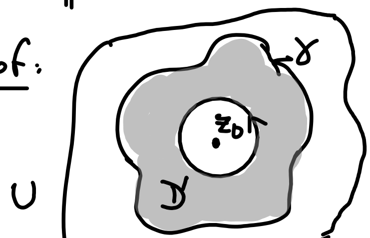
<u>Cauchy's theorem</u>: $D \subset \mathbb{C}$ bounded region with piecewise smooth boundary, $f(z)$ analytic on U open $\supset \overline{D}$: Then $\int_{\partial D} f(z)\,dz = 0$.

* proved so far under extra assumption that f' is continuous, via Stokes: $d(f(z)\,dz) = 0$.
* can allow f to be analytic on $U \setminus \{z_0\}$, $z_0 \in \mathrm{int}(D)$, s.t. $\lim_{z \to z_0} (z-z_0) f(z) = 0$.

Key consequence: <u>Cauchy's integral formula</u>:

<u>Thm</u>: $D \subset \mathbb{C}$ bounded region with piecewise smooth boundary γ, $f(z)$ analytic on an open domain containing $\overline{D}$, $z_0 \in \mathrm{int}(D)$ $\Rightarrow$ then $\boxed{f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_0}}$.

<u>Proof</u>:



Apply Cauchy's theorem to $\frac{f(z)}{z-z_0}$ on $D' = D \setminus B_r(z_0) \subset U \setminus \{z_0\}$ with boundary $\partial D' = \gamma - S^1(z_0, r)$

$$\Rightarrow \forall r>0 \text{ small, } \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi i}\int_{S^1(z_0,r)} \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta \xrightarrow[r\to 0]{\text{continuity of } f} f(z_0).$$

$$z = z_0 + re^{i\theta}, \quad \frac{dz}{z-z_0} = \frac{ire^{i\theta}\,d\theta}{re^{i\theta}} = i\,d\theta \qquad \square$$

This is magical: the values of f at <u>every point</u> inside a closed curve γ can be determined by calculating path integrals on γ!! (assuming f defined and analytic everywhere in the enclosed region, of course). In this version, to emphasize we can vary the point of evaluation, one usually rewrites (*) as: $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{w-z}$ $\forall z \in \mathrm{int}(D)$, $\partial D = \gamma$

Even better: $\boxed{\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)^{n+1}}}$ ($\Rightarrow$ all derivatives exist!!)

<u>Remark</u>: if f is given by a power series near z_0, $f(z) = \sum_{k=0}^\infty a_k (z-z_0)^k$ with $a_k = \frac{f^{(k)}(z_0)}{k!}$, then for $\gamma = S^1(z_0, r)$ small circle ($r <$ radius of convergence), uniform convergence of the series implies

$$\frac{1}{2\pi i}\int_{S^1(z_0,r)} \frac{f(w)\,dw}{(w-z_0)^{n+1}} = \sum_{k=0}^\infty \frac{a_k}{2\pi i}\underbrace{\int_{S^1(z_0,r)} \frac{(w-z_0)^k}{(w-z_0)^{n+1}}\,dw}_{\text{calc.} = 0 \text{ for } k\neq n,\ 2\pi i \text{ for } k=n} = a_n = \frac{f^{(n)}(z_0)}{n!} \checkmark$$

+ Cauchy implies $\int_\gamma = \int_{S^1(z_0,r)}$.

But the problem is ... we haven't shown yet that analytic functions are power series! in fact the proof uses Cauchy's formula ... so instead we have to work.

Prop: Suppose $\varphi(w)$ is continuous on $\gamma = \partial D$. Then $\forall n \geq 1$, $g_n(z) = \int_\gamma \frac{\varphi(w)\,dw}{(w-z)^n}$ is analytic in the interior of D, and $g_n'(z) = n\int_\gamma \frac{\varphi(w)\,dw}{(w-z)^{n+1}} = n g_{n+1}(z)$.

Proof: We first prove that g_n is continuous on $\mathrm{int}(D)$.

Fix $z_0 \in \mathrm{int}(D)$, with $B_{2\delta}(z_0) \subset D$, and let $z \in B_\delta(z_0)$ (so z and z_0 are farther than δ away from all points of γ). Calculate:

$$\frac{1}{(w-z)^n} - \frac{1}{(w-z_0)^n} = \sum_{k=1}^n \frac{1}{(w-z)^{n-k}(w-z_0)^{k-1}}\left(\frac{1}{w-z} - \frac{1}{w-z_0}\right) = \sum_{k=1}^n \frac{z-z_0}{(w-z)^{n+1-k}(w-z_0)^k}$$

So:
$$g_n(z) - g_n(z_0) = \int_\gamma \varphi(w)\left(\frac{1}{(w-z)^n} - \frac{1}{(w-z_0)^n}\right)dw$$
$$= (z-z_0)\int_\gamma \varphi(w)\left(\sum_{k=1}^n \frac{1}{(w-z)^{n+1-k}(w-z_0)^k}\right)dz$$

Since each term in the sum has $|\cdot| \leq \frac{1}{\delta^{n+1}}$, this implies

$$\Rightarrow \quad |g_n(z) - g_n(z_0)| \leq |z-z_0| \cdot \left(\sup_{w\in\gamma} |\varphi(w)|\right) \cdot \frac{n}{\delta^{n+1}}\,\mathrm{length}(\gamma).$$

Taking $z \to z_0$ this inequality proves that g_n is continuous at z_0, ie. g_n is continuous on $\mathrm{int}(D)$. Moreover,

$$\frac{g_n(z) - g_n(z_0)}{z-z_0} = \sum_{k=1}^n \int_\gamma \frac{\varphi(w)}{(w-z)^{n+1-k}(w-z_0)^k}\,dw \qquad (*)$$

The continuity result, now applied to $\frac{\varphi(w)}{(w-z_0)^k}$, shows that the terms in the rhs. are continuous functions of $z \in \mathrm{int}(D)$, hence the rhs. of $(*)$ is continuous, and its limit at $z = z_0$ equals $n\int_\gamma \frac{\varphi(w)}{(w-z_0)^{n+1}}\,dw = n g_{n+1}(z_0)$.

This gives the existence of $g_n'(z_0) = \lim_{z\to z_0} \frac{g_n(z) - g_n(z_0)}{z-z_0} = \lim_{z\to z_0} (\text{rhs of } (*)) = n g_{n+1}(z_0)$.

This holds $\forall z_0 \in \mathrm{int}(D)$, hence g_n is analytic as claimed and $g_n'(z) = n g_{n+1}(z)$. $\square$

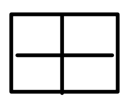
• Now if f is analytic in $U \supset \bar{D}$ then by Cauchy's integral formula, $2\pi i f(z) = \int_\gamma \frac{f(w)\,dw}{w-z}$ is the expression denoted $g_1(z)$ in the proposition, for $\varphi = f_{|\gamma}$.

The proposition then shows that f is infinitely differentiable, all derivatives are analytic, and $2\pi i f^{(n)}(z) = n!\, g_{n+1}(z)$, ie. $\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)^{n+1}}$. $\square$.

* This also lets us lift the extra assumption we've made so far in all proofs using Cauchy's theorem, that f' is continuous.

Prop: If f is analytic then f' is continuous.

Pf: If f is analytic in a disc $D \ni z_0$, define $F(z) = \int_{z_0}^{z} f(w)\,dw$ where we choose a path consisting of horizontal & vertical line segments.
We don't have the full strength of Stokes' theorem (don't know f' continuous), but we claim it holds for <u>rectangles</u>: R $\int_{\partial R} f(w)\,dw = 0$. (see below).
Given this, our def^n of F makes sense & doesn't depend on path. We claim F is analytic and $F' = f$. Indeed: $F(z+h) - F(z) = \int_\gamma f(w)\,dw$ where $\gamma =$ $z \to z+h$
Using continuity of f, as $h \to 0$ we have $\sup_{w \in \gamma} |f(w) - f(z)| \to 0$,
hence $F(z+h) - F(z) = h f(z) + o(|h|)$, hence $F'(z) = f(z)$.
So now F is analytic with continuous derivative $F' = f$, so we can apply Cauchy's integral formula and the above argument to F, so F has derivatives to all orders.
In particular $F''(z) = f'(z)$ is continuous. $\square$

<u>Cauchy's theorem on rectangles</u> (without assuming f' continuous):
Assume $R = R_0$ is a rectangle, f analytic, and $I = \int_{\partial R} f(z)\,dz \neq 0$
Cut R into 4 equal rectangles  then $\int_{\partial R} =$ sum of 4 path integrals, so
$\exists R_1 \subset R_0$ of $\operatorname{diam}(R_1) = \frac{\operatorname{diam}(R_0)}{2}$ st. $\left|\int_{\partial R_1} f(z)\,dz\right| \geq \frac{1}{4}|I|$. Repeat this process,
$R_0 \supset R_1 \supset R_2 \supset \dots$ with $\operatorname{diam}(R_n) = \frac{\operatorname{diam}(R_0)}{2^n}$ and $\left|\int_{\partial R_n} f(z)\,dz\right| \geq \frac{1}{4^n}|I|$.
$\bigcap_{n \in \mathbb{N}} R_n = \{z_0\}$ (a decreasing seq. of nonempty closed subsets in a compact space has a non-empty intersection: else complements would be an open cover w/out a finite subcover).
But now, $f(z) = f(z_0) + f'(z_0)(z - z_0) + r(z)$, $r(z) = o(|z - z_0|)$
$\Rightarrow \left|\int_{\partial R_n} f(z)\,dz\right| = \left|\int_{\partial R_n} r(z)\,dz\right| \leq \operatorname{length}(\partial R_n) \cdot \sup_{\partial R_n} |r(z)| = o\left(\frac{1}{4^n}\right)$. Contradiction. $\square$

Returning to Cauchy's integral formula for derivatives,

$f(z)$ analytic on $U \subset \mathbb{C} \Rightarrow f$ has derivatives to all orders in U, all derivatives are analytic, and for $z \in \operatorname{int}(D) \subset \bar{D} \subset U$, $\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)\,dw}{(w - z)^{n+1}}$.

From this we get, by bounding the integral in the r.h.s.

<u>Thm:</u> (<u>Cauchy's bound</u>) ‖ If f is analytic in $U \supset \overline{B_R(z_0)}$, then $\displaystyle \left|\frac{f^{(n)}(z_0)}{n!}\right| \le \frac{1}{R^n} \sup_{w \in S^1(z,R)} |f(w)|$.

(By considering $r<R$, $r \to R$, the result still holds under the weaker assumption that f is continuous on $\overline{B_R(z_0)}$ and analytic in $B_R(z_0)$).

* Cauchy's bound has important consequences for <u>entire functions</u>, ie. analytic on all of $\mathbb{C}$.

<u>Corollary:</u> ‖ If f is analytic on all of $\mathbb{C}$ ("entire function") and bounded, then f is constant.

(apply Cauchy's bound with $R \to \infty$ to get $f' = 0$.)

<u>Corollary:</u> ‖ A nonconstant entire function $f : \mathbb{C} \to \mathbb{C}$ has dense image $\overline{f(\mathbb{C})} = \mathbb{C}$.

<u>Pf:</u> if $c \notin \overline{f(\mathbb{C})}$, then $\exists\, \varepsilon > 0$ st. $|f(z) - c| \ge \varepsilon$ $\forall z \in \mathbb{C}$, and then $\dfrac{1}{f(z)-c}$ is a bounded entire function hence constant. □

* There are even more important consequences for <u>Taylor series</u> of analytic functions.

<u>Corollary:</u> ‖ The power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ (= the Taylor series of f at z_0) has radius of convergence $\ge R$, if f is analytic in $B_R(z_0)$.

(since Cauchy's bound implies $\displaystyle\left|\frac{f^{(n)}(z_0)}{n!}\right|^{1/n} \le \frac{C(r)^{1/n}}{r}$ $\forall r < R$, so $\limsup \le \frac{1}{r} \Rightarrow \le \frac{1}{R}$)

<u>Theorem:</u> ‖ If f is analytic in $B_R(z_0)$ then $\displaystyle f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $\displaystyle a_n = \frac{f^{(n)}(z_0)}{n!}$, over $B_R(z_0)$.

<u>Pf:</u> By change of variables $z - z_0$, we assume $z_0 = 0$. We prove the equality over slightly smaller discs $B_r = \{|z| < r\}$ $\forall r < R$; the Taylor series converges by the previous Corollary. For $z \in B_r$, write $\displaystyle f(z) = \frac{1}{2\pi}\int_{S^1(r)} \frac{f(w)\,dw}{w-z}$ and note $\displaystyle \frac{1}{w-z} = \frac{1}{w(1-z/w)} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n$.

For fixed $z \in B_r$ this series of functions of $w \in S^1(r)$ converges uniformly (Weierstrass M-test, $\sum \left(\frac{|z|}{r}\right)^n$ converges since $|z| < r$).

So $$\frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)\,z^n}{w^{n+1}}\,dw \underset{\substack{\uparrow \\ \text{Cauchy integral formula}}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n. \qquad \square$$