Last time: If f is analytic on $D(r)-\{0\}$, 3 possibilities:

Type of singularity	Behavior of f near 0	Laurent series
1) Removable sing.: f extends to $D(r)$	f bounded	$\sum_{k\geq 0} a_k z^k$
2) pole of order n: $f(z)=g(z)/z^n$, g analytic, $g(0)\neq 0$	$\lvert f\rvert\to\infty$ as $z\to 0$	$\sum_{k\geq -n} a_k z^k$, $a_{-n}\neq 0$
3) essential singularity	$f(D(\varepsilon)-\{0\})$ dense in $\mathbb{C}$	$\sum_{k\in\mathbb{Z}} a_k z^k$, ∞ negative part

Def: If f is analytic in $U\setminus\{p_1,\dots,p_n\}$ and has poles at $p_1\cdots p_n$ (no essential sings) then we say f is meromorphic in U.

- If $f: U\setminus\{p_i\}\to\mathbb{C}$ is meromorphic with poles at p_i, then $|f(z)|\to\infty$ as $z\to p_i$, so $1/f$ has a removable singularity at p_i, where it has a zero (of order = pole order of f)

  Hence f extends to $\hat{f}: U\to S=\mathbb{C}\cup\{\infty\}$ Riemann sphere by setting $\hat{f}(p_i)=\infty$, and $\hat{f}$ is continuous and analytic, in the sense that
  - away from the poles $\{p_i\}=\hat{f}^{-1}(\infty)$, $\hat{f}$ takes values in $\mathbb{C}$ and is analytic
  - away from the zeros, $\frac{1}{\hat{f}(z)}$ takes values in $\mathbb{C}$ and is analytic (= analytic extension of $\frac{1}{f}$ over the removable sings. at p_i).

- Hence: zeros and poles of (non-identically zero) meromorphic functions are isolated.

  → if f and g are analytic on U, $g\not\equiv 0$, then f/g is meromorphic on U.
  if f and g have no common zeros, f/g has zeros = zeros of f, poles = zeros of g.
  if there's a common zero, highest order wins (factor out powers of $(z-z_0)$).

  → the converse is actually true (won't prove): every meromorphic function is a quotient f/g of analytic functions.

- Assume f is meromorphic on all $\mathbb{C}$ (ie. f analytic on $\mathbb{C}-\{p_i\}$, w/ poles at p_i).

  If $|f(z)|$ is either bounded or $\to\infty$ as $|z|\to\infty$, then the function $g(w)=f(1/w)$ has a removable singularity or a pole at $w=0$, so it is meromorphic near 0.

  ⇒ can extend $\hat{f}$ to the Riemann sphere by setting $\hat{f}(\infty)=\hat{g}(0)$.

  Thus: if $f(z)$ and $f(\frac{1}{z})$ are meromorphic, get an analytic extension $\hat{f}: S\to S$ to the whole Riemann sphere!

- In fact, such $\hat{f}$ is necessarily a rational function. This follows from

Theorem: (homework) If f is an analytic entire function $f:\mathbb{C}\to\mathbb{C}$ and $|f(z)|\leq M|z|^n$ near $|z|\to\infty$ then f is a polynomial of degree at most n.

Corollary: If $f: S \to S$ is analytic (ie. $f(z)$ and $f(\frac{1}{z})$ are meromorphic) then f is a rational function.

Proof: • the fact that $g(w) = f(\frac{1}{w})$ is meromorphic near 0 gives a bound of the form
$$|g(w)| \le \frac{C}{|w|^n} \text{ for } w \to 0, \text{ ie. } |f(z)| \le C\,|z|^n \text{ for } z \in \mathbb{C},\ z \to \infty$$

• f isn't an entire function, it does have poles - but only finitely many of them (poles of f are zeros of $1/f$ hence isolated, and S is compact).

so: $\exists$ polynomial $P(z) = \prod (z-p_i)^{n_i}$ (p_i poles of f, n_i order) st. $P(z)f(z)$ extends to an entire function on $\mathbb{C}$, also satisfying a bound $|P(z)f(z)| \le C'\,|z|^{n+\deg P}$ as $z \to \infty$. By the previous thm, $P(z)f(z)$ is a polynomial. □

Local behavior of analytic functions: maximum principle and open mapping principle.

* Cauchy's integral formula can be viewed as a mean value formula:

Thm: If f is analytic on $U \supset \overline{B_r(z)}$ then $f(z)$ is the average value of f on $S^1(z,r)$.

Pf: by Cauchy, $f(z) = \frac{1}{2\pi i}\int_{S^1(z,r)} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}}\, d(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$. □

* **Corollary:** the maximum principle

Thm: If f is analytic on $U \subset \mathbb{C}$ (open connected) & nonconstant, then $|f|$ doesn't achieve its maximum anywhere in U. In particular, if f is analytic on U and continuous on $\overline{U}$, $\overline{U}$ compact, then the maximum of $|f|$ on $\overline{U}$ is achieved on the boundary of U.

Pf: Given $z_0 \in U$, $r > 0$ small so $\overline{B_r(z_0)} \subset U$, we have
$$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right| \underset{(*)}{\le} \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \le \max_{S^1(z_0,r)} |f|.$$

If $|f|$ has a (local) max. at z_0 then $\max_{S^1(z_0,r)} |f| = |f(z_0)|$ and these inequalities are equalities.

This implies $|f(z)| = |f(z_0)|\ \forall z \in S^1(z_0,r)$. In fact $f(z) = f(z_0)$: if $\arg(f(z))$ varies then $(*)$ is $<$.

(eg. rescale so $f(z_0) = 1$, then $|f(z)| \le 1$ so $Re(f(z)) \le 1$, and $Re\, f(z_0) = Re\left(\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right) \le 1$, equality implies $Re(f(z)) = 1\ \forall z \in S^1(z_0,r)$, and since $|f(z)| \le 1$, this gives $f(z) = 1\ \forall z \in S^1(z_0,r)$).

Since f is analytic, $f(z) - f(z_0) = 0\ \forall z \in S^1(z_0,r) \Rightarrow$ zeros of $f(z)-f(z_0)$ aren't isolated (zeros of nontrivial analytic f^ns are isolated) $\Rightarrow f(z) - f(z_0) = 0$ on $U \Rightarrow f = $ constant on U. □

(**Rmk:** This also implies max principle for $Re(f)$, since $|e^f| = e^{Re(f)}$ has no (local) max.)

• One nice (non-local) consequence is a contraction principle: the <u>Schwarz Lemma</u>.

Thm: f analytic on $D = \{|z|<1\}$, and $|f(z)|<1$ $\forall z \in D$ (ie. $f: D \to D$), and $f(0)=0$, then $|f'(0)| \le 1$, and $|f(z)| \le |z|$ $\forall z \in D - \{0\}$.
Moreover if equality holds in either of these then $f(z) = e^{i\theta} z$ for some $e^{i\theta} \in S^1$.

Pf: Write $f(z) = \sum_{n=1}^{\infty} a_n z^n = z F(z)$ where $F(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ analytic
($f(0)=0 \Rightarrow$ no constant term)

For $|z| = r \in (0,1)$, we have $|F(z)| = \left|\frac{f(z)}{z}\right| \le \frac{1}{r}$, hence by the maximum principle, $|F(z)| \le \frac{1}{r}$ whenever $|z| \le r$. Taking $r \to 1$, $|F(z)| \le 1$ $\forall z \in D$.
Hence the bounds on $f'(0) = F(0)$ and $f(z) = zF(z)$. Moreover, if $|F| = 1$ is achieved anywhere inside D then F is constant $= e^{i\theta}$, so $f(z) = e^{i\theta} z$. $\square$

<u>Note</u>: • the bound on $|f'(0)|$ is the same as the bound one gets from Cauchy's integral formula. The Schwarz lemma is a strengthening to pointwise bounds $|f(z)| \le |z|$ globally on the disc.
• by composing f with fractional linear transformations, we can get Schwarz-type bounds for all sorts of other situations, eg. if f maps a disc to a half-plane, etc.

There is another important class of functions which satisfy mean value & max. principle:

<u>Def</u>: A C^2 function $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is <u>harmonic</u> if $\underbrace{\Delta}_{\text{Laplacian}} f = \sum \frac{\partial^2 f}{\partial x_i^2} = 0$.

(Physically important! eg. electric & gravitational potentials in vacuum are harmonic, so is temperature distribution at thermal equilibrium; etc.).

Real analysis gives general methods for studying harmonic functions, but the case of 2 real variables $f(x,y)$ is closely related to complex analysis.

• $u: U \subset \mathbb{C} \xrightarrow[C^2]{} \mathbb{R}$ is harmonic if $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

<u>Thm</u>: If $f = u + iv$ is analytic, then $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$ are harmonic.

<u>Pf</u>: Cauchy-Riemann eqn $\frac{\partial f}{\partial \bar z} = 0$ says $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, so

$$\Delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = 0. \ \square$$

What is unique about harmonic functions in 2 variables is that we have a converse:

<u>Thm</u>: If u is harmonic on a simply-connected open $U \subset \mathbb{C}$, then there exists an analytic function $f: U \to \mathbb{C}$ s.t. $u = \mathrm{Re}(f)$.

ie. there exists a harmonic $v: U \to \mathbb{R}$ ("harmonic conjugate of u") st. $u+iv$ is analytic.

Ex: $u = \log|z| = \mathrm{Re}(\log z)$ on domain not enclosing origin $\rightsquigarrow v = \arg(z)$.
This shows the assumption on U is necessary (v not single-valued on $\mathbb{C}^*$).

Pf: Observe: to find v st. $u+iv$ analytic, need to solve Cauchy-Riemann $\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} \end{cases}$.

Let $\alpha = -\dfrac{\partial u}{\partial y}\,dx + \dfrac{\partial u}{\partial x}\,dy$

Then $d\alpha = \underbrace{\left(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial x^2}\right)}_{=\Delta u = 0.} dx \wedge dy = 0$, so α is closed

Since U is simply connected, closed 1-forms are exact, ie. $\exists\, v$ s.t. $dv = \alpha$
(can define $v(z) = \int_{z_0}^{z} \alpha$ by integrating over any path in U;
the integral is path-independent by Stokes since $d\alpha = 0$ and U simply connected)

$f = u + iv$ then satisfies Cauchy-Riemann, ie. it is analytic on U. $\square$

Now we know that harmonic functions are secretly real parts of analytic functions, we get:

Corollaries: (also true in n variables?)
- any C^2 harmonic function is actually C^∞
- harmonic functions satisfy the mean value thm: $u(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(z + re^{i\theta})\,d\theta$.
- ——— " ——— maximum principle

Another pair of deep results (which we won't prove) are about existence of analytic mappings & harmonic functions:

* The Riemann mapping theorem:

Thm: if $U \subset \mathbb{C}$ is a non-empty simply connected open subset, $U \neq \mathbb{C}$, then there exists a biholomorphism $\varphi: U \xrightarrow{\sim} D = \{|z| < 1\}$ ie. analytic bijection w/ analytic inverse.

(+ Carathéodory: if ∂U is a Jordan curve, then φ extends to a continuous map $\overline{U} \to \overline{D}$ giving a homeomorphism on the boundary)

Ex: can you find explicit biholom's

quarter disc $\simeq$ half disc $\simeq$ $(-\infty, 0) \times (0,1)$?

or ... unit disc $\simeq$ half-plane $\simeq$ $\mathbb{R} \times (0,1)$? (see HW).

* The existence of solutions to Dirichlet's problem (harmonic functions with prescribed values at the boundary of a domain) can be thought of as an analogue for harmonic functions:

Thm: if $U \subset \mathbb{C}$ is a simply connected bounded open subset with suff. nice boundary (eg. ∂U piecewise smooth) and $f \in C^0(\partial U, \mathbb{R})$ any continuous function
$\Rightarrow \exists$ unique $u \in C^0(\overline{U}, \mathbb{R})$ st. $u_{|\partial U} = f$ and u is harmonic inside U.

(uniqueness follows easily from the max-principle: $u-v=0$ at ∂U, $u-v$ harmonic $\Rightarrow u-v\equiv 0$). 5

(One way to prove this is actually to first establish it for the unit disc, using Fourier series to reduce to trigonometric polynomials; $\sum c_n e^{in\theta} \rightsquigarrow \sum_{n\geq 0} c_n z^n + \sum_{n<0} c_n \bar{z}^{|n|}$ then use Riemann mapping theorem (+ Carathéodory) to map $U \xrightarrow[\simeq]{\varphi} D$; u is harmonic iff $u\circ\varphi$ is.)

- Back to analytic f^{ns}, there's a stronger local result: the <u>open mapping principle</u> ($\Rightarrow$ max. principle).

<u>Thm:</u> A nonconstant analytic function is an open mapping, ie. U open $\Rightarrow f(U)$ open

in other terms: f analytic non-constant at $z_0 \Rightarrow \forall r>0,\ \exists \varepsilon>0$ st. $f(B_r(z_0)) \supset B_\varepsilon(f(z_0))$

($\Rightarrow |f(z)|$, $\mathrm{Re}\, f(z)$, ... can't have local max)

First we prove

<u>Prop:</u> if $f(z)$ has an isolated zero at $z=z_0$, then $\exists$ analytic function g defined near z_0, with $g(z_0)=0$, $g'(z_0)\neq 0$, and $n\geq 1$, st. $f(z)=g(z)^n$.

<u>Pf:</u> let n := order of the zero of f, ie. write $f(z)=\sum_{k=n}^{\infty} a_k (z-z_0)^k = a_n (z-z_0)^n (1+h(z))$ with $h(z_0)=0$. $\exists V\ni z_0$ st. $|h(z)|<1$ $\forall z\in V$; over V we can define $g(z)=a_n^{1/n}(z-z_0)(1+h(z))^{1/n}$, where $(1+h(z))^{1/n}=\exp\left(\frac{1}{n}\log(1+h(z))\right)$ well def^d for $|h|<1$. $\square$

<u>Pf. thm:</u> for $z_0\in U$, write $f(z)-f(z_0)=g(z)^n$ for some $n\geq 1$, $g(z_0)=0$, $g'(z_0)\neq 0$.
By inverse function thm, g is a local diffeo at z_0 (since $g'(z_0)\neq 0$), hence an open mapping near z_0 ($\exists$ continuous, actually analytic, inverse mapping), so $\forall V\ni z_0$ open ($\subset$ domain of g), $g(V)\ni 0$ contains some ball $B_\varepsilon(0)$, hence taking n^{th} power, $f(V)\supset B(f(z_0),\varepsilon^n)$. $\square$