Last time: harmonic functions $U \subset \mathbb{R}^2 \xrightarrow{u} \mathbb{R}$ $\longleftrightarrow$ analytic functions $U \xrightarrow{f} \mathbb{C}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u = \mathrm{Re}\, f \ (\text{or } \mathrm{Im}\, f) \qquad \frac{\partial f}{\partial \bar z} = 0$$

both have:
- → guaranteed derivatives to all orders
- → mean value formula $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\, d\theta$
- → maximum principle for $|f|$: no local maxima unless f is constant.

Another pair of deep results (which we won't prove) are about existence of analytic mappings & harmonic functions:

* The Riemann mapping theorem:

Thm: if $U \subset \mathbb{C}$ is a non-empty simply connected open subset, $U \neq \mathbb{C}$, then there exists a biholomorphism $\varphi : U \xrightarrow{\sim} D = \{|z|<1\}$ ie. analytic bijection w/ analytic inverse.

(+ Carathéodory: if ∂U is a Jordan curve, then φ extends to a continuous map $\bar U \to \bar D$ giving a homeomorphism on the boundary)

Ex: can you find explicit biholom's

quarter disc $\simeq$ half disc $\simeq$ $(-\infty, 0) \times (0,1)$?

or ... unit disc $\simeq$ half-plane $\simeq$ $\mathbb{R} \times (0,1)$? (see HW).

* The existence of solutions to Dirichlet's problem (harmonic functions with prescribed values at the boundary of a domain) can be thought of as an analogue for harmonic functions:

Thm: if $U \subset \mathbb{C}$ is a simply connected bounded open subset with suff. nice boundary (eg. ∂U piecewise smooth) and $f \in C^0(\partial U, \mathbb{R})$ any continuous function
$\Rightarrow \exists$ unique function $u \in C^0(\bar U, \mathbb{R})$ st. $\begin{cases} u_{|\partial U} = f \\ u \text{ is harmonic inside } U. \end{cases}$

(uniqueness follows easily from the max-principle: $u - v = 0$ at ∂U, $u-v$ harmonic $\Rightarrow u - v \equiv 0$).

(One way to prove this is actually to first establish it for the unit disc, using Fourier series to reduce to trigonometric polynomials; $\sum c_n e^{in\theta} \rightsquigarrow \sum_{n\geq 0} c_n z^n + \sum_{n<0} c_n \bar z^{|n|}$ then use Riemann mapping theorem (+ Carathéodory) to map $U \underset{\sim}{\xrightarrow{\varphi}} D$; u is harmonic iff $u \circ \varphi$ is.)

• Back to analytic fns, there's a stronger local result: the open mapping principle ($\Rightarrow$ max. principle).

Thm: A nonconstant analytic function is an open mapping, ie. U open $\Rightarrow f(U)$ open

in other terms: f analytic at z_0, non-constant $\Rightarrow \forall r > 0, \exists \varepsilon > 0$ st. $f(B_r(z_0)) \supset B_\varepsilon(f(z_0))$

($\Rightarrow |f(z)|$, $\mathrm{Re}\, f(z)$, ... can't have local max)

uses **Lemma:** if $f(z)$ has an isolated zero at $z = z_0$, then $\exists$ analytic function g defined near z_0, with $g(z_0) = 0$, $g'(z_0) \neq 0$, and $n \geq 1$, st. $f(z) = g(z)^n$.

Pf: let n = order of the zero of f, ie. write $f(z)=\sum_{k=n}^{\infty} a_k(z-z_0)^k = a_n(z-z_0)^n(1+h(z))$
with $h(z_0)=0$. $\exists V\ni z_0$ st. $|h(z)|<1$ $\forall z\in V$; over V we can define
$g(z)=a_n^{1/n}(z-z_0)(1+h(z))^{1/n}$, where $(1+h(z))^{1/n}=\exp\left(\frac{1}{n}\log(1+h(z))\right)$ well def^d for $|h|<1$. $\square$

Pf. Thm: for $z_0\in U$, write $f(z)-f(z_0)=g(z)^n$ for some $n\geq 1$, $g(z_0)=0$, $g'(z_0)\neq 0$.
By inverse function thm, g is a local diffeo at z_0 (since $g'(z_0)\neq 0$), hence an open mapping near z_0 ($\exists$ continuous, actually analytic, inverse mapping), so $\forall V\ni z_0$ open ($\subset$ domain of g), $g(V)\ni 0$ contains some ball $B_\varepsilon(0)$, hence taking n^{th} power, $f(V)\supset B(f(z_0),\varepsilon^n)$. $\square$

<u>The argument principle</u>: Our proof of open mapping principle actually shows: near z_0, f takes every value near $f(z_0)$ n times where n is the order of the zero of $f(z)-f(z_0)$ at $z=z_0$.
More generally, we can estimate the number of zeros of f (or $\#f^{-1}(c)$) in a domain D:

<u>Thm</u>: If $f:U\to\mathbb{C}$ is analytic, D bounded domain with $\bar{D}\subset U$, $\partial D=\gamma$ piecewise smooth, assume f is nonzero at every point of γ. Then the number of zeros of f inside D, counted with multiplicity = order of each zero, is $n(\gamma,0)=\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$.

Observe: $\frac{f'(z)}{f(z)}=\frac{d}{dz}(\log f(z))$ - the <u>logarithmic derivative</u>.
(NB: $\log f$ is only def^d locally up to $+2\pi i\mathbb{Z}$, but this doesn't matter for the derivative!).

Let $z_1,\dots,z_k$ be the zeros of f inside D, with multiplicities $m_1,\dots,m_k$.
(isolated, hence finitely many since $\bar{D}$ is compact).

Then we can write $f(z)=(z-z_1)^{m_1}\dots(z-z_k)^{m_k}g(z)$ where g is analytic and nowhere zero in D (check this makes sense & works near each z_i).

Properties of log (or calculation) $\Rightarrow$ $\frac{f'(z)}{f(z)}=\frac{m_1}{z-z_1}+\dots+\frac{m_k}{z-z_k}+\frac{g'(z)}{g(z)}$.

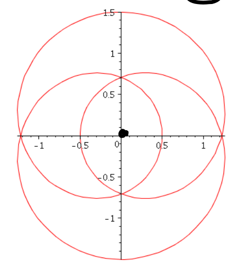
Now $\frac{g'(z)}{g(z)}$ is analytic in D (g has no zeroes) so $\int_\gamma \frac{g'(z)}{g(z)}\,dz=0$,
while $\frac{1}{2\pi i}\int_\gamma \frac{m_j}{z-z_j}\,dz=m_j$ (Cauchy formula) $\Rightarrow$ $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz=\sum m_j$. $\square$

* <u>Topological/geometric interpretation</u>:
view f as a mapping $U\to\mathbb{C}$, it maps the loop $\gamma\subset U$ to $f_*(\gamma)=f\circ\gamma$ loop in $\mathbb{C}$. (may self-intersect). We've assumed $f\neq 0$ on γ, so $f\circ\gamma$ is actually a loop in $\mathbb{C}^*$.

$$n(\gamma,0)=\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} \underset{\text{set } w=f(z)}{=} \frac{1}{2\pi i}\int_{f_*(\gamma)}\frac{dw}{w}$$
(pullback formula, or more concretely, change of var's in path integral/chain rule)

$$= \text{change in } \frac{1}{2\pi i}\log(w), \text{ ie. } \frac{1}{2\pi}\arg(w) \text{ around } f_*(\gamma)$$

$=$ winding number of $f\circ\gamma$ around the origin in 0. ③



Ex: $f(z) = z^3 - \frac{1}{2} z$ on unit circle : winding number around origin is 3 (3 roots in unit disc)

Generalization: if $c \notin f(\gamma)$ then $n(\gamma, c) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-c}\, dz$ = winding number of $f(\gamma)$ around $c \in \mathbb{C}$ gives the number of times $f(z)=c$ inside D (with multiplicities).

This quantity varies continuously with c, & is an integer $\Rightarrow$ locally constant (indept of c) as long as $c \notin f(\gamma)$. (Note: γ is compact, so $f(\gamma)$ as well $\Rightarrow \mathbb{C} - f(\gamma)$ is open.

Rmk: Applying to $\gamma = S^1(z, \delta)$, $n(\gamma, f(z)) > 0$ (isolation of zeros $\Rightarrow$ for $\delta > 0$ small, $f(z) \notin f(\gamma)$)

$\Rightarrow n(\gamma, w) > 0 \quad \forall w \in B_\varepsilon(f(z)) \subset \underset{\text{open}}{\mathbb{C} \setminus f(\gamma)}$, i.e. $f(B_\delta(z)) \supset B_\varepsilon(f(z))$. (in fact the whole connected component of $f(z)$ in $\mathbb{C} \setminus f(\gamma)$).

This gives another proof of the open mapping principle.

* Another immediate generalization is to the case where f is meromorphic in D, rather than analytic: similarly write $f(z) = \frac{(z-a_1)^{m_1} \cdots (z-a_k)^{m_k}}{(z-b_1)^{n_1} \cdots (z-b_\ell)^{n_\ell}}\, g(z)$, where

a_j are the zeros of f in D (with order m_j)
b_j —"— poles —"— n_j

$\Rightarrow \text{winding}(f\circ\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)\,dz}{f(z)} = \sum_{j=1}^{k} m_j - \sum_{j=1}^{\ell} n_j.$

* A useful consequence of the argument principle is

Rouché's thm: if f and g are analytic in $U \supset \bar{D}$, $\partial D = \gamma$ simple closed curve, and $|f(z) - g(z)| < |f(z)| \quad \forall z \in \gamma$, then f and g have the same number of zeroes in D, counting with multiplicities.

Proof: $\left| \frac{g(z)}{f(z)} - 1 \right| < 1$ on γ, so $\frac{g}{f}$ maps γ to the open disc $B_1(1)$, which doesn't enclose the origin. So the winding number $=$ #zeros $-$ #poles $= \# g^{-1}(0) - \# f^{-1}(0) = 0$. $\square$

Rouché's thm is a good way of estimating the number of zeros of g in D by reducing to an easier calculation.

Ex: $g(z) = z^3 - 4z^2 + 1$: the fundamental thm of algebra says g has 3 roots, but how many of these are in the unit disc?
Answer: on S^1, $|z^3 + 1| < |4z^2|$, so we can compare to $f(z) = -4z^2$ and conclude 2 of the 3 roots are in the unit disc.

<u>Residue calculus</u>: instead of using Cauchy's integral formula to study the behavior of analytic functions, let's now use it to evaluate integrals!

Assume we want to evaluate $\int_\gamma f(z)\,dz$, where $\gamma = \partial D$ and f is analytic in $U \supset \overline{D} - \{p_1, \dots, p_n\}$. (or, later, a definite integral whose value can be related to $\int_\gamma$).

* <u>Def</u>: The <u>residue</u> of f at p is $\mathrm{Res}_p(f) = \frac{1}{2\pi i}\int_{S^1(p,\varepsilon)} f(z)\,dz$.
(for $\varepsilon > 0$ small so f is analytic in $D^*(p,\varepsilon) = D(p,\varepsilon) - \{p\}$.

Expressing f as a Laurent series $\sum_{-\infty}^{\infty} a_n (z-p)^n$ in $D^*(p,\varepsilon)$, $\boxed{\mathrm{Res}_p(f) = a_{-1}}$

So: the residue is easiest to calculate if f has a simple pole (ie. order 1) at p, in this case $\mathrm{Res}_p(f) = \lim_{z\to p} (z-p)f(z)$. Otherwise, need to calculate, usually by determining part of the Laurent series for f. (eg. for rational functions, partial fraction decomposition will accomplish this).

* Now, Cauchy's theorem for $D \setminus \bigcup D(p,\varepsilon)$ gives:

<u>Residue theorem</u>: $\overline{D}$ compact domain with piecewise smooth boundary $\gamma = \partial D$, $P \subset \mathrm{int}(D)$ finite set, f analytic on $U \supset \overline{D} - P$, then $\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{p\in P} \mathrm{Res}_p(f)$.

We now explore how to use this to evaluate various kinds of definite integrals.

<u>Example 1</u>: $\int_0^{2\pi} R(\sin\theta, \cos\theta)\,d\theta$ (or $R(e^{i\theta})$) where R is a rational function (w/o. poles on S^1).

<u>e.g.</u> let's calculate $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$, where $a > 1$.

Set $z = e^{i\theta}$ to turn this into a path integral on S^1. Then $d\theta = \frac{1}{i}\,d\log z = \frac{dz}{iz}$

and $\cos\theta = \frac{z+z^{-1}}{2}$. $\Rightarrow \int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = \int_{S^1} \frac{dz/z}{\frac{i}{2}(z+2a+z^{-1})} = -2i\int_{S^1}\frac{dz}{z^2+2az+1}$

The poles are at $p_\pm = -a \pm \sqrt{a^2-1}$; of these, only $p_+ = -a+\sqrt{a^2-1}$ is inside the unit circle. How do we calculate the residue?

→ partial fractions: $f(z) = \frac{1}{(z-p_+)(z-p_-)} = \frac{1}{p_+-p_-}\left(\frac{1}{z-p_+} - \frac{1}{z-p_-}\right)$, so $\mathrm{Res}_{p_+}(f) = \frac{1}{p_+-p_-} = \frac{1}{2\sqrt{a^2-1}}$

→ since this is a simple pole: $\mathrm{Res}_{p_+}(f) = \lim_{z\to p_+}(z-p_+)f(z) = \lim_{z\to p_+}\frac{(z-p_+)}{(z-p_+)(z-p_-)} = \frac{1}{p_+-p_-}$ = same.

Hence $\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = -2i\int_{S^1} f(z)\,dz = 4\pi\,\mathrm{Res}_{p_+}(f) = \frac{2\pi}{\sqrt{a^2-1}}$.