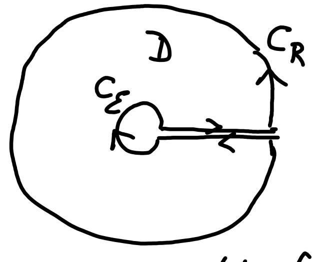
Lecture 34 - Wed. April 20 - Calculating definite integrals using residues, continued

Consider for example: $\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx$ for $0<\alpha<1$. ($\Rightarrow$ converges at ∞).

Difficulty: $\frac{z^\alpha}{1+z^2}$ isn't a single-valued analytic function $\Rightarrow$ how to use residues?

Trick: "keyhole" region of integration: $\varepsilon \le |z| \le R$, with a slit along real positive axis.



With a better behaved integrand, the two portions along $[\varepsilon, R]$ would cancel out! But here they don't; we can define $\frac{z^\alpha}{1+z^2}$ as an analytic function over $\mathrm{int}(D)$, but its values on either side of the real axis don't match!

Explicitly: we take $z^\alpha = e^{\alpha \log z}$ to be the branch with $\mathrm{Im}(\log z) \in (0, 2\pi)$.
Going around the origin, $\log x \rightsquigarrow \log x + 2\pi i$, so x^α gets multiplied by $e^{2\pi i \alpha}$.

So $\int_{\partial D} \frac{z^\alpha}{1+z^2}\,dz = \int_\varepsilon^R \frac{x^\alpha}{1+x^2}\,dx + \int_{C_R} \frac{z^\alpha}{1+z^2}\,dz - \int_\varepsilon^R \frac{e^{2\pi i\alpha} x^\alpha}{1+x^2}\,dx - \int_{C_\varepsilon} \frac{z^\alpha}{1+z^2}\,dz$

$\int_{C_R} \to 0$ as $R \to \infty$ $\left(\left|\frac{z^\alpha}{1+z^2}\right| \le \frac{C}{R^{2-\alpha}}, \text{ length} = 2\pi R, \text{ and } 2-\alpha > 1 \right)$.

$\int_{C_\varepsilon} \to 0$ as $\varepsilon \to 0$ (integrand and length $\to 0$).

So: $\lim_{\varepsilon\to 0,\, R\to\infty} \int_{\partial D} \frac{z^\alpha}{1+z^2}\,dz = (1-e^{2\pi i\alpha}) \int_0^\infty \frac{x^\alpha}{1+x^2}\,dx$.

while the residue formula equates this with $2\pi i \left(\mathrm{Res}_{z=i}\left(\frac{z^\alpha}{1+z^2}\right) + \mathrm{Res}_{z=-i} \right)$

at $z=i$: $\lim_{z\to i} \frac{z^\alpha (z-i)}{z^2+1} = \left(\frac{z^\alpha}{z+i}\right)_{|z=i} = \frac{1}{2i} e^{\alpha \log(i)} = \frac{1}{2i} e^{i\frac{\pi}{2}\alpha}$

at $z=-i$: similarly get $-\frac{1}{2i} e^{3i\frac{\pi}{2}\alpha}$.

Hence: $\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \pi \frac{e^{i\pi\alpha/2} - e^{3i\pi\alpha/2}}{1-e^{2\pi i\alpha}} = \frac{\pi \sin(\frac{\pi\alpha}{2})}{\sin(\pi\alpha)} = \frac{\pi}{2\cos(\pi\alpha/2)}$.

Our next topic is <u>infinite sum & product expansions</u> (Ahlfors ch. 5.1-5.2)

We've seen: if f is analytic in a disc or annulus then we can write as an infinite sum = power series or Laurent series. But ... often there are more insightful ways to represent a function. Eg, for rational functions, Laurent series are less useful than factoring as a product or using partial fractions to express as a sum:

$R(z) = c \cdot \frac{\prod (z-a_i)^{n_i}}{\prod (z-b_i)^{m_i}} = \frac{C_1(z)}{(z-b_1)^{m_1}} + \dots + \frac{C_\ell(z)}{(z-b_\ell)^{m_\ell}} + S(z)$ where $C_1 \dots C_\ell, S$ polynomials, $\deg(C_i) \le m_i - 1$.

(such expressions tell us directly about poles & residues!)

We now want to find similar expressions for general meromorphic functions.

* Starting point: if $f(z)$ is meromorphic with a pole of order m at $b \in \mathbb{C}$, then we can write $f(z) = \dfrac{g(z)}{(z-b)^m}$ with $g(z)$ analytic in a nbd. of b.

Or, expressing $g(z)$ as a power series in $(z-b)$, $g(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$

we have a Laurent series for f with finite negative part, as already noticed:

$$f(z) = \underbrace{\left[\frac{a_0}{(z-b)^m} + \frac{a_1}{(z-b)^{m-1}} + \dots + \frac{a_{m-1}}{z-b}\right]}_{\text{THE POLAR PART OF } f \text{ at } z=b} + h(z), \quad h(z) = \sum_{n=0}^{\infty} a_{m+n}(z-b)^n$$

analytic near b.

This looks a lot like partial fractions, and in fact, for rational functions, it <u>is</u> partial fractions: if f is meromorphic with finitely many poles $b_1 \dots b_\ell$, by induction on #poles (observe: remainder $h(z)$ has one fewer pole than f), we get

$$f(z) = \frac{c_1(z)}{(z-b_1)^{m_1}} + \dots + \frac{c_\ell(z)}{(z-b_\ell)^{m_\ell}} + g(z), \quad c_i(z) \text{ polynomials of degree} < m_i,$$

where $g(z)$ is now analytic everywhere. <u>What if there's ∞ many poles?</u>

* Given $f(z)$ meromorphic on all of $\mathbb{C}$, with infinitely many (isolated) poles $b_1, b_2, \dots$ we have near each b_j the polar part (=(finite) negative part) of the Laurent expansion,

$$P_j\left(\frac{1}{z-b_j}\right) = \frac{a_{-m}}{(z-b_j)^m} + \dots + \frac{a_{-1}}{z-b_j}$$

(a polynomial without constant term in the variable $\frac{1}{z-b_j}$).

and we hope to be able to write $f(z) = \sum_{j=1}^{\infty} P_j\left(\frac{1}{z-b_j}\right) + g(z)$

where $g(z)$ no longer has any poles hence is an entire function.

<u>Questions</u>:
- → when do these kinds of sums converge? uniformly?
- → what meromorphic functions can be represented in such a way?
- → existence: given a discrete set of poles b_j and orders m_j, does there exist a meromorphic function with exactly those poles? can we prescribe the polar parts $P_j\left(\frac{1}{z-b_j}\right)$ arbitrarily?

(Apparent problem: expressions like $\sum_{n\in\mathbb{Z}} \frac{1}{z-n}$ don't seem to make sense?)

<u>Example</u>: let's consider the function $f(z) = \dfrac{\pi^2}{\sin^2(\pi z)}$, with poles (of order 2) exactly at the integers.

The polar part at 0 can be found by expanding

$$\sin \pi z = \pi z - \frac{\pi^3}{6} z^3 + \dots \Rightarrow \sin^2(\pi z) = \pi^2 z^2 - \frac{\pi^4}{3} z^4 + \dots = \pi^2 z^2 \left(1 - \frac{\pi^2}{3} z^2 + \dots\right)$$

so $\frac{\pi^2}{\sin^2(\pi z)} = \frac{1}{z^2}\left(1 + \frac{\pi^2}{3}z^2 + \dots\right)$ $\Rightarrow$ the polar part at 0 is just $\frac{1}{z^2}$.

Since f is periodic $(f(z+1)=f(z))$, the polar part at $z=n\in\mathbb{Z}$ is $\frac{1}{(z-n)^2}$.

Observe: the sum $h(z)=\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ is convergent $\forall z\in\mathbb{C}-\mathbb{Z}$

and the convergence is uniform over compact subsets of $\mathbb{C}-\mathbb{Z}$ (prove it!)

(key observation: $K\subset\mathbb{C}-\mathbb{Z}$ compact $\Rightarrow$ $K\subset B(0,R)$, so for $|n|$ large $>>R$ the terms are bounded by $\sum_{|n|>n_0} \frac{1}{(|n|-R)^2}$, which converges. Apply M-test.)

so the sum is an analytic function on $\mathbb{C}-\mathbb{Z}$, easily checked to have the correct behavior (pole of order 2 with polar part $\frac{1}{(z-n)^2}$) at each $n\in\mathbb{Z}$:

indeed $h(z)-\frac{1}{(z-n)^2} = \sum_{m\neq n}\frac{1}{(z-m)^2}$ converges uniformly near $z=n$ hence analytic at n.

Hence: $g(z)=f(z)-h(z)$ is an entire analytic function (the polar parts cancel at each $z=n$)

$\rightarrow$ can write
$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2} + g(z)$$

where $g(z)$ is an entire function, periodic: $g(z+1)=g(z)$. What is g?

Observe: for $\operatorname{Im}(z)\to+\infty$, $|e^{i\pi z}| = e^{-\pi \operatorname{Im} z} << e^{\pi \operatorname{Im} z} = |e^{-i\pi z}|$, so $|f(z)| \approx \frac{4\pi^2}{e^{2\pi \operatorname{Im} z}} \to 0$.

& similarly for $\operatorname{Im} z\to-\infty$.

Meanwhile, for $h(z)$: if $z=x+iy$, $y\to+\infty$, $x\in[0,1]$ wlog by periodicity,

then $\left|\frac{1}{(z-n)^2}\right| = \frac{1}{|z-n|^2} = \frac{1}{(n-x)^2+y^2}$ $\Rightarrow$ terms with $|n|<y$ are $\leq 1/y^2$

$|n|>y$ are $\leq 1/(n-1)^2$

$\Rightarrow |h(z)| \leq 2y\cdot\frac{1}{y^2} + 2\sum_{n\geq y}\frac{1}{n^2} \leq \frac{C}{y}$. Similarly for $y\to-\infty$.

So: $g(z)$ is an entire function, $g(z+1)=g(z)$, $|g(z)|\to 0$ as $|\operatorname{Im} z|\to\infty$ (uniformly $\forall \operatorname{Re} z$)

$\rightarrow$ g is bounded on $\mathbb{C}$, hence constant!! and since $g\to 0$ as $y\to\infty$, the constant is 0.

Conclusion:
$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}$$

Q: what about simple poles? can we find $f(z)$ with simple poles at all integers, and residue 1 at each? and can we express it as a partial fraction type of sum?

In terms of partial fractions, the natural guess would be $\sum_{n\in\mathbb{Z}}\frac{1}{z-n}$ ... but this series doesn't converge!

Solution: add to each term an analytic function of z to cancel the divergence.
In this case: just subtract from each term its value at 0, ie. $-1/n$:

$$f(z) = \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \frac{1}{z} + \sum_{n\neq 0} \frac{z}{n(z-n)}$$

This series now converges $\forall z \in \mathbb{C} - \mathbb{Z}$, uniformly over compact subsets, and has the desired polar part at each integer point.

Can we use a similar trick to build meromorphic functions with arbitrary poles and polar parts at each pole? Answer: yes, but we may need to add more complicated counter-terms to achieve convergence.

Thm: Let $\{b_j\}$ be an arbitrary set of complex numbers with no limit points, and for each j, P_j an arbitrary polynomial without constant term. Then there exists a meromorphic function $f(z)$ on all of $\mathbb{C}$, analytic on $\mathbb{C} - \{b_j\}$, and whose polar part at b_j is $P_j\left(\frac{1}{z-b_j}\right)$ $\forall j$.

Pf: The proof uses the same idea as above, except to achieve convergence we subtract from each $P_j\left(\frac{1}{z-b_j}\right)$ (for $b_j \neq 0$) a polynomial in z: given $m_j \geq 0$ integer, let $q_j(z)$ = sum of the terms of degree $\leq m_j$ of the Taylor series of $P_j(1/(z-b_j))$ at $z=0$. The point (see Ahlfors §5.2.1) is that we can choose the m_j's so that the series $f(z) = \sum_j \left(P_j\left(\frac{1}{z-b_j}\right) - q_j(z) \right)$ converges on $\mathbb{C} - \{b_j\}$.

How does one show this? First observe: $\{b_j\}$ no limit points $\Rightarrow$ only finitely many inside any compact subset of $\mathbb{C}$ $\Rightarrow |b_j| \to \infty$. Next, we need explicit bounds on the remainder $P_j\left(\frac{1}{z-b_j}\right) - q_j(z)$ of the Taylor series of $P_j\left(\frac{1}{z-b_j}\right)$. Base case:

$$\frac{1}{z-b_j} = -\frac{1}{b_j}\,\frac{1}{1-\frac{z}{b_j}} = -\frac{1}{b_j}\left(1 + \frac{z}{b_j} + \left(\frac{z}{b_j}\right)^2 + \dots\right) \text{ with remainder } \left(\frac{z}{b_j}\right)^{m_j+1} \frac{1}{z-b_j},$$

$\Longrightarrow$ (after more work...) $\left| P_j\left(\frac{1}{z-b_j}\right) - q_j(z) \right| \leq C_j \left(\frac{|z|}{|b_j|}\right)^{m_j+1}$ whenever $|z| \leq \frac{|b_j|}{2}$, where C_j depends on $P_j(\dots)$ but <u>not</u> on m_j.

Now, pick m_j's suff$^{\text{tly}}$ large, eg. so $\frac{C_j}{2^{m_j+1}} \leq \frac{1}{j^2}$. Then $|z| \leq \frac{|b_j|}{2} \Rightarrow \left| P_j\left(\frac{1}{z-b_j}\right) - q_j(z) \right| \leq \frac{1}{j^2}$.

Since $|b_j| \to \infty$, this implies uniform convergence over compact subsets of $\mathbb{C}$.
(since all but finitely many terms of the series are bounded by $\sum \frac{1}{j^2}$) $\square$

- Back to our function with simple poles at all integers,

$$f(z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n\neq 0}\frac{z}{n(z-n)}$$

Since convergence is uniform on compact subsets of $\mathbb{C}-\mathbb{Z}$, using analyticity, we can differentiate term by term. (recall: f_n analytic, $f_n \to f$ uniformly $\Rightarrow f_n' \to f'$ uniformly on compacts).

We find $f'(z) = -\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \frac{-\pi^2}{\sin^2(\pi z)}$!

Recall: $\cot(t) = \frac{\cos(t)}{\sin(t)}$ has derivative $\cot'(t) = \frac{\sin\cdot\cos' - \cos\cdot\sin'}{\sin^2(t)} = \frac{-1}{\sin^2 t}$

$\Rightarrow$ hence: $f(z) = \pi\cot(\pi z) + C$.

Since both sides are odd functions of z $(f(-z) = -f(z))$, necess. $C=0$.

Hence: $\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)$.

Remark: there's another way to achieve convergence in this case, instead of the general method of polynomial counter-terms: combining the terms for $\pm n$,

$\frac{1}{z-n} + \frac{1}{z+n} = \frac{2z}{z^2-n^2}$ which form a convergent series. (while $+\frac{1}{n} - \frac{1}{n}$ cancel).

Hence: $\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\geq 1}\frac{2z}{z^2-n^2}$.