Munkres ch.2 (§12–22) Lec. 2

1 Topological spaces: $(X,\mathcal{T})$, $\mathcal{T}=\{U\subset X \mid U \text{ open}\}$.

Axioms: $\emptyset$, X, arbitrary unions, finite intersections of open subsets are open.

$F\subset X$ closed $\iff$ F^c open.

- Basis for a topology:
  - open sets = unions of elements of $\mathcal{B}$. U open $\iff \forall x\in U\ \exists B\in\mathcal{B}$ st. $x\in B\subset U$.
  - axioms for basis: $\bigcup\mathcal{B}=X$; $x\in B_1\cap B_2 \Rightarrow \exists B'\in\mathcal{B}$ st. $x\in B'\subset B_1\cap B_2$.
  - Ex: open balls $B_r(x)$ in a metric space (X,d) basis for the metric topology.
- if $\mathcal{T}\subset\mathcal{T}'$ say $\mathcal{T}'$ finer / $\mathcal{T}$ coarser.
- $f: X\to Y$ is continuous if $\forall\, U\subset Y$ open, $f^{-1}(U)\subset X$ is open.
  (for metric spaces, this is $\iff \forall p\in X, \forall\varepsilon>0, \exists\delta>0$ st. $d_X(p,x)<\delta \Rightarrow d_Y(f(p),f(x))<\varepsilon$).

Lec. 3
- closure: $\bar{A}=\bigcap$ all closed subsets $\supset A$, interior $\mathrm{int}(A)=\bigcup$ all open subsets $\subset A$.
  $\bar{A}=A\cup\{\text{limit pts of } A\}$. $x\in\bar{A}\iff$ every nbd of x intersects A.
- limit points of subsets (x limit pt of A $\iff \forall U\ni x$ neighborhood, $(U-\{x\})\cap A\neq\emptyset$)
  $\neq$ limits of sequences ($x_n\to x \iff \forall U\ni x$ neighborhood, all but finitely many $x_n\in U$.)
- subspace topology on $A\subset X$: $\{U\cap A \mid U\in\mathcal{T}_X\}$

Lec. 4
- product topology: basis $\left\{\prod_{i\in I} U_i \mid U_i\subset X_i \text{ open}, \ U_i=X_i \text{ for all but finitely many } i\right\}$
  ↑ if omit this, get box topology (finer).
  For products of metric spaces, the uniform topology ($d_\infty(\vec{x},\vec{y})=\sup d_i(x_i,y_i)$ up to truncation) is inbetween.
  $f=(f_i): Z\to X=\prod X_i$ is continuous in product top. iff each $f_i=\pi_i\circ f: Z\to X_i$ is continuous.
- quotient topology on $Y=X/\sim$: $U\subset Y$ is open $\iff q^{-1}(U)=\{x\in X \mid [x]\in U\}$ is open in X.

Lec. 10–11
  $\bar{f}: Y\to Z$ continuous $\iff f=\bar{f}\circ q: X\to Z$ continuous & compatible with $\sim$ ($[x]=[x'] \Rightarrow f(x)=f(x')$).

Lec. 9–10 Munkres ch.4 §30–34
- X is Hausdorff if $\forall x\neq y$, $\exists U\ni x, V\ni y$ open st. $U\cap V=\emptyset$.
  Stronger separation axioms (regular, normal) separate points from closed sets / closed sets from each other by disjoint opens. Metric spaces are normal ($\Rightarrow$ Hausdorff).
  Urysohn's thm: normal (or regular) spaces with countable basis are metrizable.

Munkres ch.3 §23–24 26–29

2 Connectedness & compactness:

- X is connected if $X=U\cup V$, U,V open disjoint $\Rightarrow$ one is X and the other is $\emptyset$.

Lec. 5
- $f: X\to Y$ continuous, X connected $\Rightarrow f(X)$ connected. ($\Rightarrow$ intermediate value theorem) (connected subsets of $\mathbb{R}$ are intervals).
- path-connected := any two points of X can be joined by a path $f: I\to X$.
  path-conn. $\Rightarrow$ connected ($\not\Leftarrow$ in general)

Lec. 6
- X is compact if $\forall$ open cover $X=\bigcup_{i\in I}U_i$, $\exists$ finite subcover $X=U_{i_1}\cup\dots\cup U_{i_n}$.
- $f: X\to Y$ continuous, X compact $\Rightarrow f(X)$ compact ($\Rightarrow$ extreme value thm).
- X compact, $F\subset X$ closed $\Rightarrow F$ compact. $K\subset X$ Hausdorff, K compact $\Rightarrow K$ closed.
  in $\mathbb{R}^n$, compact $\iff$ closed and bounded.
- (finite) products of $\left\{\begin{matrix}\text{compact}\\ \text{connected}\end{matrix}\right\}$ spaces are $\left\{\begin{matrix}\text{compact}\\ \text{connected}\end{matrix}\right\}$.

Lec.7 • If (X,d) metric space & compact then:

- every open cover $X=\bigcup U_i$ has a Lebesgue number $\delta>0$: $\operatorname{diam}(A)<\delta \Rightarrow \exists i$ st. $A\subset U_i$.
- every continuous function $f:(X,d)\to(Y,d_Y)$ is uniformly continuous

Lec.8 • For metric spaces, compact (open covers have finite subcovers) $\Longleftrightarrow$ limit point compact (every infinite subset has a limit point) $\Longleftrightarrow$ sequentially compact (every sequence has a convergent subsequence)

(always $\Rightarrow$) (always $\Leftarrow$).

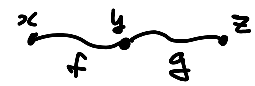
• A one-point compactification of X is a compact space Y st. $Y-\{\infty\} \underset{\text{homeo}}{\simeq} X$.

Build: $Y=X\cup\{\infty\}$, $\mathcal{T}_Y$ = opens of X + complements of compact subsets of X.

Lec.9 If X is locally compact ($\forall x\in X\ \exists$ nbd. $U\ni x$ and compact $C\supset U\ni x$) and Hausdorff then Y is Hausdorff and unique up to homeo.

Munkres ch.9 §51-55 +58-60 ③ Homotopy and fundamental group:

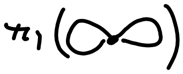
- homotopy: $f_0,f_1: X\to Y$ continuous: a homotopy is $F: X\times I\to Y$ continuous, $F_{|X\times 0}=f_0$, $F_{|X\times 1}=f_1$.

Lec.11
- paths $f_0,f_1: I\to Y$ are path-homotopic if $\exists$ homotopy $F: I\times I\to Y$ fixing end points.
- path-homotopy classes of paths in X form a groupoid for path composition $f*g$ 

Lec.12-13.
- ——— " ——— loops based at x_0 (= paths $x_0\to x_0$) = fundamental group $\pi_1(X,x_0)$

(product = composition, identity = constant loop, inverse = reverse loop)

- $x_0,x_1\in$ same path-component of $X \Rightarrow \pi_1(X,x_0)\cong\pi_1(X,x_1)$ (by attaching path $\alpha*f*\alpha^{-1}$).
- $f:(X,x_0)\to(Y,y_0)$ induces homomorphism $f_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$. Functorial ($(f\circ g)_*=f_*\circ g_*$).
- Ex: $\mathbb{R}^n$, convex subsets of $\mathbb{R}^n$, S^n $n\geq 2$ are simply connected ($\pi_1=\{1\}$).

Lec14-15 $\pi_1(S^1,b_0)\simeq\mathbb{Z}$, $\pi_1($$)\simeq\mathbb{Z}^2$, $\pi_1($$)\simeq$ free group $\langle a,b\rangle$

- Applications of $\pi_1(S^1)=\mathbb{Z}$:
  - $\nexists$ retraction $r: B^2\to S^1$ (ie. r continuous, $r_{|S^1}=id_{S^1}$).
  - every continuous $f: B^2\to B^2$ has a fixed pt ($f(x)=x$) (Brouwer)

• Deformation retraction: $r: X\to A$ retraction ($r_{|A}=id_A$) st. $i\circ r$ is homotopic to id_X

Lec.11-12 among maps that leave A fixed. ie. $H: X\times I\to X$, $H(x,0)=x$, $H(x,1)\in A\ \forall x\in X$, $H(a,t)=a\ \forall a\in A$

Then $\pi_1(A,a_0)\simeq\pi_1(X,a_0)$ (i_*,r_* inverse isoms.)

• The same holds more generally for homotopy equivalences $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, $g\circ f\simeq id_X$, $f\circ g\simeq id_Y$.

Lec.14 • Covering spaces: $p: E\to B$, $\forall b\in B\ \exists U\ni b$ evenly covered by p

($p^{-1}(U)$ = disjoint union of slices V_α, each $p_{|V_\alpha}: V_\alpha \underset{\text{homeo}}{\xrightarrow{\sim}} U$).

- every path $f: I\to B$ starting at b_0 has unique lift $\tilde{f}: I\to E$ starting at $e_0\in p^{-1}(b_0)$. (Path) homotopies lift to (path) homotopies.
- Looking at end points of lifts of loops in (B,b_0), get lifting map $\pi_1(B,b_0)\to p^{-1}(b_0)$.

- Those loops which lift to a loop in (E, e_0) form a subgroup $H \subset \pi_1(B, b_0)$, and $p_* : \pi_1(E, e_0) \xrightarrow{\sim} H \subset \pi_1(B, b_0)$.

Lec. 16

- A map $g : (Y, y_0) \to (B, b_0)$ lifts to $\tilde{g} : (Y, y_0) \to (E, e_0)$ iff $g_*(\pi_1(Y, y_0)) \subset H$.

Munkres §79-80

- Classification of covering spaces (up to equivalence) $\Leftrightarrow$ classif. of subgroups $H \subset \pi_1(B)$ (up to conjugacy).
- Universal cover: simply-connected E (ie., $H = \{1\}$).

Lec. 17

Munkres §70

- <u>Van Kampen</u>: $X = U \cup V$, U, V open, $U \cap V \ni x_0$ path-connected $\Rightarrow$
  - $\pi_1(X, x_0)$ is generated by the images of $i_{1*} : \pi_1(U) \to \pi_1(X)$, $i_{2*} : \pi_1(V) \to \pi_1(X)$
  - if $\pi_1(U \cap V) = \{1\}$ then $\pi_1(X)$ is the free product $\pi_1(U) * \pi_1(V)$
  - otherwise, quotient by smallest normal subgroup that makes $i_{1*}(g) = i_{2*}(g)$ $\forall g \in \pi_1(U \cap V)$.