Recall from last time: given a meromorphic f^n $f(z)$ with discrete set of poles $\{b_j\}$, want to write $f(z) = \sum_j P_j\left(\frac{1}{z-b_j}\right) + g(z)$, g analytic entire function.

P_j ← polar part of Laurent series at b_j: $\frac{a_{-m}}{(z-b_j)^m} + \dots + \frac{a_{-1}}{z-b_j}$

Ex: we derived $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ by checking

- the sum defines a meromorphic function (series converges locally uniformly on $\mathbb{C}-\mathbb{Z}$)
- the polar parts at each $z=n$ agree, so the difference is an entire function; we also checked the difference is bounded and $\to 0$ as $|\mathrm{Im}\, z|\to\infty$ ⇒ it's zero.

Q: what about Simple poles? can we find $f(z)$ with simple poles at all integers, and residue 1 at each, expressed as a partial fraction type of sum?

The most natural guess would be $\sum_{n\in\mathbb{Z}} \frac{1}{z-n}$ ... but this doesn't converge.

Solution: add to each term an analytic function of z to cancel the divergence.

In this case: just subtract from each term its value at 0, ie. $-1/n$:

$$f(z) = \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n\neq 0} \frac{z}{n(z-n)}$$

This series now converges $\forall z\in\mathbb{C}-\mathbb{Z}$, uniformly over compact subsets, and has the desired polar part at each integer point.

Can we use a similar trick to build meromorphic functions with arbitrary poles and polar parts at each pole? Answer: yes, but we may need to add more complicated counter-terms to achieve convergence.

Thm: Let $\{b_j\}$ be an arbitrary set of complex numbers with no limit points, and for each j, P_j an arbitrary polynomial without constant term. Then there exists a meromorphic function $f(z)$ on all of $\mathbb{C}$, analytic on $\mathbb{C}-\{b_j\}$, and whose polar part at b_j is $P_j\left(\frac{1}{z-b_j}\right)$ $\forall j$.

Pf: The proof uses the same idea as above, except to achieve convergence we subtract from each $P_j\left(\frac{1}{z-b_j}\right)$ (for $b_j\neq 0$) a polynomial in z: given $m_j\geq 0$ integer, let $q_j(z)$ = sum of the terms of degree $\leq m_j$ of the Taylor series of $P_j(1/(z-b_j))$ at $z=0$. The point (see Ahlfors §5.2.1) is that we can choose the m_j's so that the series $f(z) = \sum_j \left(P_j\left(\frac{1}{z-b_j}\right) - q_j(z)\right)$ converges on $\mathbb{C}-\{b_j\}$.

How does one show this? First observe: $\{b_j\}$ no limit points $\Rightarrow$ only finitely many inside any compact subset of $\mathbb{C}$ $\Rightarrow |b_j| \to \infty$. Next, since $P_j\left(\frac{1}{z-b_j}\right)$ is analytic in the disc of radius $|b_j|$, its Taylor series converges uniformly on $\overline{D(b_j/2)}$

$\Rightarrow$ for m_j suff. large, $q_j(z)$ = sum of terms of degree $\leq m_j$ in Taylor series satisfies

$$\left|P_j\left(\frac{1}{z-b_j}\right) - q_j(z)\right| \leq \frac{1}{j^2} \quad \forall z \text{ s.t. } |z| \leq \frac{|b_j|}{2}.$$

Since $|b_j| \to \infty$, this implies uniform convergence of the series $\sum_j \left(P_j\left(\frac{1}{z-b_j}\right) - q_j(z)\right)$ over compact subsets of $\mathbb{C}$. (since all but finitely many terms of the series are bounded by $\sum \frac{1}{j^2}$) $\square$

Back to our function with simple poles at all integers,

$$f(z) = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n \neq 0} \frac{z}{n(z-n)}$$

Since convergence is uniform on compact subsets of $\mathbb{C} - \mathbb{Z}$, using analyticity, we can differentiate term by term. (recall: f_n analytic, $f_n \to f$ uniformly $\Rightarrow f_n' \to f'$ uniformly on compacts).

We find $f'(z) = -\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} = \frac{-\pi^2}{\sin^2(\pi z)}$!

Recall: $\cot(t) = \frac{\cos(t)}{\sin(t)}$ has derivative $\cot'(t) = \frac{\sin \cdot \cos' - \cos \cdot \sin'}{\sin^2(t)} = \frac{-1}{\sin^2 t}$

$\Rightarrow$ hence: $f(z) = \pi \cot(\pi z) + C$.

Since both sides are odd functions of z $(f(-z) = -f(z))$, necess. $C = 0$.

Hence: $$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Remark: there's another way to achieve convergence in this case, instead of the general method of polynomial counter-terms: combining the terms for $\pm n$,

$$\left(\frac{1}{z-n} + \frac{1}{n}\right) + \left(\frac{1}{z+n} - \frac{1}{n}\right) = \frac{1}{z-n} + \frac{1}{z+n} = \frac{2z}{z^2 - n^2}$$ which form a convergent series.

Hence: $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n \geq 1} \frac{2z}{z^2 - n^2}$. (rearrangement of Σ is legal for the series with the counterterms, which converges absolutely. Non abs. convergent series aren't safe to rearrange.)

Next we look at infinite products. The convention/definition we want to use for products is:

Def: $\prod_{i=1}^{\infty} p_i$ converges if 1) at most finitely many terms p_i are zero, and

2) the products of the nonzero terms $\prod_{\substack{1 \leq i \leq n \\ p_i \neq 0}} p_i$ converge to a nonzero limit as $n \to \infty$.

This feels awkward, and less natural than the obvious idea ($\prod_{i=1}^{n} p_i$ converges to a limit which may be zero), but is more suitable for expressing analytic functions as products.

The requirements ensure:

→ adding/removing finitely many factors to the product doesn't affect convergence

→ when a convergent product of analytic functions vanishes, it does so to a finite order ($=\sum$ orders of the factors that equal zero) & we can factor out the zeros. (a convergent product of nonzero factors is nonzero by definition of convergence!)

→ for nonzero products, the convergence of $\prod p_i$ is equivalent to that of $\sum \log p_i$.

Since convergence forces $\log(p_i)\to 0$ ie. $p_i \to 1$, it's customary to write infinite products in the form $\prod_{n=1}^{\infty}(1+a_n)$, and convergence $\Leftrightarrow$ $\sum \log(1+a_n)$ converges

(with necessarily $a_n \to 0$, so we take our <u>preferred choice of log, with $|\mathrm{Im}(\log)|<\pi$.</u>)

Moreover: $\sum \log(1+a_n)$ converges absolutely iff $\sum a_n$ converges absolutely

(using comparison: since either implies $a_n \to 0$, for suff. large n we have $\frac{|a_n|}{2} \le |\log(1+a_n)| \le 2|a_n|$)

When this happens we say the product <u>converges absolutely</u>. However, non-absolute convergence may involve more subtle cancellations, and cannot be reduced to that of $\sum a_n$.

<u>Goal</u>: given an entire analytic function $f(z)$, express it as a product that makes the zeroes of f immediately apparent, just as we write a polynomial in the form $c\prod(z-b_i)^{m_i}$

- Since an infinite product of $(z-b_i)$'s isn't going to converge, instead we aim for a product of factors of the form $\prod_{i=1}^{\infty}\left(1-\frac{z}{b_i}\right)^{m_i}$. (For $b_i \neq 0$. If f has a zero at $z=0$, we keep that factor as z^{m_0}).
- If the infinite product converges $\forall z$, and if the convergence is <u>uniform</u> on compact subsets of $\mathbb{C}\setminus\{b_i\}$ (which, by definition, means $\sum m_i \log\left(1-\frac{z}{b_i}\right)$ converges uniformly), then it defines an analytic function with the same zeros as f. So the ratio of $f(z)$ and this function is an entire function without zeros, hence can be written as $e^{g(z)}$ for some entire analytic function $g(z)$ (cf. homework! eg: lifting lemma for $\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \mathbb{C} \arrow[ur, dashed, "g"] \arrow[r] & \mathbb{C}^* \end{tikzcd}$)
- In summary: our hope is to arrive at $f(z) = z^{m_0} e^{g(z)} \prod_{i=1}^{\infty}\left(1-\frac{z}{b_i}\right)^{m_i}$.

Just like the case of sums, the questions that come up are:

→ can we represent given functions in this way?

→ when do these expressions converge?

→ given $b_i \in \mathbb{C}$ without limit points (ie. $b_i \to \infty$), can we find an entire function with zeros of prescribed orders at b_i?

The answers to these questions parallel what we did with partial fractions.
Just like last time, we start with an example: the function $\sin(\pi z)$.

Expressing $\sin(\pi z)$ as an infinite product:

Since $\sin(\pi z)$ has zeros exactly at the integers, our naive guess is $z\prod_{n\neq 0}\left(1-\frac{z}{n}\right)$.
Unfortunately the series $\sum \log\left(1-\frac{z}{n}\right)$ diverges (just like $\sum \frac{1}{n}$).
Just like we did for partial fractions, we cancel the divergence by subtracting from each term the beginning of its Taylor series.
Here: $\log\left(1-\frac{z}{n}\right) = -\frac{z}{n} - \frac{z^2}{2n^2} - \dots$ so we can consider $\sum\left(\log\left(1-\frac{z}{n}\right)+\frac{z}{n}\right)$, which converges (comparison: $\sum \frac{z^2}{n^2}$ converges). This yields the product

$$z\prod_{n\neq 0}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right)$$

, which does converge (by convergence of $\sum \log(\dots)$).

Now we can write $\sin(\pi z) = z\,e^{g(z)}\prod_{n\neq 0}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right)$ for some analytic $g(z)$.

How do we find $g(z)$? Answer: compare logarithmic derivatives $\left(\frac{f'(z)}{f(z)}\right)$ for both sides

Note: uniform convergence of the series $\sum\left(\log\left(1-\frac{z}{n}\right)+\frac{z}{n}\right)$ over compact subsets of $\mathbb{C}-\mathbb{Z}$ implies that we can differentiate term by term $\left(\left(\sum f_n\right)' = \sum (f_n')\right)$
(remember, this is for analytic f^ns. In real analysis we need to assume the uniform convergence of $\sum(f_n')$, not just that of $\sum f_n$.)
so that the logarithmic derivative of a product is the sum of those of the factors.

Logarithmic derivatives.

$$\sin \pi z \rightsquigarrow \frac{\pi\cos\pi z}{\sin \pi z} = \pi\cot(\pi z)$$

$$z \rightsquigarrow 1/z$$

$$\prod_{n\neq 0}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right) \rightsquigarrow \sum_{n\neq 0}\left(\frac{-1/n}{1-z/n}+\frac{1}{n}\right) = \sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

$$e^{g(z)} \rightsquigarrow g'(z).$$

So: $\pi\cot(\pi z) = \frac{1}{z} + g'(z) + \sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right)$.

$\Rightarrow$ using formula seen last time, $g'(z)=0$, so $e^{g(z)}$ = constant.

To find the constant c s.t. $\sin(\pi z) = c\,z\prod_{n\neq 0}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right)$,
divide both sides by z and evaluate at $z=0$ $\Rightarrow \lim_{z\to 0}\frac{\sin \pi z}{z} = c$.

So $c=\pi$ and $\sin(\pi z) = \pi z\prod_{n\neq 0}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right)$.

Or combining the terms corresponding to $+n$ and $-n$, $\sin(\pi z) = \pi z\prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$.

The general existence theorem: analogous to what we've seen for sums,

Thm: Given a subset $\{b_1, b_2, \dots\} \subset \mathbb{C}$ with $|b_j| \to \infty$ ($\Leftrightarrow$ no limit points), and multiplicities $m_j \geq 1$, there exists an entire analytic function $f(z)$ with zeroes exactly at the points b_j, with order m_j at each.

The proof is the same as for partial fractions: we want to modify the sum $\sum m_j \log\left(1 - \frac{z}{b_j}\right)$ to achieve convergence. As before we do this by subtracting part of the Taylor series (*) $\log\left(1 - \frac{z}{b_j}\right) = -\frac{z}{b_j} - \frac{z^2}{2b_j^2} - \dots$ stopping at some degree d_j.

$\Rightarrow$ we consider the infinite product
$$z^{m_0} \prod_j \left[\left(1 - \frac{z}{b_j}\right) e^{\frac{z}{b_j} + \frac{1}{2}\left(\frac{z}{b_j}\right)^2 + \dots + \frac{1}{d_j}\left(\frac{z}{b_j}\right)^{d_j}}\right]^{m_j}$$
if $\exists\, b_0 = 0$.

The same sort of argument as for partial fractions shows that, for a suitable choice of d_j's the remainders $r_j(z)$ in (*) form a series s.t. $\sum m_j r_j(z)$ converges uniformly on compact subsets; the infinite product is then (uniformly) convergent. $\square$

Corollary: Any meromorphic function on $\mathbb{C}$ is the quotient of two analytic entire functions.

Proof: Suppose f has poles at $\{b_j\}$ with orders m_j: the above thm gives the existence of an entire function $g(z)$ with zeroes precisely at b_j with order m_j. So $h(z) = g(z) f(z)$ is everywhere analytic (zeroes of g cancel poles of f), and we have $f(z) = \frac{h(z)}{g(z)}$. $\square$