Math 55b Homework 10

Due Wednesday April 13, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.

Material covered: Cauchy's integral formula and applications, Taylor series (Ahlfors sections 4.1-4.3.2 and 5.1.2, or McMullen section 10)

1. Recall that the derivative of the arctangent function is $\arctan'(z) = 1/(z^2+1)$.

(a) Express the arctangent function as a power series. What is the radius of convergence?

(b) Use partial fractions to find an expression for $\arctan(z)$ in terms of complex logarithms (at least in a neighborhood of $z = 0$; for which values of z does your formula make sense?)

(c) Calculate $\displaystyle\int_C \frac{dz}{z^2+1}$, where C is the circle $|z| = 2$ oriented counterclockwise.

2. Suppose $f(z)$ is analytic over an open set $U \subset \mathbb{C}$, and γ is a smooth closed curve contained in U. Show that $\displaystyle\int_\gamma \overline{f(z)}\, f'(z)\, dz$ is purely imaginary.

3. For $n, m \in \mathbb{Z}$, calculate: (a) $\displaystyle\int_{|z|=1} z^{-n}\, e^z\, dz$, (b) $\displaystyle\int_{|z|=2} z^n (1-z)^m\, dz$.

4. Calculate $\displaystyle\int_{|z|=r} \frac{|dz|}{|z-a|^2}$ for $|a| \neq r$. Here $|dz|$ stands for the length element, i.e. given a parametrized path $\gamma : [a,b] \to \mathbb{C}$ we define $\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))\, |d\gamma/dt|\, dt$.

(Hint: observe that, on the circle of radius r, $\bar{z} = r^2/z$ and $|dz| = -ir\, dz/z$.)

5. Show that if f is analytic in a neighborhood of 0 and n is a positive integer then there exists an analytic function g in a neighborhood of 0 such that $f(z^n) = f(0) + g(z)^n$.

6. Let $f_n(z)$ be a sequence of analytic functions on the unit disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$, converging uniformly to $f(z)$.

(a) Show that $f_n'(z)$ converges uniformly to $f'(z)$ on $\overline{D}_r = \{z \in \mathbb{C} \,|\, |z| \leq r\}$ for all $r < 1$.

(b) Give an example showing that $f_n'(z)$ does not necessarily converge uniformly to $f'(z)$ on D.

7. (a) Show that, if $f(z)$ is analytic in the unit disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$, $f(0) = 0$, and $|f(z)| < 1$ for all $z \in D$, then $|f'(0)| \leq 1$. (Hint: Cauchy's bound).

(b) Show that, if $f(z)$ is analytic in the unit disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ and $|f(z)| < 1$ for all $z \in D$, then

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \quad \text{for all } z \in D.$$

(Hint: use fractional linear transformations to reduce to the case where $z = f(z) = 0$.)

8. (a) Show that, if $f(z)$ is analytic in the disc $D_r = \{z \,|\, |z| < r\}$ and $\operatorname{Re} f(z) < a$ for all $z \in D_r$ for some constant $a \in \mathbb{R}$, then

$$|f'(z)| \leq \frac{2r(a - \operatorname{Re} f(z))}{r^2 - |z|^2} \quad \text{for all } z \in D_r.$$

(Hint: consider $g(z) = f(rz)/(2a - f(rz))$ and use the previous problem.)

(b) Show that, if $f(z)$ is analytic in the disc $D_r = \{z \,|\, |z| < r\}$, $f(0) = 0$, and $\operatorname{Re} f(z) < a$ for all $z \in D_r$ for some constant $a > 0$, then $|f(z)| < a$ for all $z \in D_{r/5}$.

(Hint: use the result of (a) to give a bound on $|f'(z)|$ for all $z \in D_{r/5}$ at which $|f(z)| < a$. Then use the mean value inequality to show that the latter condition must hold at every point of $D_{r/5}$.)

9. (a) Prove that, if $f(z)$ is analytic in the whole complex plane and there exist constants $A, B > 0$ and an integer $n \geq 0$ such that $|f(z)| \leq A|z|^n + B$ for all $z \in \mathbb{C}$, then f is a polynomial.

(Hint: consider Cauchy's bound for $f^{(n+1)}$.)

(b) Show that, if $f(z)$ is analytic in the whole complex plane and there exist constants $A, B > 0$ and an integer $n \geq 0$ such that $\operatorname{Re} f(z) \leq A|z|^n + B$ for all $z \in \mathbb{C}$, then f is a polynomial.

(Hint: Use the result of the previous problem to find a bound on $|f(z)|$.)

10. Find all analytic functions $f(z)$ on the whole complex plane such that f never takes the value zero and there exist $c_1, c_2 > 0$ such that $|f(z)| \leq c_1 \exp(c_2|z|^2)$ for all $z \in \mathbb{C}$.

11. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?