Math 55b Homework 11

Due Wednesday April 20, 2022.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.

Material covered: Laurent series, poles and singularities; maximum principle, harmonic functions; argument principle, residues (Ahlfors sections 4.3, 4.5.1-4.5.2, 4.6.1 and 5.1, or McMullen sections 11, 12, and 13 through Corollary 13.6).

1. Find Laurent series expressions for the function $f(z) = \dfrac{1}{z(z-1)(z-2)}$:
(i) in the region $\{1 < |z| < 2\}$, (ii) in the region $\{|z| > 2\}$.

2. Let f be an analytic function over a domain U which contains the closed disc $\{z \in \mathbb{C},\ |z| \le 3\}$, and suppose that $f(1) = f(i) = f(-1) = f(-i) = 0$. Show that $|f(0)| \le \dfrac{1}{80} \max\limits_{|z|=3} |f(z)|$.

3. Let $f(z)$ be an analytic function over the annulus $R_1 < |z| < R_2$, and let $M(r) = \sup_{z \in S^1(r)} |f(z)|$. Prove that $\log M(e^s)$ is a convex function of $s \in (\log R_1, \log R_2)$.

(Hint: apply the maximum principle to $z^m f(z)^n$ for suitably chosen m, n.)

4. (a) We consider the following three domains in $\mathbb{C}$: $D = \{|z| < 1\}$, $H = \{\mathrm{Re}(z) > 0\}$, and $S = \{0 < \mathrm{Im}(z) < 1\}$ (the unit disc, the right half-plane, and an infinite horizontal strip), and their closures in $\mathbb{C}$. Find explicit homeomorphisms $\overline{D} - \{\pm 1\} \simeq \overline{H} - \{0\} \simeq \overline{S}$ whose restrictions to the interior are biholomorphisms (i.e. analytic maps with analytic inverses) $D \simeq H \simeq S$.

(b) Use this to find a continuous function $u : \overline{D} - \{\pm 1\} \to \mathbb{R}$ such that u is harmonic in D, $u(z) = 1$ on the upper half of the unit circle ($|z| = 1$ and $\mathrm{Im}(z) > 0$), and $u(z) = 0$ on the lower half of the unit circle ($|z| = 1$ and $\mathrm{Im}(z) < 0$).

Optional (extra credit): show that there is a unique *bounded* continuous function on $\overline{D} - \{\pm 1\}$ with these properties, but that uniqueness fails if we don't assume u to be bounded.

5. How many roots of the equation $z^7 + 7z^4 - 3z^2 + 2 = 0$ satisfy $1 < |z| < 2$?

(Hint: use Rouché's theorem).

6. Prove or disprove: if $f(z)$ is analytic on the unit disc D and has n zeros in D, then $f'(z)$ has at least $n - 1$ zeros in D. What happens if "at least" is replaced by "at most"?

7. Find all the singularities of $f(z) = z/(e^{z^2} - 1)$, and determine the residues at each of its poles.

8. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?