Recall: the residue of f (analytic in $\overline{D^*(p,\varepsilon)}$) at p is $\mathrm{Res}_p(f) = \frac{1}{2\pi i}\int_{S^1(p,\varepsilon)} f(z)\,dz$.
$=$ coeff^t of $(z-p)^{-1}$ in Laurent series of f near p ($= \lim (z-p)f(z)$ if simple pole)

Residue theorem: $\overline{D}$ compact domain with piecewise smooth boundary $\gamma = \partial D$, $P \subset \mathrm{int}(D)$ finite set, f analytic on $U \supset \overline{D} - P$, then $\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{p\in P} \mathrm{Res}_p(f)$.

We now explore how to use this to evaluate various kinds of definite integrals.

Example 1: $\int_0^{2\pi} R(\sin\theta, \cos\theta)\,d\theta$ (or $R(e^{i\theta})$) where R is a rational function (w/o. poles on S^1).

e.g. let's calculate $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$, where $a > 1$.

Set $z = e^{i\theta}$ to turn this into a path integral on S^1: then $d\theta = \frac{1}{i}\, d\log z = \frac{dz}{iz}$ and $\cos\theta = \frac{z+z^{-1}}{2}$.

$$\Rightarrow \int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = \int_{S^1} \frac{dz/z}{\frac{i}{2}(z+2a+z^{-1})} = -2i\int_{S^1}\frac{dz}{z^2+2az+1}$$

The poles are at $p_\pm = -a \pm \sqrt{a^2-1}$; of these, only $p_+ = -a+\sqrt{a^2-1}$ is inside the unit circle. How do we calculate the residue?

$\to$ partial fractions: $f(z) = \frac{1}{(z-p_+)(z-p_-)} = \frac{1}{p_+-p_-}\left(\frac{1}{z-p_+} - \frac{1}{z-p_-}\right)$, so $\mathrm{Res}_{p_+}(f) = \frac{1}{p_+-p_-} = \frac{1}{2\sqrt{a^2-1}}$

$\to$ since this is a simple pole: $\mathrm{Res}_{p_+}(f) = \lim_{z\to p_+}(z-p_+)f(z) = \lim_{z\to p_+}\frac{(z-p_+)}{(z-p_+)(z-p_-)} = \frac{1}{p_+-p_-}$ = same.

Hence $\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = -2i\int_{S^1} f(z)\,dz = 4\pi\,\mathrm{Res}_{p_+}(f) = \frac{2\pi}{\sqrt{a^2-1}}$.

Example 2: $\int_{-\infty}^{\infty} f(x)\,dx$ where f is a rational function $\frac{P(x)}{Q(x)}$

(assume Q has no real roots, and $\deg Q \geq \deg P + 2$, so the integral converges).

The trick here is to recall $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$; and complete the segment $[-R,R]$ to a closed curve in $\mathbb{C}$ by adding a semicircle of radius R in the upper half plane:

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{\substack{\mathrm{Im}\,p>0 \\ \text{and } |p|<R}} \mathrm{Res}_p(f).$$

Now, since $f = \frac{P}{Q}$ with $\deg Q \geq \deg P + 2$, $|f(z)| \leq \frac{C}{|z|^2}$, so $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$.

Hence: taking $R\to\infty$, we get $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\mathrm{Im}(p)>0} \mathrm{Res}_p(f)$.

We can use $\lim_{z\to p}(z-p)f(z)$ to find $\mathrm{Res}_p(f)$ if all poles are simple, else partial fractions. (Of course, the method of partial fractions already allowed us to integrate f!)

<u>Ex</u>: $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^2+1} = 2\pi i\,\mathrm{Res}_i\left(\frac{1}{z^2+1}\right)=\pi$ (which we already knew using arctan)

using $\displaystyle\mathrm{Res}_{z=i}\left(\frac{1}{z^2+1}\right)=\lim_{z\to i}\frac{z-i}{z^2+1}=\lim_{z\to i}\frac{1}{z+i}=\frac{1}{2i}$.

<u>Example 3</u>: mixed rational & exponential functions (now we get to something new!)

Assume $f(z)=\frac{P(z)}{Q(z)}$ is a rational function as before (no real poles, $\deg Q\geq \deg P+2$).

Then we can use the same method as above to calculate $\int_{-\infty}^{\infty} f(z)e^{iz}\,dz$ by considering a large disc in the upper half plane.

The key point is that $|e^{iz}|=e^{-\mathrm{Im}(z)}\leq 1$ in the upper half plane, so the path-integral along the semicircle still goes $\to 0$.

(whereas if integrand has e^{-iz} we'd want to consider the lower half-plane instead)

<u>Ex</u>: $\displaystyle\int_{-\infty}^{\infty}\frac{e^{iz}}{1+z^2}\,dz = 2\pi i\,\mathrm{Res}_{z=i}\left(\frac{e^{iz}}{1+z^2}\right)=2\pi i\cdot e^{-1}.\ \mathrm{Res}_{z=i}\left(\frac{1}{1+z^2}\right)=\frac{2\pi i}{2ie}=\frac{\pi}{e}$.

Taking real and imaginary parts:

$$\int_{-\infty}^{\infty}\frac{\cos x\,dx}{1+x^2}=\frac{\pi}{e},\quad \int_{-\infty}^{\infty}\frac{\sin x\,dx}{1+x^2}=0 \quad \text{(this was expected, by symmetry)}$$

<u>Example 3'</u>: we can actually handle the case $\deg Q=\deg P+1$ (still assuming $\nexists$ real poles)

Then $\int_{-\infty}^{\infty} f(z)e^{iz}\,dz$ still converges, but not absolutely!

(example: $\displaystyle\int_{n\pi}^{(n+1)\pi}\frac{\sin x}{x}\,dx \sim (-1)^n\frac{2}{n}$ convergent series, even though not abs. convergent).

Closing the path in $\mathbb{C}$ also requires some care, to show the integrals along the portions we add do $\to 0$ as radius $\to\infty$: bounding the integrand by $C/|z|$ isn't good enough.

One popular choice = large rectangle, but semicircle is actually fine! The point is that:

- over the portion where $\mathrm{Im}(z)>A$, $|e^{iz}|<e^{-A}$, so $\left|\int f(z)e^{iz}\,dz\right|\lesssim Ce^{-A}\to 0$ as $A\to\infty$
- the portion where $\mathrm{Im}(z)<A$ has length $\lesssim A$, and $|z|\gtrsim R$, so we have a bound by $\frac{CA}{R}$.

If we use eg. $A=\sqrt{R}$ to split things, we still get $\to 0$ as $R\to\infty$.

<u>Eg</u>: $(a>0)$ $\displaystyle\int_{-\infty}^{\infty}\frac{ze^{iz}}{a^2+z^2}\,dz = 2\pi i\,\mathrm{Res}_{z=ia}\left(\frac{ze^{iz}}{a^2+z^2}\right)=2\pi i\left(\frac{ze^{iz}}{z+ia}\right)_{|z=ia}=i\pi e^{-a}$.

Taking imaginary part, $\displaystyle\int_{-\infty}^{\infty}\frac{x\sin x}{a^2+x^2}\,dx=\pi e^{-a}$.

How about... $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$? (G.H. Hardy's note in the Mathematical Gazette, 1909, scores various methods (!)).

→ this one again converges, though not absolutely.

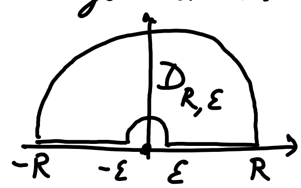
→ $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$ is analytic in the whole plane, so... what residues ??

→ $\frac{\sin z}{z} = \frac{e^{iz} - e^{-iz}}{2iz}$ → ∞ both in upper & lower half plane, so how do we use our trick of closing to a half-disc?

→ however... $\frac{\sin x}{x} = \lim_{a\to 0} \frac{x \sin x}{a^2 + x^2}$, and in fact after a painful discussion of the convergence as $a \to 0$ and interchange of limits, one can check $a \to 0$ is legitimate.

But it is more instructive to see how we can adjust the previous argument to handle $a = 0$.

→ the actual issue: for $x \in \mathbb{R}$, $\frac{\sin x}{x} = \mathrm{Im}\left(\frac{e^{ix}}{x}\right)$, but $\frac{e^{iz}}{z}$ has a pole at 0, on the path of integration. And in fact, $\int_0^{\infty} \frac{e^{ix}}{x}\, dx$ is divergent at 0.



Solution: modify the contour of integration to avoid 0, to carve out a small half-disc from our large semidisc or rectangle.

Observe:

- $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{\sin x}{x}\, dx = \lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \mathrm{Im} \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{e^{iz}}{z}\, dz$
- $\int_{\partial D_{R,\varepsilon}} \frac{e^{iz}}{z}\, dz = 0$ by Cauchy (no poles in $D_{R,\varepsilon}$)
- the integral on the semicircle of radius R tends to 0 as $R \to \infty$ as before
  $\left(\left|\frac{e^{iz}}{z}\right| = \frac{e^{-\mathrm{Im}\, z}}{R} \Rightarrow \text{consider separately regions } \mathrm{Im}(z) < A \text{ and } > A \text{ for } 1 \ll A \ll R. \right)$
- on the semicircle of radius ε: $\mathrm{Res}_0\left(\frac{e^{iz}}{z}\right) = 1$, so we can write
  $\frac{e^{iz}}{z} = \frac{1}{z} + g(z)$ where $g(z)$ is analytic near 0 $\left(g(z) = \frac{e^{iz} - 1}{z}\right)$.

Since g is bounded, $\int_{C_\varepsilon} g(z)\, dz \to 0$ as $\varepsilon \to 0$, whereas $\int_{C_\varepsilon} \frac{1}{z}\, dz = i\pi$ (half of our usual $2\pi i$!).

Combining: $\partial D_{R,\varepsilon} = ([-R,-\varepsilon] \cup [\varepsilon, R]) + C_R - C_\varepsilon$

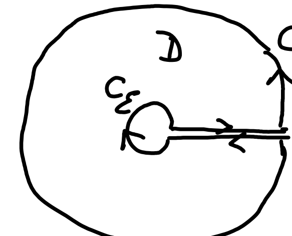
$$\Rightarrow \lim_{\varepsilon\to 0,\, R\to\infty} \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{e^{iz}}{z}\, dz = i\pi \quad \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi .$$

- One more class of examples: <u>non-integer powers of z</u>.

  Consider for example: $\int_0^{\infty} \frac{x^\alpha}{1+x^2}\, dx$ for $0 < \alpha < 1$. (⇒ converges at ∞).

If $\alpha = \frac{p}{q} \in \mathbb{Q}$ we can evaluate by substitution $x = u^q$ to get a rational function (but still not much fun). But here's a more general approach.

Difficulty: $\dfrac{z^\alpha}{1+z^2}$ isn't a single-valued analytic function $\Rightarrow$ how to use residues?



Trick: do something similar to our last example, use a "keyhole" region of integration: $\varepsilon \le |z| \le R$, with a slit along real positive axis.

With a better behaved integrand, the two portions along $[\varepsilon, R]$ would cancel out! But here they don't; we can define $\dfrac{z^\alpha}{1+z^2}$ as an analytic function over $\mathrm{int}(D)$, but its values on either side of the real axis don't match!

Explicitly: we take $z^\alpha = e^{\alpha \log z}$ to be the branch with $\mathrm{Im}(\log z) \in (0, 2\pi)$. Going around the origin, $\log x \rightsquigarrow \log x + 2\pi i$, so x^α gets multiplied by $e^{2\pi i\alpha}$.

So $$\int_{\partial D} \frac{z^\alpha}{1+z^2}\,dz = \int_\varepsilon^R \frac{x^\alpha}{1+x^2}\,dx + \int_{C_R} \frac{z^\alpha}{1+z^2}\,dz - \int_\varepsilon^R \frac{e^{2\pi i\alpha} x^\alpha}{1+x^2}\,dx - \int_{C_\varepsilon} \frac{z^\alpha}{1+z^2}\,dz$$

$\int_{C_R} \to 0$ as $R \to \infty$ $\left(\left|\dfrac{z^\alpha}{1+z^2}\right| \le \dfrac{C}{R^{2-\alpha}},\ \text{length} = 2\pi R,\ \text{and } 2-\alpha > 1 \right)$.

$\int_{C_\varepsilon} \to 0$ as $\varepsilon \to 0$ (integrand and length $\to 0$).

So: $$\lim_{\varepsilon \to 0,\, R \to \infty} \int_{\partial D} \frac{z^\alpha}{1+z^2}\,dz = \left(1 - e^{2\pi i\alpha}\right) \int_0^\infty \frac{x^\alpha}{1+x^2}\,dx.$$

while the residue formula equates this with $2\pi i \left(\mathrm{Res}_{z=i}\left(\dfrac{z^\alpha}{1+z^2}\right) + \mathrm{Res}_{z=-i} \right)$

at $z = i$: $$\lim_{z\to i} \frac{z^\alpha (z-i)}{z^2+1} = \left(\frac{z^\alpha}{z+i}\right)_{|z=i} = \frac{1}{2i} e^{\alpha \log(i)} = \frac{1}{2i} e^{i\frac{\pi}{2}\alpha}$$

at $z = -i$: similarly get $-\dfrac{1}{2i} e^{3i\frac{\pi}{2}\alpha}$.

Hence: $$\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \pi \frac{e^{i\pi\alpha/2} - e^{3i\pi\alpha/2}}{1 - e^{2\pi i\alpha}} = \frac{\pi \sin\left(\frac{\pi\alpha}{2}\right)}{\sin(\pi\alpha)} = \frac{\pi}{2\cos(\pi\alpha/2)}.$$

Our next topic is infinite sum & product expansions (Ahlfors ch. 5.1–5.2)

We've seen: if f is analytic in the annulus $\{R_1 < |z| < R_2\}$ then it has a Laurent series expansion $f(z) = \sum_{-\infty}^{\infty} a_n z^n$, which may or may not have finite negative part.

But ... often there are better representations. E.g., for rational functions, we may factor into products of deg 1 terms to evidence the zeros & poles, or express as sum of $\dfrac{1}{(z-a_i)^k}$ (partial fractions). We'll learn how to do analogues of this for general meromorphic functions.