

* Last time we saw several remarkable consequences of Cauchy's formula, including:

- every analytic $f: U \rightarrow \mathbb{C}$ can be expressed as a power series on any disc $\subset U$, namely its Taylor series at the center of the disc
- this implies that the zeros of f are isolated in U . (unless $f \equiv 0$).
- Cauchy's formula implies bounds $\frac{f^{(k)}(z_0)}{k!} \leq \frac{1}{r^k} \sup_{S^1(z_0, r)} |f|$.
- this in turn implies that if f_n are analytic and $f_n \rightarrow f$ uniformly on a disc, then f is analytic and $f'_n \rightarrow f'$ (& $f''_n \rightarrow f''$, etc.) uniformly on a slightly smaller disc.

→ in other terms: analytic functions are a closed subspace of $C^0(U, \mathbb{C})$ with C^0_{loc} topology of (local) uniform convergence, and moreover the $C^0_{loc}, C^1_{loc}, C^2_{loc} \dots$ topologies all coincide when we restrict them to the subspace of analytic functions

(whereas in real analysis C^1 is strictly finer than C^0 , etc.).

(also contrast with Stone-Weierstrass, which says various classes of real functions are dense in C^0 , hence very far from closed...)

Ex: $\frac{1}{n} \sin(nx) \rightarrow 0$ uniformly on $\mathbb{R}$, but derivatives $\cos(nx)$ don't converge. (so $\nrightarrow 0$ in C^1_{loc}). Note the corresponding analytic functions $\frac{1}{n} \sin(nz)$ don't converge on any disc centered at 0 (grows exponentially whenever $z \notin \mathbb{R}$).

• There are also some less surprising facts (with some caveats): antiderivatives

• Thm: If $f(z)$ is analytic on a simply connected open $U \subset \mathbb{C}$ then $\exists$ analytic function $F: U \rightarrow \mathbb{C}$ st. $F'(z) = f(z)$.

This is because we can define $F(z) = \int_{z_0}^z f(z) dz$, Cauchy's thm implies that the choice of path doesn't matter: given any piecewise differentiable closed loop γ in U , $\int_{\gamma} f(z) dz = 0$. In fact, over discs $B_r(z_0) \subset U$ we can define F by term-by-term integration of the power series expression for f .

Simply connected is necessary! eg. $f(z) = \frac{1}{z}$ on $\mathbb{C}^* = \mathbb{C} - \{0\}$, can only integrate to $F(z) = \log z$ over a simply connected subset (not allowing paths that enclose 0).

• Thm: If f is analytic near a , with $f(a) = b$ and $f'(a) \neq 0$, then $\exists$ analytic inverse function g defined on a neighborhood of b , st $g(b) = a$ & $g'(b) = 1/f'(a)$.

(This is a direct consequence of the inverse function theorem for $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, together with observation that $f'(a) \neq 0 \Rightarrow Df(a)$ is invertible, and its inverse is also complex-linear.)

• We can use these to get things like the power series expansion of $\log(1+z)$ (the antiderivative of $\frac{1}{1+z}$, due to $\exp' = \exp$ & inverse f' -theorem)

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We'll now study the behavior of analytic functions at an isolated singularity, i.e. st. f is defined on $U - \{z_0\}$, $z_0 \in \text{int}(U)$, but this won't handle $\log z$ or $z^{1/2}$ which aren't analytic on a whole $D(r) = D(r) - \{0\}$.

Laurent series: these are power series with positive and negative exponents!

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n. \quad \text{Convergence is best understood by splitting into}$$

$$\sum_{n \geq 0} a_n z^n \quad (\text{usual power series}) \quad \text{converges for } |z| < R_2 = \frac{1}{\limsup_{n \rightarrow +\infty} |a_n|^{1/n}}$$

and $\sum_{n < 0} a_n z^n$ (power series in $\frac{1}{z}$) converges for $|z| > R_1 = \frac{1}{\limsup_{n \rightarrow -\infty} |a_n|^{1/|n|}}$

$\Rightarrow$ we have an annulus of convergence $\{R_1 < |z| < R_2\}$. (may be empty).

Beware: general (formal) Laurent series don't form a ring. The issue is that the coefficient of z^n in $(\sum a_k z^k)(\sum b_k z^k)$ should be $\sum_{k \in \mathbb{Z}} a_k b_{n-k}$, which may not be a convergent series. (Things are fine if annuli of convergence have non-empty intersection). A better-behaved class of Laurent series are those with only finitely many negative powers of z , i.e. $\sum_{-N}^{\infty} a_n z^n$ ($= \frac{1}{z^N} \cdot (\text{power series})$).

These are actually a field - the field of fractions of the ring of power series.

Thm: If $f(z)$ is analytic in $A_{R_1, R_2} = \{R_1 < |z| < R_2\}$ then we can express it as a Laurent series $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ which converges on A_{R_1, R_2} .

Pf: we show this on slightly smaller annuli $\{r_1 \leq |z| \leq r_2\} \quad \forall R_1 < r_1 < r_2 < R_2$.

Then the Cauchy formula for A_{r_1, r_2} and its boundary $S^+(r_2) - S^+(r_1)$ gives

$$f(z) = \frac{1}{2\pi i} \int_{S^+(r_2)} \frac{f(w) dw}{w-z} - \frac{1}{2\pi i} \int_{S^+(r_1)} \frac{f(w) dw}{w-z} \quad \text{for } r_1 < |z| < r_2.$$

$$\text{On } S^+(r_2) \text{ we have } \frac{1}{w-z} = \frac{w^{-1}}{1-z/w} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} \quad \text{converging uniformly} \\ (|z/w| < 1)$$

$$\text{On } S^+(r_1) \text{ we have } \frac{1}{z-w} = \frac{z^{-1}}{1-w/z} = \sum_{k=0}^{\infty} \frac{w^k}{z^{k+1}} = \sum_{n \leq -1} \frac{z^n}{w^{n+1}}, \quad \text{also converging uniformly} \\ (|w/z| < 1) \quad (n = -k-1)$$

Uniform convergence allows us to move the sum outside of the integrals, giving

$$f(z) = \sum_{n \geq 0} \frac{1}{2\pi i} z^n \int_{S^+(r_2)} \frac{f(w) dw}{w^{n+1}} + \sum_{n \leq -1} \frac{1}{2\pi i} z^n \int_{S^+(r_1)} \frac{f(w) dw}{w^{n+1}}.$$

$$= \sum_{n \in \mathbb{Z}} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_{S^+(r)} \frac{f(w) dw}{w^{n+1}} \quad \begin{array}{l} \text{(for any } r \in (R_1, R_2), \\ \text{since this is indep't} \\ \text{of } r \text{ by Cauchy).} \end{array}$$

(compare with our earlier result about Taylor series).

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Corollary: || any analytic function on $\{R_1 < |z| < R_2\}$ can be written as the sum of an analytic function on $\{|z| < R_2\}$ and an analytic function on $\{|z| > R_1\}$.

Singularities and removability: Assume f is analytic on $D^*(R) = D(R) - \{0\}$, and express it as a Laurent series $\sum_{n \in \mathbb{Z}} a_n z^n$. Let $N = \inf \{n \in \mathbb{Z} / a_n \neq 0\}$ (if exists)

1) If $N \geq 0$ (ie. $a_n = 0 \ \forall n < 0$), f is a power series and the singularity at 0 is removable, ie. can extend f to an analytic function on $D(R) \ni 0$.

• $N = \infty$ ie. $a_n = 0 \ \forall n$: then $f \equiv 0$.

• $N > 0$, then $f(z) = z^N (a_N + \dots)$ has an isolated zero of order N at 0.

• $N = 0$, $f(0) = a_0 \neq 0$

The new cases are when the negative part of the Laurent series isn't zero.

2) If $N < 0$ finite, ie. there are finitely many negative powers of z in the series:

$$\text{then } f(z) = \frac{1}{z^{|N|}} (a_N + \dots) = \frac{g(z)}{z^{|N|}}, \quad g \text{ analytic with } g(0) = a_N \neq 0.$$

We say f has a pole of order $|N|$ at 0.

3) If $N = -\infty$, ie. the negative part of the series has ∞ many terms: we say f has an essential singularity at 0 (= non-removable singularity other than a pole).

Ex: $\exp(1/z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$ essential singularity at 0.

The qualitative differences between the 3 cases can also be understood without involving Laurent series.

Then: || f analytic on $D^*(R)$:

1) the singularity at 0 is removable iff $f(z)$ is bounded on a neighborhood of 0.

2) f has a pole at 0 iff $|f(z)| \rightarrow \infty$ as $z \rightarrow 0$

3) f has an essential singularity iff $\forall \varepsilon > 0$, $f(D^*(\varepsilon))$ is dense in $\mathbb{C}$

(equivalently: $\forall y \in \mathbb{C} \cup \{\infty\}$, $\exists z_n \rightarrow 0$ st. $f(z_n) \rightarrow y$).

Pf (without using Laurent series!)

i) assume f bounded on $D^*(r)$. Since f is continuous on $S^*(r)$, we have seen that

$$g(z) = \frac{1}{2\pi i} \int_{S^*(r)} \frac{f(w) dw}{w-z}$$
 is analytic in $D(r)$. By Cauchy's formula, if

$$0 < \varepsilon < \frac{|z|}{2}, \text{ then } \frac{1}{2\pi i} \int_{\partial D} \frac{f(w) dw}{w-z} = \frac{1}{2\pi i} \left(\int_{S^*(r)} - \int_{S^*(z, \varepsilon)} - \int_{S^*(0, \varepsilon)} \right)$$

$$= g(z) - f(z) - \frac{1}{2\pi i} \int_{S^*(0, \varepsilon)} \frac{f(w) dw}{w-z} = 0$$



but the last integral $\rightarrow 0$ as $\varepsilon \rightarrow 0$ since the integrand is bounded and $\text{length}(S'(\varepsilon)) \rightarrow 0$. (4)

So: g is analytic in $D(r)$ and $g(z) = f(z) \quad \forall z \in D(r) - \{0\}$.

ie. the singularity at 0 is removable.

(Conversely, it is clear that f is bounded near 0 if the sing. is removable).

2) assume $|f| \rightarrow \infty$ as $z \rightarrow 0$, then $h(z) = \frac{1}{f(z)}$ is analytic and bounded in a neighborhood of 0, hence has a removable singularity, ie. $\exists$ analytic extension which we denote again by h . Since $|h| \rightarrow 0$ as $z \rightarrow 0$, h has an (isolated) zero at $z=0$, where it vanishes to finite order: $\exists n \geq 1$ and $k(z)$ analytic, $k(0) \neq 0$ st. $h(z) = z^n k(z)$.

Hence $f(z) = \frac{1}{h(z)} = \frac{g(z)}{z^n}$ where $g(z) = \frac{1}{k(z)}$ is analytic on a nbd. of 0:

f has a pole of order n .

Conversely if $f(z) = \frac{g(z)}{z^n}$, $n \geq 1$, g analytic, $g(0) \neq 0$ then $\exists c > 0$ st.

$|g(z)| \geq c > 0$ over a neighborhood of 0, and $|f(z)| \geq \frac{c}{|z|^n} \rightarrow \infty$ as $z \rightarrow 0$.

3) if $f(D^*(\varepsilon))$ isn't dense in $\mathbb{C}$, then $\exists c$ st. $h(z) = \frac{1}{f(z) - c}$ is bounded near 0, hence has a removable singularity; we denote the extension over 0 by h again.

If $h(0) = 0$ then, as in the previous case, h has a zero of finite order $n \geq 1$, $\frac{1}{h(z)}$ has a pole of order n , and $f(z) = c + \frac{1}{h(z)}$ also has a pole of order n .

If $h(0) \neq 0$ then $f(z) = c + \frac{1}{h(z)}$ extends over 0, the singularity is removable.

So: essential singularity $\Rightarrow f(D^*(\varepsilon))$ is dense in $\mathbb{C} \quad \forall \varepsilon > 0$.

(the converse is clear too: $f(D^*(\varepsilon))$ dense $\Rightarrow f$ isn't bounded and $|f| \nrightarrow \infty$, so neither removable nor pole). $\square$

Def: || If f is analytic in $U - \{p_1, \dots, p_n\}$ and has poles at $p_1, \dots, p_n$ (no essential sings) then we say f is meromorphic in U .

• If $f: U - \{p_i\} \rightarrow \mathbb{C}$ is meromorphic with poles at p_i , then $|f(z)| \rightarrow \infty$ as $z \rightarrow p_i$, so

$1/f$ has a removable singularity at p_i , where it has a zero (of order = pole order of f)

Hence f extends to $\hat{f}: U \rightarrow S = \mathbb{C} \cup \{\infty\}$ Riemann sphere by setting $\hat{f}(p_i) = \infty$, and $\hat{f}$ is continuous and analytic, in the sense that

• away from the poles $\{p_i\} = \hat{f}^{-1}(\infty)$, $\hat{f}$ takes values in $\mathbb{C}$ and is analytic

• away from the zeros, $\frac{1}{\hat{f}(z)}$ takes values in $\mathbb{C}$ and is analytic
(= analytic extension of $\frac{1}{f}$ over the removable sing. at p_i).