

Riemann surfaces were historically introduced to deal with the multivalued nature of certain algebraic functions and their integrals.

* Eg. to evaluate $\int_{z_0}^{z_1} \frac{dz}{\sqrt{z^2+1}}$, you might use trig. substitutions (here: $z = \sinh(w)$)

but a more elegant approach is to think of this as a path integral on a Riemann surface:

since $\sqrt{z^2+1}$ isn't single-valued (2 choices whenever $z \notin \{\pm i\}$), its graph is rather a 2-sheeted covering space of $\mathbb{C} \setminus \{\pm i\}$, $w = \pm \sqrt{z^2+1}$. If we vary z along eg. a circle around one of $\pm i$, the lift of this path to the covering changes sheets: starting at w we come back to $-w$.

So: we introduce $\Sigma = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z^2 + 1\}$ and now view z and w as single-valued analytic functions on Σ rather than multivalued functions on $\mathbb{C}$.

Σ is an example of a complex manifold - near each point of Σ we can use one of w or z as local coordinate and express all quantities as analytic functions of it.

In particular, our integral is now best understood as $\int_{p_0}^{p_1} \frac{dz}{w}$ between points $p_0, p_1 \in \Sigma$.
 $p_0 = (z_0, w_0), p_1 = (z_1, w_1)$.

This is sometimes a pointless complication if you already had a clear mind about what to do with the integral but in general it can bring considerable insight.

* Here, the remarkable fact is that Σ is biholomorphic to a domain in the complex plane.

Explicitly, in terms of the Riemann sphere $S = \mathbb{C} \cup \{\infty\}$, we have inverse analytic bijections

$$S \setminus \{\pm i\} \xrightarrow{\cong} \Sigma = \{(z, w) \mid w^2 = z^2 + 1\}$$

$$\lambda \longmapsto \left(\frac{2\lambda}{1-\lambda^2}, \frac{1+\lambda^2}{1-\lambda^2} \right)$$

$$\frac{w-1}{z} \longleftarrow (z, w)$$

So: we can transform our path integral on Σ into one on $S \setminus \{\pm i\}$, by the

change of variables $w = \frac{1+\lambda^2}{1-\lambda^2}$, $dz = d\left(\frac{2\lambda}{1-\lambda^2}\right) = \frac{2(1+\lambda^2)}{(1-\lambda^2)^2} d\lambda$.

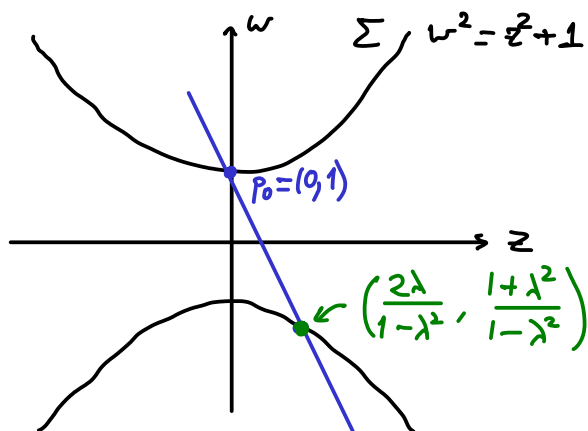
So $\int_{p_0}^{p_1} \frac{dz}{w} = \int_{\lambda_0}^{\lambda_1} \frac{2 d\lambda}{1-\lambda^2}$ which is now easy to deal with by partial fractions.

* What's the geometry behind this change of variables - how does one come up with it?

$\Sigma = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z^2 + 1\}$ is an algebraic equation of degree 2, so its intersection with a (complex) line in $\mathbb{C}^2$ consists of (usually) 2 points.

So: we can project it along the family of lines through a point $p_0 \in \Sigma$ (here $(0,1)$), ② each of these lines meets Σ at p_0 and (usually) one other point.

(Compare w/ the idea of stereographic projection! $S^2 \subset \mathbb{R}^3$ degree 2 eqⁿ; it's the same conceptual idea, in $\mathbb{C}^2$ instead of $\mathbb{R}^3$)



L_λ line with slope λ through $(0,1)$:
 $w = \lambda z + 1$.

The line of slope λ through p_0 has eqⁿ $w = \lambda z + 1$. Plugging this into $w^2 = z^2 + 1$ gives a degree 2 equation in z (with coeffs depending on λ), which always has $z=0$ as one of its roots, so it's especially easy to find the other root!

$$(\lambda z + 1)^2 = z^2 + 1$$

$$\rightarrow (\lambda^2 - 1)z^2 + 2\lambda z = 0$$

$$\rightarrow z = 0 \text{ or } z = \frac{2\lambda}{1 - \lambda^2}$$

Also, every point $p \in \Sigma$ ($p \neq p_0$) arises from this construction, by taking the line (p_0, p) .

(Special cases: $\therefore$ for $\lambda = 0$, L_λ is tangent to Σ at p_0 so we get double root $z=0$

$\therefore$ for $\lambda = \pm 1$, the other intersection of L_λ and Σ goes missing ("at ∞ ")

$\therefore$ to obtain the point $(0, -1) \in \Sigma$, need to allow slope $\lambda = \infty$).

$\rightarrow$ this gives a biholom. $S - \{\text{finite set}\} \xrightarrow{\sim} \Sigma$ given by rational functions

(say Σ is a rational curve; "curve" because it's complex 1-dimensional) even though it looks actually like a surface... (real 2-dimensional).

• This process allows us to evaluate path integrals on algebraic curves $\Sigma \subset \mathbb{C}^2$ defined by any quadratic polynomial $Q(z, w) = 0$... but then it breaks down.

Q: calculate the arclength of a portion of ellipse $x^2 + \frac{y^2}{2} = 1$ between (x_0, y_0) & (x_1, y_1)

If you write $y = \pm \sqrt{2(1-x^2)}$ and use $\int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ you end up with something

like $\int_{x_0}^{x_1} \sqrt{\frac{1+x^2}{1-x^2}} dx$. If you instead use parametric length, you end up with $\int_{\theta_0}^{\theta_1} \sqrt{1 + \cos^2 \theta} d\theta$.

Managing this further we can reduce to e.g. $\int \frac{dx}{\sqrt{1-x^4}}$

But none of these "elliptic integrals" can be expressed in terms of known functions.

So early 19th century mathematicians remained stuck until Riemann, Abel, ... provided the right view point - Riemann surfaces are needed to make sense of what's going on.

(This is a topic at the intersection of complex analysis, topology, and algebraic geometry!)

So we now look at the graph of $\sqrt{1-z^4}$: $\Sigma = \{(z, w) \in \mathbb{C}^2 / w^2 = z^4 - 1\}$

③

Claim: the reason this one is so different from the previous one is that it's not an open subset of the Riemann sphere, but rather an open subset of a torus (an "elliptic curve" - the name comes from the problem of elliptic integrals & has stuck)

How do we see this? Ans: project to the z -coordinate: $(z, w) \mapsto z$.

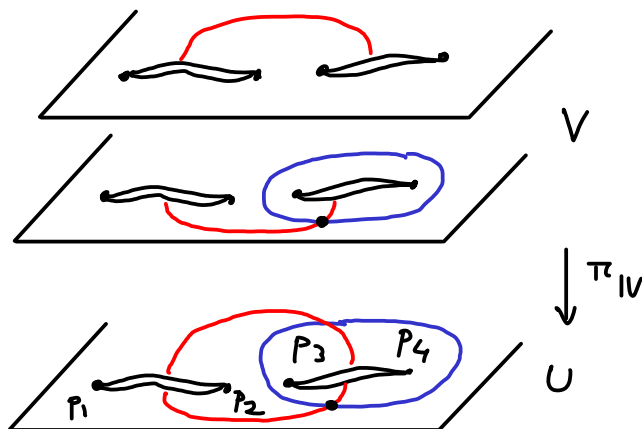
This map is a "branched covering" - a two-sheeted covering map after we remove the roots p_i of the polynomial in z (here $z^4 - 1 \Rightarrow \pm 1, \pm i$) from $\mathbb{C}$, and $q_i = (p_i, 0)$ from Σ .
 $\Rightarrow$ the map $\Sigma - \{q_i\} \xrightarrow{\pi} \mathbb{C} - \{p_i\}$ is a 2:1 covering.
 $(z, w) \mapsto z$

The p_i are branch points: the lift of a small circle around p_i is a path that ends up on the opposite sheet of where it started ($w \mapsto -w$); in general, a loop in $\mathbb{C} - \{p_i\}$ lifts to a loop in $\Sigma - \{q_i\}$ iff the sum of its winding numbers around $p_1 \dots p_4$ is even.

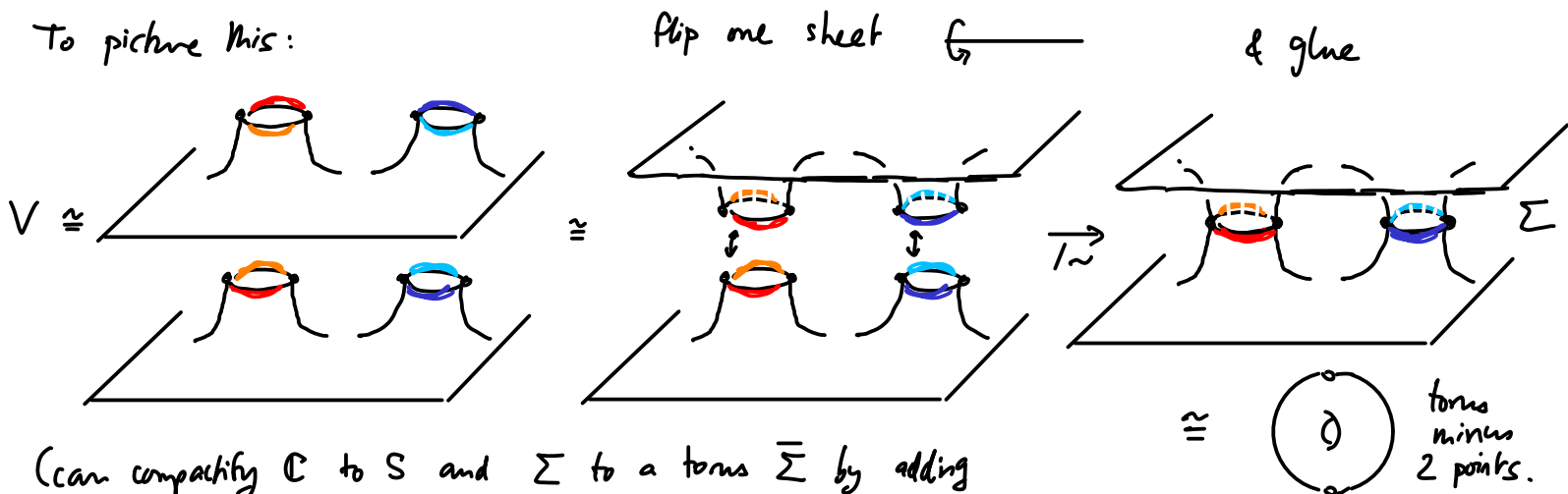
Draw two arcs γ, γ' in $\mathbb{C}$ connecting p_1 to p_2 and p_3 to p_4 (for example), and let $U = \mathbb{C} - (\gamma \cup \gamma')$. Then any loop in U has even total winding number, so lifts to a loop in Σ . Hence the restricted covering map from $V = \pi^{-1}(U)$ to U is trivial: $V = V_+ \sqcup V_-$, $\pi|_{V_{\pm}}: V_{\pm} \xrightarrow{\sim} U$.

making the slits in these planes more visible:

Adding back in the missing arcs $\gamma \cup \gamma'$, the lift of a path in $\mathbb{C}$ jumps between the two sheets $V_{\pm}$ each time it crosses $\gamma \cup \gamma'$, so Σ is obtained from V by attaching one side of each slit in each sheet to the other side of the same slit in the other plane.



To picture this:



(can compactify $\mathbb{C}$ to S and Σ to a torus $\bar{\Sigma}$ by adding 2 preimages of ∞).

The implication for complex analysis is that, since $\bar{\Sigma}$ isn't simply connected, path integrals on it depend on the path of integration.

Returning to our integral $\int \frac{dz}{w}$ on $\Sigma = \{w^2 = z^4 - 1\}$ (or another polynomial of deg. 3 or 4 with simple roots) (4)

- $\frac{dz}{w}$ is actually an analytic 1-form on $\bar{\Sigma}$, without poles or zeroes

(at $(z, w) = (p_i, 0)$, the local coordinate on Σ is actually w , not z ,

but $w^2 = P(z) \leadsto 2w dw = P'(z) dz \Rightarrow \frac{dz}{w} = \frac{2 dw}{P'(z)}$ no pole).

- the integral $\int_{p_0}^{p_1} \frac{dz}{w}$ is invariant under path homotopy (Cauchy) but depends on homotopy class.

Choose loops α_1, α_2 generating $\pi_1(\bar{\Sigma}) \simeq \mathbb{Z}^2$, then a change of homotopy class modifies the value of $\int$ by an integer linear combination of the periods $\omega_1 = \int_{\alpha_1} \frac{dz}{w}$, $\omega_2 = \int_{\alpha_2} \frac{dz}{w}$.

given 2 paths , $[\gamma - \gamma'] = m_1[\alpha_1] + m_2[\alpha_2]$ for some $m_1, m_2 \in \mathbb{Z}$

$$\Rightarrow \int_{\gamma} - \int_{\gamma'} = m_1 \omega_1 + m_2 \omega_2.$$

- $\int_{p_0}^p \frac{dz}{w} = F(p)$ defines an analytic mapping $\bar{\Sigma} \xrightarrow{F} \mathbb{C} / \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ which

- cannot be expressed in terms of elementary functions

- has everywhere nonzero derivative, so F is a local homeomorphism, and in fact a covering map.

- by winding number arguments (if you're a complex analyst) or studying the map on fundamental groups (if you're a topologist), $\#F^{-1}(c) = 1 \forall c$, so in fact F is a biholomorphism.

- What is the inverse of F ? Ans: a doubly periodic function! ($\approx$ Weierstrass P-function)

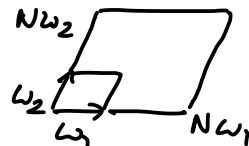
The Weierstrass P-function: look for doubly periodic functions $f(z+\omega_1) = f(z+\omega_2) = f(z)$?

If f is analytic then it's bounded hence constant, so the only interesting such functions are meromorphic. Residue formula integrating around a large parallelogram

$\Rightarrow \sum$ of residues in fundamental domain must be zero

(since path $\int$ linear in N vs. $\sum \text{Res } q(z) dz$ in N)

$\Rightarrow$ can't have just a single pole of order 1 in the fundamental domain.



The simplest of all either have one pole of order 2, or 2 poles of order 1, in ω_2 

Weierstrass' starting point has a pole of order 2, with vanishing residue

$\Rightarrow$ up to translation $z \mapsto z - a$ we can place the pole at 0, polar part $\frac{1}{z^2}$.

Following our study of infinite sums and how to achieve convergence, this leads to

$$P(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \quad (\omega = n_1\omega_1 + n_2\omega_2, (n_1, n_2) \in \mathbb{Z}^2 - \{(0,0)\})$$

the Weierstrass P-function.

This series converges uniformly on compact sets (using: $\sum_{\omega \neq 0} \frac{1}{|\omega|^3} < \infty$) (5)

$$P'(z) = -2 \sum_{\omega} \frac{1}{(z-\omega)^3} \text{ is doubly periodic, so } P(z+\omega_1) - P(z) = \text{const.}$$

$$P(z+\omega_2) - P(z) = \text{const.}$$

But clearly $P(z)$ is an even function $P(-z) = P(z) \Rightarrow$ take $z = \frac{\omega_1}{2}, z = \frac{\omega_2}{2}$ in $\uparrow$
to get P is periodic too.

Working out the Laurent expansions at $z=0$

$$P(z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \dots$$

(constant term vanishes;
odd terms vanish since P even)

for some constants $g_2, g_3 \in \mathbb{C}$ (depending on ω_1, ω_2).

$$P'(z) = \frac{-2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \dots$$

$$\Rightarrow P'(z)^2 = \frac{4}{z^6} - \frac{2g_2}{5z^2} - \frac{4g_3}{7} + \dots$$

$$\text{vs. } 4P(z)^3 = \frac{4}{z^6} + \frac{3g_2}{5z^2} + \frac{3g_3}{7} + \dots$$

$\Rightarrow P'(z)^2 = 4P(z)^3 - g_2 P(z) - g_3$
(polar parts match, so equal up to
entire periodic function = constant,
but constant terms match too)

Outcome: $z \mapsto (P(z), P'(z))$ gives a biholomorphism

$$\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \xrightarrow{\cong} \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\} \cup \{\infty\}$$

(another elliptic curve!)

$$dP(z) = P'(z) dz \Rightarrow dz = \frac{dP(z)}{P'(z)} = \frac{dx}{y}, \text{ i.e. the inverse function is } \int \frac{dx}{y} = \int \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}$$

(This is almost what we had in the other direction, except this one has
one of the 4 branch points at ∞ , unlike our previous example where all 4 $p_i \in \mathbb{C}$.
Simple coordinate transformations by rational functions let us switch between the two.)