s_n \to f \\ s_n' \to g \end{array}\right\}$ uniformly $\Rightarrow f' = g$.
Unfortunately the proof used mean value thm., which doesn't hold here. But for power series there's an easier proof using mean value inequalities, thanks to... bounds on s_n'', which also converges uniformly on $\overline{D}_r$ hence $\exists$ uniform bound $|s_n''(z)| \le M$ $\forall n \in \mathbb{N}$ $\forall z \in \overline{D}_r$.
So: for $z, z+h \in D_r$, mean value inequalities (for $s_n'(z+th)$, $t \in [0,1]$) give
first $|s_n'(z+th) - s_n'(z)| \le M t |h|$, and so $|s_n(z+h) - s_n(z) - s_n'(z) h| \le \frac{1}{2} M |h|^2$.
Taking limit as $n \to \infty$ we get $|f(z+h) - f(z) - g(z) h| \le \frac{1}{2} M |h|^2 \rightsquigarrow f'(z) = g(z)$. □

Ex: $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$ has $R = 1$. For $|z| = 1$ the series is always divergent (the terms don't $\to 0$), but the right hand side makes sense as soon as $z \ne 1$.

There are in fact expressions as power series over any disc not containing the pole $z = 1$. Eg, around $z_0 = -1$: $\dfrac{1}{1-z} = \dfrac{1}{2 - (z+1)} = \sum_{n=0}^{\infty} \dfrac{(z+1)^n}{2^{n+1}}$ $R = 2$



around $z_0 = 2$: $\dfrac{1}{1-z} = \dfrac{-1}{1 + (z-2)} = \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n$ $R = 1$



...

* Starting from $\sum z^n$, this process of extending past the disc of convergence is called <u>analytic continuation</u>; here it yields a rational function defined on $\mathbb{C}-\{1\}$.
Similarly for all rational functions! (on $\mathbb{C}-\{\text{poles}\}$). (eg. use partial fractions + case of $\frac{1}{(z-a)^k}$).
(but some power series have more serious divergences on ∂D_R and can't be continued).

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* <u>Def:</u> $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad R=\infty$, converges $\forall z \in \mathbb{C}$.

By algebraic manipulations, $\exp(z+w)=\exp(z)\exp(w)$. In particular $e^{-z}=\frac{1}{e^z}$
(remember: can multiply absolutely convergent series). $\quad e^z \neq 0 \quad \forall z \in \mathbb{C}$

$e^{x+iy} = e^x e^{iy}$ has $|\cdot| = e^x$ and $\arg = y$.

- Define $\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$, $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, ...
and usual properties follow (watch out if $z \notin \mathbb{R}$! $\cos(iy) = \cosh(y)$ ...).
- $\exp'(z) = \exp(z) \neq 0$ so $\exp$ is a local diffeomorphism near each point!
Globally, $\exp: \mathbb{C} \to \mathbb{C}^*$ is the universal covering map!

* What about logarithm? For $w \in \mathbb{C}^*$, want to define $\log(w) = z$ st. $e^z = w$.
Such z exists, but isn't unique: can add integer multiples of $2\pi i$.
$\mathrm{Re}\,\log(w)$ is well defined though, and equal to $\log|w|$ (for usual log on $\mathbb{R}_+$).
In general "$\log(w) = \log|w| + i\arg(w)$" not well def^d & continuous on $\mathbb{C}^*$;
but ok over simply connected subsets of $\mathbb{C}^*$ (so can't go around $0 \Rightarrow \arg$ well defined).
This is consistent with what we've seen about lifting problem for

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ U \arrow[r, "i"'] \arrow[ur, dashed] & \mathbb{C}^* \end{tikzcd}$$

The same issue comes up with defining z^a for $a \notin \mathbb{Z}$:
would like to define it as $z^a = \exp(a \log z)$, but this only works on suitable domains. Eg. $\sqrt{z}$ is multivalued ($\pm\sqrt{z}$) and we can't define a continuous function on a domain that encloses the origin.
There are still power series expressions away from origin. Eg:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots, \qquad \sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \dots \quad (R=1)$$

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* Now we consider <u>path integrals</u> of <u>complex 1-forms</u> $\omega = f(z)\,dz$:
given a continuous function $f: U \to \mathbb{C}$ and a (piecewise) differentiable path $\gamma: [0,1] \to \mathbb{C}$,

$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$ (or: pick points $z_i = \gamma(t_i)$ along the path, with $\operatorname{diam} \gamma([t_i,t_{i+1}]) < \varepsilon$, then $\int = \lim_{\varepsilon\to 0} \sum_i f(z_i)(z_{i+1}-z_i)$)

<u>Ex:</u> $a \xrightarrow{\gamma} b$ $\quad \int_\gamma z^n\,dz = \int_0^1 \gamma(t)^n \gamma'(t)\,dt = \frac{1}{n+1}(b^{n+1} - a^{n+1})$

→ for a power series $f(z)=\sum a_n z^n$, if γ is entirely contained in the disc of convergence, it follows that $\int_\gamma f(z)\,dz = F(b)-F(a)$, where $F(z)=\sum \frac{a_n}{n+1} z^{n+1}$: indeed $F'=f$ and so the equality follows from fundamental thm of calculus.

In general, a 1-form on $\mathbb{R}^2$ need not be exact & their path integrals need not be path-independent. One of the miracles is that things are much simpler in the analytic setting:

<u>Key result</u>: <u>Cauchy's theorem</u>:

‖ $D\subset\mathbb{C}$ bounded region with piecewise smooth boundary, $f(z)$ analytic on U open $\supset\overline{D}$
‖ Then $\int_{\partial D} f(z)\,dz = 0$.

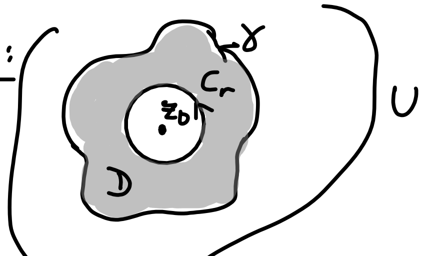
<u>Proof</u> <u>assuming f' is continuous</u>: the 1-form $\omega = f(z)\,dz$ is C^1, and
$d\omega = df\wedge dz = f'(z)\,dz\wedge dz = 0$. Stokes thm $\Rightarrow \int_{\partial D}\omega = \int_D d\omega = 0$. □

↳ let's check this more carefully, just to be safe:
$\left(\omega = f(z)\,dz = f(z)\,dx + i f(z)\,dy \Rightarrow d\omega = \left(-\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial x}\right) dx\wedge dy = 0 \text{ by Cauchy-Riemann}\right)$

We'll see later how to show that f analytic $\Rightarrow f'$ continuous. In the meantime we add the continuity of f' to our working assumptions.

* This holds not just for a simply connected region bounded by a simple closed curve! We can also allow holes in the region D, eg. around points where f isn't defined.

<u>Ex</u>:



f analytic on $U-\{z_0\}$, γ enclosing z_0 as shown
$$\Rightarrow \int_\gamma f(z)\,dz = \int_{C_r = S^1(z_0,r)} f(z)\,dz.$$
(by Cauchy's theorem: $\partial D = \gamma - C_r$.)

* Now assume f is analytic on $U-\{z_0\}$ <u>and</u> $\lim_{z\to z_0} (z-z_0) f(z) = 0$.

(eg enough for f to be bounded near z_0).

Then $\left|\int_{C_r} f(z)\,dz\right| \le \sup_{z\in C_r} |f(z)|\cdot \text{length}(C_r) = 2\pi r \sup_{z\in C_r}|f(z)| = 2\pi \sup_{z\in C_r} |(z-z_0) f(z)|$

Since this quantity $\to 0$ as $r\to 0$, and the path integral is independent of r, we get:

<u>Thm</u>: ("improved Cauchy") ‖ Cauchy's theorem $\left(\int_{\partial D} f(z)\,dz = 0\right)$ remains true under weaker assumption that
‖ f is defined & analytic in $D-\{z_0\}$, $z_0\in \text{int}(D)$, and $\lim_{z\to z_0}(z-z_0)f(z)=0$.

◦ However, we can't get rid of all assumptions about the behavior of f at z_0.

<u>Example</u>: $\int_{S^1(z_0,r)} (z-z_0)^n\,dz = \int_0^{2\pi} (re^{i\theta})^n\, ire^{i\theta}\,d\theta = \begin{cases} 0 & \text{if } n\neq -1 \\ 2\pi i & \text{if } n=-1. \end{cases}$

or any γ going once around z_0, by Cauchy! $z = z_0 + re^{i\theta}$

(cf. fundamental thm. / multivalued nature of log)

Using this, we get to <u>Cauchy's integral formula</u>:

<u>Thm</u>: $D \subset \mathbb{C}$ bounded region with piecewise smooth boundary γ, $f(z)$ analytic on an open domain containing $\overline{D}$, $z_0 \in \mathrm{int}(D)$ $\Rightarrow$ then
$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_0}.} \quad (*)$$

<u>Proof</u>: • since $\int_\gamma \frac{dz}{z-z_0} = 2\pi i$, the formula is equivalent to:
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$$

• The differentiability of f at z_0 implies: as $z \to z_0$, $\frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$,
and in particular $(z-z_0)\frac{f(z)-f(z_0)}{z-z_0} \to 0$. (+ analytic for $z \neq z_0$).
The result thus follows from improved Cauchy. □

<u>Alt. proof</u>: Cauchy's thm gives $\frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi i}\int_{S^1(z_0,r)} \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta \underset{r\to 0}{\longrightarrow} f(z_0)$. □

This is magical: the values of f at <u>every point</u> inside a closed curve γ can be determined by calculating path integrals on γ!! (assuming f defined and analytic everywhere in the enclosed region, of course). In this version, to emphasize we can vary the point of evaluation, one usually rewrites $(*)$ as: $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{w-z}$ $\forall z \in \mathrm{int}(D)$, $\partial D = \gamma$

* Next time we'll do even better: $\boxed{\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)^{n+1}}}$ $(**)$
($\Rightarrow$ all derivatives exist !!)

<u>Remark</u>: if f is given by a power series near z_0, $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ with $a_k = \frac{f^{(k)}(z_0)}{k!}$, then for $\gamma = S^1(z_0,r)$ small circle ($r <$ radius of convergence), uniform convergence of the series implies
$$\frac{1}{2\pi i}\int_{S^1(z_0,r)} \frac{f(w)\,dw}{(w-z_0)^{n+1}} = \sum_{k=0}^{\infty} \frac{a_k}{2\pi i}\int_{S^1(z_0,r)} \frac{(w-z_0)^k}{(w-z_0)^{n+1}}\,dw = a_n = \frac{f^{(n)}(z_0)}{n!} \checkmark$$
calc. $= 0$ for $k \neq n$, $2\pi i$ $k = n$

+ Cauchy implies $\int_\gamma = \int_{S^1(z_0,r)}$.

But the problem is ... we haven't shown yet that analytic functions are power series! in fact the proof uses Cauchy's formula... so instead we have to work.

<u>Prop</u>: suppose $\varphi(w)$ is continuous on $\gamma = \partial D$. Then $\forall n \geq 1$, $g_n(z) = \int_\gamma \frac{\varphi(w)\,dw}{(w-z)^n}$ is analytic in the interior of D, and $g_n'(z) = n\int_\gamma \frac{\varphi(w)\,dw}{(w-z)^{n+1}} = n g_{n+1}(z)$.

Pf + how to derive $(**)$ from this: next time.