A Model of the Reserve Asset*

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February 15, 2015

Abstract

A portion of the global wealth portfolio is directed towards a safe and liquid reserve asset, which recently has been the US Treasury bond. Our model links the determination of reserve asset status to relative fundamentals and relative debt sizes, by modeling two countries that issue sovereign bonds to satisfy investors' reserve asset demands. A sovereign's debt is more likely to be the reserve asset if its fundamentals are strong relative to other possible reserve assets, but not necessarily strong on an absolute basis. Debt size can enhance or detract from reserve asset status. If global demand for the reserve asset is high, a large-debt sovereign which offers a savings vehicle with better liquidity is more likely to be the reserve asset. If demand for the reserve asset is low, then large debt size is a negative as it carries more rollover risk, leading to a riskier vehicle for saving. When global demand is high, countries may make fiscal/debt-structuring decisions to enhance their reserve asset status. These actions have a tournament feature, and are self-defeating: countries may over-expand debt size to win the reserve asset tournament. Coordination can generate benefits. We use our model to study the benefits of "Eurobonds" – i.e. a coordinated common Europe-wide sovereign bond design. Eurobonds deliver welfare benefits only when they make up a sufficiently large fraction of countries' debts. Small steps towards Eurobonds may hurt countries and not deliver welfare benefits.

*Preliminary; references are missing; please do not circulate or quote.

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1 Introduction

US government debt has been the world's reserve asset for over half a century. German government debt is the reserve asset within Europe. US and German debt appear to have high valuations relative to the debt of other countries with similar fundamentals, measured in terms of debt or deficit to income ratios. Moreover, as fundamentals in the US and Germany have deteriorated, these high valuations have persisted. Finally, as evident in the financial crises over the last five years, during times of turmoil, the value of these countries' bonds rise relative to the value of other countries' bonds in a "flight to quality."

This paper develops a model that helps understand these facts and think about what drives the value of a reserve asset. We study a model with many investors and two countries, each which issues government bonds. The investors have a pool of savings that they must invest in government bonds – there are no alternative savings vehicles. Thus the bonds of one, or possibly both of the countries, will hold these savings and serve as a reserve asset. However, the debts are subject to rollover risk. The countries differ in their fundamentals, which measures their ability to service their debt and factors into their rollover risk; and debt size, which proxies for the financial depth or liquidity of the country's debt market. Our model links fundamentals and debt size to the valuation and equilibrium determination of the reserve asset.

An important assumption is that there are no savings vehicles other than the countries' sovereign debts. That is, all savings needs are satisfied by sovereign debt that is subject to rollover risk. There is no "gold" in the model, nor are there any corporations/banks that are able to honor commitments of repaying debts. Alternatively, the model can be interpreted as one where such vehicles do exist, but their supplies are small relative to the reserve asset needs of investors. Thus, substantially all of the world's reserve asset needs must be satisfied by debt that is subject to rollover risk.

In the model, an investor's valuation of a bond depends on the number of other investors who purchase that bond. If only a few investors demand a country's bond, the bond auction fails and the country defaults on the bond. For a country's bonds to be safe, the number of investors who invest in the bond must exceed a threshold, which is decreasing in the country's fundamentals (i.e., its fiscal surplus) and increasing in the size of the debt . Our modeling of rollover risk is similar to Cole and Kehoe (2000). Investor actions are complements – as more investors invest in a country's bonds, other investors are incentivized to follow suit. Valuation has a coordination aspect which we consider to be an important feature of the economics of a reserve asset.

Besides the above strategic complementarity, the model also has a strategic substitutability force, as is common in models of competitive financial markets. Once the number of investors who invest in the bonds exceeds the threshold required to rollover debts, then investor actions become substitutes. Beyond the threshold, more demand for the bond drives up the bond price, leading to lower returns. Our model links the debt size to this strategic substitutability: for the same investor demand, a smaller debt size leads to a smaller return to investors. In other words, countries with lower financial depth are at a disadvantage in accommodating investors' demand for a reserve asset.

The model predicts that relative fundamentals more so than absolute fundamentals are an important component of debt valuation. Relative fundamentals matter because of the coordination aspect of valuation. Investors expect that other investors will invest in the country with better fundamentals, and thus relative valuation determines which country's bonds is the reserve asset. This prediction helps understand the observations we have made regarding the valuation of US debt in a time of deteriorating fiscal fundamentals. In short, all countries' fiscal conditions have deteriorated along with the US, so that US debt has maintained and perhaps strengthened its reserve asset status. The same logic can be used to understand the value of the German Bund (as a reserve asset within Europe) despite deteriorating German fiscal conditions. The Bund has retained/enhanced its value because of the deteriorating fiscal conditions of other Euro area countries.

The model also predicts that debt size is an important determinant of reserve asset status. If the global demand for reserve assets is high, then large debt size enhances reserve asset status. Consider an extreme example with large debt country and a small debt country – in fact, infinitesimally small debt. If investors coordinate on this small debt as the reserve asset, then the return on the reserve asset will be infinitesimal. That is the quantity of world demand concentrating on a small float of bonds will drive bond prices up to a point that investors' incentives in equilibrium will be to coordinate investment in the large debt. On the other hand, if global demand for reserve assets is low, then investors will be concerned that the large debt will not attract sufficient demand to rollover the debt. That is, when funding conditions are tight, rollover risk for a large debt size is high. In this case, investors will tend to coordinate on the small debt size as the reserve asset.

Our model offers some guidance on when the US may lose its reserve asset status. Many academics have argued that we are and have been in a global savings glut, which in the model will correspond to a high global demand for reserve assets. In this case, the US is likely to continue its reserve asset status unless US fiscal fundamentals deteriorate significantly relative to other countries, or if another sovereign debt can compete with the US Treasury in terms of size. Eurobonds seem like the only possibility of the latter, although there is considerable uncertainty whether such bonds will exist and will have better fundamentals than the US debt. However, if the savings glut ends and the world moves to a low demand for reserve assets, then our model predicts that an alternative high fundamentals country with a relatively low supply of debt may become the reserve asset. The German Bund is a leading example of such an asset.

The size effect of our model also identifies a novel *contagion* channel. In the high savings regime, a larger debt size of the large debt country adversely affects the small debt country by increasing the small country's probability of default and its expected bond yield. This is because the expansion of the supply of the reserve asset creates safe "parking spots" for funds that may otherwise have been invested in small country debt. From 2007Q4 to 2009Q4, the supply of US Treasury bonds increased by $2.7 trillion (the money stock increased another $1.3 trillion). Our model hence hints a causal link from the increase in US Treasury bond supply/Fed QE and the eruption of the European sovereign debt crisis in 2010.

We use our model to study incentives to change debt size, when doing so may enhance reserve asset status. We study a case where two countries have a "natural" debt size, determined for example by their GDP, but can deviate from its natural debt size by some adjustment cost. Two interesting cases emerge. When countries are roughly symmetric – similar natural debt size – and when global demand for reserve assets is high, countries will engage in a rat race to become the reserve asset. Starting from the natural debt sizes, and holding fixed the size decision of one country, the other country will have an incentive to increase its debt size since the larger debt size can confer a reserve status. But then the first country will have an incentive to respond in a similar way, and so on so forth. In equilibrium, both countries will expand in a self-defeating manner beyond their natural debt size. This prediction of the model can help to shed some light on the expansion of relatively safe stocks of debt in the US (GSE debt) and Europe (sovereign debt) in the build-up to the crisis. These expansions have ultimately ended badly.

The model identifies a second case, when countries are asymmetric and one country is the natural "top dog." In this case, the larger debt country will have an incentive to reduce debts to the point that balances rollover risk and retaining reserve asset status, while the smaller country will have an incentive to expand its debt size. The model is suggestive that asymmetry leads to better outcomes than symmetry.

We use our model to investigate the benefits of creating "Eurobonds." We are motivated by the Eurobond proposals that have been floated over the last few years (see Claessens, Mody and Vallee, 2012, for a review of various proposals). A shared feature of the many proposals is to create a common Europe-wide reserve asset. Each country receives proceeds from the issuance of the "common bond" which is meant to serve as the reserve asset, in addition to proceeds from the sale of an individual country-specific bond. By issuing a common Euro-wide reserve asset, all countries benefit from investors' need for a reserve asset, as opposed to just one country (Germany) which is the de-facto reserve asset in the absence of a coordinated security design. Our model, in which the determination of the reserve asset is endogenous, is well-suited to analyze these issues formally. Suppose that countries issue α share of common bonds and $1-\alpha$ share as individual bonds. We ask, how does varying α affect welfare, and the probability of safety for each country? Our main finding is that welfare is only unambiguously increased for α above a threshold. Above this threshold, the common-bond structure enhances the safety of both common bonds and individual bonds, and all these bonds develop some reserve asset status. Thus, a higher α makes both countries safer.

However, below the threshold, welfare can be increasing or decreasing, depending on the assumed equilibrium; and one country may be made worse off while another may be made better off by increasing α. Thus we conclude that a successful Eurobond proposal requires a significant amount of coordination and fiscal union.

LITERATURE REVIEW: TO BE COMPLETED

2 Model

2.1 The Setting

Consider a two-period model with two countries, indexed by i, and a continuum of homogeneous risk-neutral investors, indexed by j. At date 0 each investor is endowed with one unit of consumption good, which is the numeraire in this economy. Investors invest in the bonds offered by these two countries to maximize their expected date 1 consumption, and there is no other storage technology available. This latter restriction is important to the analysis as will be clear.

There is a large country, called country 1, and a small country, called country 2. We normalize the size of the large country to be one (i.e., $s_1 = 1$), and denote the size of the small country by $s_2 \equiv s \in (0, 1]$. Each country sells bonds at date 0 promising repayment at date 1. The country size determines the total face value (in terms of promised repayment) of bonds that each country sells: the large (small) country offers 1 (s) units of sovereign bonds. Hence the aggregate bond supply is $1 + s$. All bonds are zero coupon bonds.

The aggregate measure of investors, which is also the aggregate demand for bonds, is $1 + f$, where $f > 0$ is a constant parameterizing the aggregate savings need. To save, we assume that investors place *market orders* to purchase sovereign bonds. In particular, since purchases are via *market orders*, the aggregate investor demand does not depend on the equilibrium price.[1] Denote by p_i the equilibrium price of the bond issued by country i. Since there is no storage technology

[1] Market orders avoid the thorny theoretical issue of investors using the information aggregated by the market clearing price to decide which country to invest in, a topic extensively studied in the literature of Rational Expectation Equilibrium.

available to investors, all savings of investors go to buy these sovereign bonds. This implies via the market clearing condition that

$$s_1 p_1 + s_2 p_2 = p_1 + s p_2 = 1 + f.$$

Country i has fundamentals denoted θ_i, which can be interpreted as the country's fiscal surplus. The surplus is proportional to country size, i.e., for country i it is $s_i\theta_i$. We assume that each country has an existing debt obligation, assumed to be equal to the country size s_i.[2] The country has total resources consisting of fundamentals $s_i\theta_i$ and the proceeds from newly issued bonds $s_i p_i$,

$$s_i\theta_i + s_i p_i.$$

We assume that a country defaults if and only if

$$\underbrace{s_i\theta_i + s_i p_i}_{\text{total funds available}} < \underbrace{s_i}_{\text{debt obligations}}.$$

which we rewrite as,

$$s_i p_i < s_i(1 - \theta_i).$$

If the country defaults at date 0, there is zero recovery and any investors who purchased the bonds of that country receive nothing.[3] If the country does not default, then each investor in that country receives one at date 1. For simplicity, there is no default possibility at date 1, e.g., this assumption can be justified by a sufficiently high fundamental in period 1.

We note that our model of sovereign debt features a multiple equilibrium crisis, in the sense of

[2] One can think of the timing, as discussed in the text, as s_i is past debts that must be rolled over. This is a rollover risk interpretation, where we take the past debt as given. Here is another interpretation. The bonds are auctioned at date 0 with investors anticipating repayment at date 1. The date 0 proceeds of $s_i p_i$ are used by the country in a manner that will generate $s_i\theta_i + s_i p_i$ at date 1 which is then used to repay the auctioned debt of s_i.

[3] For the situation of positive recovery, see discussion in Remark3.

Cole and Kehoe (2000). If investors conjecture that other investors will not invest in the debt of a given country, then p_i is low which means the country is more likely to default, which rationalizes the conjecture that other investors will not invest in the debt of the country.

We follow the global games approach to link equilibrium selection to fundamentals. We assume that there is a publicly observable world-level fundamental index θ lying in the interval $(0,1)$. Our analysis focuses on a measure of relative strength between country 1 and country 2, which we denote by $\tilde{\delta}$. Specifically, conditional on the relative strength $\tilde{\delta}$, the fundamentals of these two countries satisfy

$$1-\theta_1 = (1-\theta)\exp\left(-\tilde{\delta}\right); \tag{1}$$

$$1-\theta_2 = (1-\theta)\exp\left(+\tilde{\delta}\right). \tag{2}$$

Recall that $1-\theta_i$ is the funding need of a country. Given $\tilde{\delta}$, the higher the θ, the greater the surplus of both countries and therefore the lower their funding need. And, given θ, the higher the $\tilde{\delta}$, the better are country 1 fundamentals relative to country 2, and therefore the lower is country 1's relative funding need.[4] Finally, the above specification implies that the funding need for each country is always positive.

We assume that the relative strength of country 1, $\tilde{\delta}$, follows a uniform distribution

$$\tilde{\delta} \sim \mathbb{U}\left[-\overline{\delta},\overline{\delta}\right] \tag{3}$$

where $\mathbb{U}\left[-\overline{\delta},\overline{\delta}\right]$ denotes the uniform distribution over the interval $\left[-\overline{\delta},\overline{\delta}\right]$. For much of the analysis we set $\overline{\delta} < \ln\frac{1+f}{s(1-\theta)}$, which ensures that for the worse case scenario, financing need of either country exceeds the total savings $1+f$. This gives us the usual dominance regions when the fundamentals take extreme values.

As we will use the global games technique to pin down the unique threshold strategy equilibrium,

[4] The scale of $1-\theta$ and exponential noises $e^{\tilde{\delta}}$ in (1) and (2) help in obtaining a simple close-form solution. The Appendix A.1 considers an additive specification $\theta_i = \theta + (-1)^i\,\tilde{\delta}$ and solves the case for $\sigma > 0$; we show that the main qualitative results hold in that setting.

we assume that the country 1 relative strength $\tilde{\delta}$ is not publicly observable. Instead, each investor $j \in [0,1]$ receives a private signal

$$\delta_j = \tilde{\delta} + \epsilon_j,$$

where $\epsilon_j \sim \mathbb{U}[-\sigma, \sigma]$ and ϵ_j are independent across all investors $j \in [0,1]$. Following the global games literature a la Morris and Shin (2003) we will focus on the limit case where noise $\sigma \to 0$.

2.2 Equilibrium Characterization and Properties

We focus on symmetric threshold equilibria in this section. More specifically, we assume that all investors adopt the same threshold strategy in which each investor purchases country 1 bonds if and only if his private signal about country 1's relative strength is above a certain threshold, i.e. $\delta_j > \delta^*$; otherwise he purchases country 2 bonds. The corner portfolio decisions are for simplicity and are consistent with risk neutrality.

Deriving the equilibrium threshold. In equilibrium, the marginal investor who receives the threshold signal $\delta_j = \delta^*$ must be indifferent between investing his money in either country. Based on this signal, the marginal investor forms belief about other investors' signals and hence their strategies. Denote by x the fraction of investors who receive signals that are above his own signal $\delta_j = \delta^*$, and as implied by threshold strategies will invest in country 1. It is well-known (e.g., Morris and Shin, 2003) that in the limit of diminishing noise $\sigma \to 0$, the marginal investor forms a "diffuse" view about other investors' strategies, in that he assigns a uniform distribution for $x \sim \mathbb{U}[0,1]$.

Given that there is a fraction x of investors who purchase the bonds of country 1, the total funds going to country 1 and 2 are $(1+f)\,x$ and $(1+f)\,(1-x)$, respectively. The resulting bond prices are thus

$$p_1 = (1+f)\,x \text{ and } p_2 = \frac{(1+f)\,x}{s}.$$

We now calculate the expected return from investing in bond i, Π_i.

Expected return from investing in country 1. Given x and its fundamental θ_1, country 1 does not default if and only if

$$p_1 - 1 + \theta_1 \quad = \quad (1+f)\,x - 1 + \theta_1 \geq 0 \Rightarrow x \geq \frac{1-\theta_1}{1+f}. \tag{4}$$

This is intuitive: country 1 does not default only when there are sufficient investors who receive favorable signals about country 1 and place their funds in country 1's bonds accordingly. The survival threshold $\frac{1-\theta_1}{1+f}$ is lower when the country 1 fundamental, θ_1, is higher and when the total funds available for savings, f, are higher.

Of course, the country 1 fundamental $1-\theta_1 = (1-\theta)\,e^{-\tilde{\delta}}$ in (1) is uncertain. We take the limit as $\sigma \to 0$, so that the signal is almost perfect and the threshold investor who receives a signal δ^* will be almost certain that[5]

$$1 - \theta_1 = (1-\theta)\,e^{-\delta^*}. \tag{5}$$

Hence, in the limiting case of $\sigma \to 0$, plugging (5) into (4) we find that the large country 1 survives if and only if

$$x \geq \frac{1-\theta_1}{1+f} = \frac{(1-\theta)\,e^{-\delta^*}}{1+f}. \tag{6}$$

Here, either higher average fundamentals θ or a higher threshold δ^* make country 1 more likely to not default.

Now we calculate the investors' return by investing in country 1. Conditional on survival, the realized return is

$$\frac{1}{p_1} = \frac{1}{(1+f)\,x};$$

while if default occurs the realized return is 0. From the point of view of the threshold investor with

[5]In equilibrium, θ_1 depends on the realization of x, which is the fraction of investors with signals above δ^*. Given that the signal noise ϵ_j is drawn from a uniform distribution over $[-\sigma, \sigma]$, we have

$$x = \Pr\left(\tilde{\delta} + \epsilon_j > \delta^*\right) = \frac{\tilde{\delta} + \sigma - \delta^*}{2\sigma} \Rightarrow \tilde{\delta} = \delta^* + (2x-1)\,\sigma.$$

which implies that $\theta_1 = \theta + (1-\theta)\left(1 - e^{-\delta^* - (2x-1)\sigma}\right)$. Taking $\sigma \to 0$ we get (5).

signal δ^*, the chance that country 1 survives is simply the integral w.r.t. to the uniform density dx from $\frac{(1-\theta)e^{-\delta^*}}{1+f}$ to 1:

$$\Pi_1 = \int_{\frac{(1-\theta)e^{-\delta^*}}{1+f}}^{1} \frac{1}{(1+f)\,x} dx = \frac{1}{1+f}\left(\ln\frac{1+f}{1-\theta} + \delta^*\right).$$

The higher the threshold δ^*, the greater the chance that country 1 survives, and hence the higher the return by investing in country 1 bonds.

Expected return from investing in country 2. Denote the measure of investors that are investing in country 2 by $x' \equiv 1 - x$, which also follows a uniform distribution over $[0, 1]$. If the investor instead purchases country 2's bonds, he knows that country 2 does not default if and only if

$$sp_2 - s + s\theta_2 = (1+f)\,x' - s + s\theta_2 \geq 0 \Leftrightarrow x' \geq \frac{s\,(1-\theta_2)}{1+f}, \tag{7}$$

Country 2 survives if the fraction of investors investing in country 2, x', is sufficiently high. The threshold is lower if the country is smaller, fundamentals are better, and the total funds available for savings are higher.

Similar to the argument in the previous section, in the limiting case of almost perfect signal $\sigma \to 0$, country 2 fundamental θ_2 in (7) is almost certain from the perspective of the threshold investor with signal δ^* (recall (2)):

$$1 - \theta_2 = (1-\theta)\,e^{\delta^*}. \tag{8}$$

Plugging equation (8) into equation (7), we find that country 2 survives if and only if

$$x' \geq \frac{s\,(1-\theta)\,e^{\delta^*}}{1+f}. \tag{9}$$

Relative to (6), country size s plays a role. All else equal, the lower size s and the smaller country 2, the more likely that the country 2 survives.

Given survival, the investors' return of investing in country 2, conditional on x', is

$$\frac{1}{p_2} = \frac{s}{(1+f)\,x'}; \tag{10}$$

while the return is zero if country 2 defaults. As a result, using (10), the expected return from investing in country 2 is

$$\begin{aligned} \Pi_2 &= \int_{\frac{s(1-\theta)e^{\delta^*}}{1+f}}^{1} \frac{s}{(1+f)\,x'} dx'. \\ &= \frac{1}{1+f} \cdot \underbrace{s}_{\text{payout}} \left(\underbrace{-\ln s}_{\text{survival prob.}} + \underbrace{\ln \frac{1+f}{1-\theta} - \delta^*}_{\text{country 2 fundamental}} \right) \end{aligned}$$

Here, "country 2 fundamental" essentially comes from turning off the size effect by setting $s = 1$. Then, we see that this term is the same as for country 1.

We decompose the source of the country 2 return in three parts. The third part "country 2 fundamental" is the most transparent: for the threshold investor who observes aggregate fundamental θ and receives a signal δ^* about the relative strength between country 1 and country 2, $\ln \frac{1+f}{1-\theta} - \delta^*$ represents the country 2 fundamental – that is, when setting $s = 1$, this is what remains, analogous to country 1. The first part, "payout", and the second part, "survival prob", are affected by country size. The first part s indicates that the total bond payment is of size s; all else equal, the smaller the country size the lower the return, as can be seen in (10). The second part $-\ln s$, which is decreasing in s, captures the effect of a country's size on its survival probability. This is because all else equal, a smaller country is more likely to survive for a given amount of investment in its bonds, driving up the total return, as can be seen in (9).

Expected return of investing in country 1 versus country 2. Figure 1 plots the return to investing in each country as a function of x (x') which is the measure of investors investing in country 1 (country 2), from the perspective of the marginal investor. Consider the solid green curve

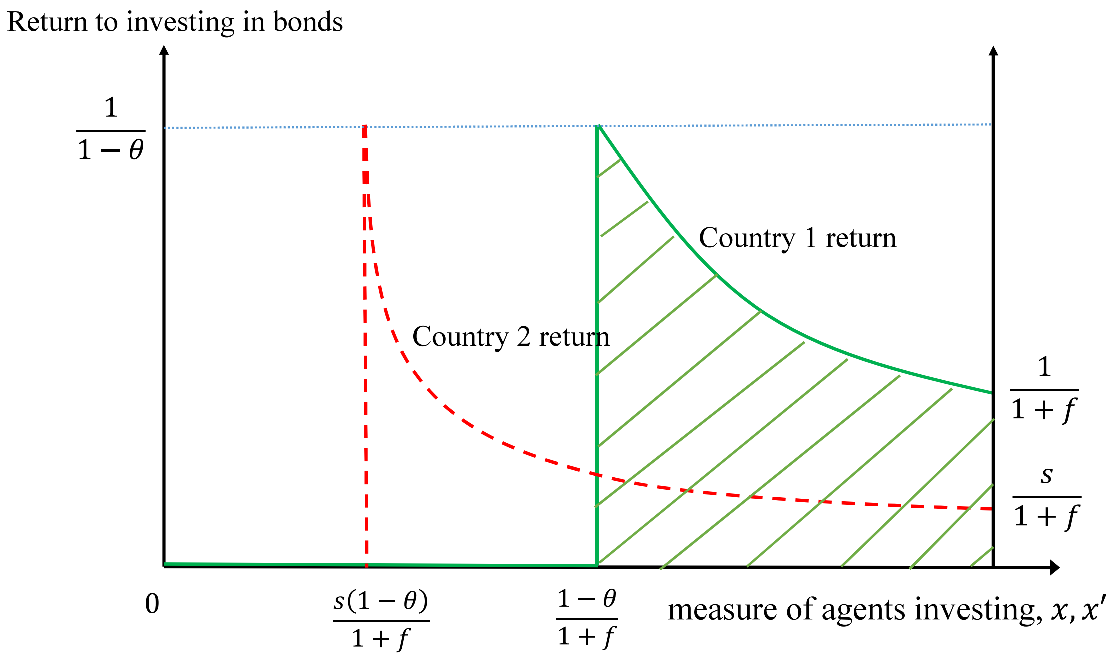

Figure 1: **Returns of the marginal investor when investing in country 1 (country 2) as a function of x (x').** The return to investing in country 1 is in green solid line, while the return to investing in country 2 is in red dashed line. The figure assumes $\delta^* = 0$ so that the marginal investor believes that both countries have the same fundamentals. The bonds issued by the large country 1 only pay when $x > \frac{1-\theta}{1+f}$ while country 2's bonds only pay when $x' > \frac{s(1-\theta)}{1+f}$. The return to country 1's bonds falls to $\frac{1}{1+f}$ when $x = 1$, while for country 2's bonds the return falls more rapidly to $\frac{s}{1+f}$ when $x' = 1$.

first which is the return to investing in country 1. For x below the default threshold $\frac{1-\theta}{1+f}$, the return is zero. This default threshold is relatively high, since country 1 is large and hence it needs a large number of investors to buy bonds to ensure a successful auction. Across the threshold $\frac{1-\theta}{1+f}$, investor actions are strategic complements – i.e., if a given investor knows that other investors are going to invest in country 1, the investor wants to follow suit. Past the threshold, the return falls as the face value of bonds is constant and investors' demand simply bids up the price of the bonds. In this region, investor actions are strategic substitutes. The marginal investor's expected return from investing in country 1 is the integral of shaded area beneath the green solid line.

The dashed red curve plots the return to investing in country 2, as a function of x' which is the measure of investors investing in country 2. The default threshold, which is $\frac{s(1-\theta)}{1+f}$, is lower for this country ($\frac{1-\theta}{1+f}$) than for country 1 given that country 2 only auctions off a small number of bonds.

When $\delta^* = 0$, i.e., the marginal investor with signal $\delta^* = 0$ believes that both countries share the same fundamentals, the return to investing in country 2 is $\frac{1}{1-\theta}$. This is the same as the threshold return to investing in country 1, as shown in Figure 1. While country 2 has a lower default threshold which implies a smaller strategic complementarity effect, past the threshold the return to investing in country 2 falls off quickly. That is, the strategic substitutes effect is more significant for country 2 than country 1. This is because country 2 has a small bond issue and hence an increase in demand for country 2 bonds increases the bond price (decreases return) more than the same increase in demand for country 1 bonds. We see this most clearly at the boundary where $x = x' = 1$, where the return to investing in the large country 1 is $\frac{1}{1+f}$, while the return to investing in country 2 is $\frac{s}{1+f}$.

To sum up, because the large country auctions off more bonds, it needs more investors to participate to ensure no-default. However, the very fact that the large country sells more bonds makes the large country a deeper financial market that can offer a higher return on investment. This tradeoff – size features more rollover risk but provides a more liquid savings vehicle – is at the heart of our analysis.

Equilibrium threshold δ^* The equilibrium threshold δ^* is determined by the indifference condition for the threshold investor between investing in these two countries, i.e.,

$$0 = \Pi_1 - \Pi_2 = \frac{1}{1+f}\left(\ln\frac{1+f}{1-\theta} + \delta^*\right) - \frac{s}{1+f}\left(-\ln s + \ln\frac{1+f}{1-\theta} - \delta^*\right).$$

Solving for the equilibrium threshold signal δ^* yields (recall that $s \in (0, 1]$)

$$\delta^* = \underbrace{-\frac{1-s}{1+s}}_{\text{negative, liquidity}} \times z + \underbrace{\frac{-s\ln s}{1+s}}_{\text{positive, rollover risk}}. \tag{11}$$

where we define

$$z \equiv \ln\frac{1+f}{1-\theta} > 0.$$

Here, z measures aggregate funding conditions, which is greater if either more aggregate funds f are available or there is a higher aggregate fundamental θ. The "savings glut" which many have argued to characterize the world economy for the last decade is a case where z is high.

From (11) we see that there are two effects of size. The first term is negative (for $s \in (0,1)$) and reflects the liquidity benefit that accrues to the larger country, making country 1 safer all else equal. The second term is positive and reflects the rollover risk for country 1, whereby a larger size makes country 1 less safe. The benefit term is modulated by the aggregate funding condition z. We next discuss implications of our model based on the equation (11).

2.3 Model Implications

2.3.1 Probability of default and reserve asset

The equilibrium threshold δ^* tells us about the ex-ante likelihood of default in country 1 and country 2. That is, suppose there is an ex-ante stage before signals are drawn. Then the probability that country 1 is safe is the probability that the country 1's relative strength is above δ^*, i.e. $\Pr\left(\tilde{\delta} > \delta^*\right)$. Thus the safety of country 1 is decreasing in the threshold δ^*, while the safety of country 2 is increasing in δ^*. Also, in the benchmark case with equal country sizes ($s = 1$), we have $\delta^* = 0$ so that it is equally likely that country 1 or country 2 is safe (recall that we assume that $\tilde{\delta}$ is distributed around zero symmetrically).

The following proposition gives the properties of the equilibrium threshold $\delta^*(s,z)$, as a function of country 2's relative size s and the aggregate funding condition z.

Proposition 1 *We have the following results for the equilibrium threshold $\delta^*(s,z)$.*

1. *The equilibrium threshold $\delta^*(s,z)$ is decreasing in the aggregate funding conditions $z = \ln \frac{1+f}{1-\theta}$, i.e., $\frac{\partial}{\partial z}\delta^*(s,z) < 0$. Hence, the probability that country 1's bonds become the reserve asset is higher if the aggregate fundamental θ or aggregate saving f is higher.*

2. *The equilibrium threshold $\delta^*(s,z) \leq 0$ for all $s \in (0,1]$, if and only if $z \geq 1$. Hence, when the aggregate funding $z \geq 1$, the bonds issued by the larger country 1 are more likely to be the*

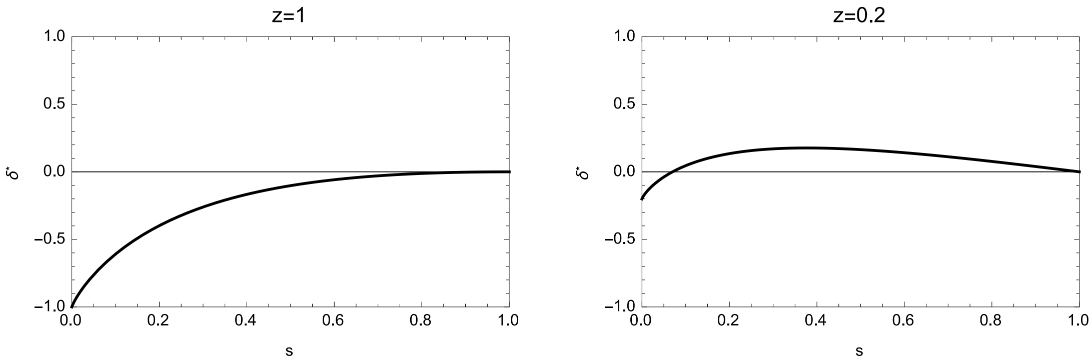

Figure 2: Equilibrium threshold δ^* as a function of country 2 size s. The left panel is for the case of strong aggregate funding conditions with $z = 1$, and the right panel is for the case of low aggregate funding conditions with $z = 0.2$.

reserve asset than those issued by the smaller country 2.

3. *When $s \to 0$ the equilibrium threshold $\delta^*(s,z)$ approaches its minimum, i.e., $\lim_{s\to 0}\delta^*(s,z) = \inf_{s\in(0,1]}\delta^*(s,z) = -z < 0$. This implies that all else equal, country 1 has the highest likelihood of survival when country 2 is smallest.*

Proof. Result (1.) follows because of $\frac{\partial}{\partial z}\delta^*(s,z) = -\frac{1-s}{1+s} < 0$, and $\Pr\left(\tilde{\delta} > \delta^*\right)$ is decreasing in z. To show result (2.), note that when $z = 1$ we have $\delta^*(s, z = 1) = \frac{s - s\ln s - 1}{1+s} < 0$ for $s \in (0,1]$. This inequality can be shown by observing 1) $[s - s\ln s - 1]' > 0$ and 2) $[s - s\ln s - 1]_{s=1} = 0$. Result (3.) holds because

$$\delta^*(s,z) = -\frac{1-s}{1+s}z + \frac{-s\ln s}{1+s} > -\frac{1-s}{1+s}z > -z,$$

where the last inequality is due to $-\frac{1-s}{1+s}z$ being increasing in s for $z > 0$. ■

We illustrate these effects in Figure 2. The left panel of Figure 2 plots δ^* as a function of s for the case of $z = 1$, which corresponds to strong aggregate funding conditions with abundant savings and/or good fundamentals. In this case, the equilibrium threshold $\delta^*(s)$ is always negative, and is monotonically increasing in the small country size s. For small s close to zero, the large country is safest, and this probability falls towards 0.5 as s rises towards 1. The probability is highest when $s = 0$, because in this case country 2 does not exist as an investment alternative. Then because all investors have no choice but to invest in country 1, the bonds issued by country 1 is always the reserve asset, minimizing its default probability. If we assume that the aggregate savings $1+f$ are enough to cover country 1's financing shortfall $1-\theta_1\left(\tilde{\delta}\right)$ even for the worst realization of $\tilde{\delta}=-\overline{\delta}$ then the $\Pr\left(\tilde{\delta}>\delta^*\right)$ will equal one in this case. This $s=0$ case offers one perspective on why Japan has been able to sustain a large debt without suffering a rollover crisis. Since many of the investors in Japan are so heavily invested in Japanese government, eschewing foreign alternative investments, Japan's debt is safe.

The right panel in Figure 2 plots δ^* for a case of weak aggregate funding conditions ($z=0.2$), with insufficient savings and/or low fundamentals. Consistent with the first result in Proposition 1 we see that in this case the large country can be at a disadvantage. For medium levels of s (around 0.4), investors are concerned that there will not be enough demand for the large country bonds, exposing the large country to rollover risk. As a result, investors are more likely to coordinate on the small country's debt as the reserve asset, making the large country relatively more likely to default. For small s, the size disadvantage of the small country becomes a concern: the probability of the large country being safe rises as s goes towards zero (the third result in Proposition 1). For s large, we are back in the symmetric case and the probability of safety goes towards half. Comparing the right panel with $z=1$ to the left panel with $z=0.2$ highlights that the large country's debt size is a clear advantage only when the aggregate funding conditions are strong; as the pool of savings shrink, and in equilibrium interest rates rise, the large debt size triggers rollover risk fears so that investors coordinate on the small country as the reserve asset.

2.3.2 Relative fundamentals

Our model emphasizes relative fundamentals as being a central ingredient in debt valuation. To clarify this point, consider a standard model without coordination elements and without the reserve asset saving need. In particular, suppose that the world interest rate is R^* and consider any two countries in the world with surpluses given by θ_1 and θ_2. Suppose that investors purchase these countries' bonds for $p_i s_i$ and receive repayment of $s_i \min(\theta_i, 1)$. Then,

$$p_1 = \frac{E[\min(\theta_1, 1)]}{1+R^*} \quad \text{and} \quad p_2 = \frac{E[\min(\theta_2, 1)]}{1+R^*},$$

so that bond prices depend on fundamentals, but not particularly on relative fundamentals $\theta_1 - \theta_2$. In contrast, in our model when the country 1's relative fundamentals are high $\tilde{\delta} > \delta^*$, bonds issued by country 1 attract all the savings so that

$$p_1 = 1 + f \quad \text{and} \quad p_2 = 0. \tag{12}$$

Similarly, if the country 1's relative fundamentals are low $\tilde{\delta} < \delta^*$, investors only invest in bonds issued by country 2 so that

$$p_1 = 0 \quad \text{and} \quad p_2 = 1 + f. \tag{13}$$

Valuation in our model becomes sensitive to relative fundamentals, as investors are endogenously coordinating to buy bonds issued by relatively stronger country.

The importance of relative fundamentals helps us to understand why, despite deteriorating US fiscal conditions, US Treasury bond prices have continued to be high: In short, all countries' fiscal conditions have deteriorated along with the US, so that US debt has maintained and perhaps strengthened its reserve asset status. The same logic can be used to understand the value of the German Bund (as a reserve asset within Europe) despite deteriorating German fiscal conditions. The Bund has retained/enhanced its value because of the deteriorating general European fiscal conditions.

Remark 2 *We have analyzed a case where θ reflects the overall fundamentals of the global economy, and the relative fundamental $\widetilde{\delta}$ is "almost perfectly" observable by investors, thus discussing safety of the two countries in terms of the realization of $\widetilde{\delta}$. In practice, it common knowledge that US or Germany have stronger fundamentals than many other countries. We can easily modify our model to capture this effect, by specifying the surpluses of countries to be*

$$1-\theta_1 = (1-\theta)\exp\left(-\tilde{\delta}-\Delta\right),\ \textit{and}\ 1-\theta_2 = (1-\theta)\exp\left(+\tilde{\delta}+\Delta\right),$$

where Δ is a component of the relative fundamental that is common knowledge. The equilibrium threshold now becomes

$$\delta^* = -\frac{1-s}{1+s}z + \frac{-s\ln s}{1+s} - \Delta.$$

Consistent with intuition, common knowledge that a country has better fundamentals makes the country's debt more likely to be the reserve asset.

Remark 3 *Admittedly, the zero valuation of the non-reserve asset in equations (13) and (12) is unrealistic. The zero is because we assume that the bond recovery in default is zero. If we introduce a small but positive recovery in default that is increasing in the country i's surplus, say $\beta\theta_i$ with $\beta > 0$, then even the non-reserve asset would carry a positive value. In equilibrium, similar to the case of $\beta = 0$, the reserve asset will carry an extra value because of its higher relative fundamentals. One could carry out a formal analysis for this case; the only caveat is that we need to allow for strategies that are non-monotone in the investor's signal to derive the equilibrium, similar to the one discussed in Section 2.5.*

2.3.3 Size and aggregate funding conditions

Our model highlights the importance of debt size in determining reserve asset status, and its interactions with the aggregate funding conditions. In the high savings regime, which the literature on the global savings glut has argued to be true of the world in recent history (see, e.g., Bernanke, 2005, Caballero, Farhi, Gourinchas, 2007, and Caballero and Krishnamurthy, 2009), higher debt size increases reserve asset status. The US is the world reserve asset in part because it has maintained large debt issues that can accommodate the world's reserve asset demands.

These predictions of the model also offer some insight into when the US Treasury bond may be displaced as reserve asset. If the world continues in the high savings regime, the US will only be displaced if another country can offer a large debt size and/or good relative fundamentals. This seems unlikely in the foreseeable future. On the other hand, if the world switches to the low savings regime, it is possible that another country with a smaller debt size and good fundamentals, such as the German Bund, will take on the reserve asset role.

The size effect of our model also identifies a novel *contagion* channel. In the high savings regime, increasing the debt size of the large debt country reduces δ^* and thus increases the probability of default and expected bond yield of the smaller country.[6] We can see this from Figure 2, left panel with $z = 1$. Suppose that we decrease the relative size of country 2, s, away from 1; it is equivalent to increasing the size of the large country's debt. We see that the default probability of small country, which is just one minus the default probability of the large country, goes up; this comparative static result will be derived formally in equation (15) later. Linking this observation to data, from 2007Q4 to 2009Q4, the supply of US Treasury bonds increased by \$2.7 trillion (the money stock increased another \$1.3 trillion). Our model suggests that this increase should have increased default probabilities of other country's debts. That is, our model suggests a causal link from the increase in US Treasury bond supply/Fed QE and the eruption of the European sovereign debt crisis in 2010. Intuitively, the expansion of US debt supply created safe "parking spots" for funds that may otherwise have been invested in European sovereign debt.

The sovereign debt contagion channel is a novel prediction for which there is no systematic empirical evidence. However, there is a closely related contagion channel for which there is evidence. Let us reinterpret the small country debt of the model as the short-term debt issued by a private bank that can be used to store value, but is subject to a coordination/rollover problem (i.e., bank runs). Then our model suggests that as the supply of country 1 sovereign debt rises, the safety of the private bank debt should fall. Krishnamurthy and Vissing-Jorgensen (2013) provide evidence of this sort of crowding out, documenting a positive relation between the supply of US government

[6]The yield on the large country debt is ambiguous because the direct effect of increasing debt size, for fixed f, would cause yields to rise, but since the probability of default falls, there is a countervailing effect that reduces bond yields.

debt and the yields on private short-term bank debt.[7]

2.4 Switzerland, Denmark and gold

There are no savings vehicles in the model other than the countries' sovereign debts. That is, all savings needs are satisfied by sovereign debt that is subject to default risk. There is no "gold" in the model, nor are other companies, banks or other governments that are able to honor commitments to repay debts. In practice such assets do exist. Switzerland and Denmark have been prominent in the news recently because of safe-haven flows into these countries, perhaps because these countries can commit to repay their relatively small outstanding supply of bonds. It is easy to accommodate this factor into the model.

Suppose that there exists a quantity of full-commitment sovereign bonds. The supply of these bonds is $\underline{s}$, that is, these bonds pay in total $\underline{s}$ at the final date. Investors invest $f - \hat{f}$ in these bonds, with a return of $\frac{\underline{s}}{f-\hat{f}}$. Let us focus on the symmetric case with $s = 1$ and $\delta^* = 0$. Investing in sovereign bonds of country 1 or country 2 depending on the signal $\tilde{\delta}$ gives a return of $\frac{1}{1+\hat{f}}$ as the small noise assumption implies that investors are perfectly able to pick the "winner". Thus in equilibrium it must be that the full-commitment bond also offers a return of $\frac{1}{1+\hat{f}}$, which then implies that

$$\frac{\underline{s}}{f-\hat{f}} = \frac{1}{1+\hat{f}} \Rightarrow \hat{f} = \frac{f-\underline{s}}{1+\underline{s}}.$$

We assume that the supply of full-commitment bonds $\underline{s}$ satisfies $\underline{s} < f$ so that $\hat{f} > 0$. We then can solve our model following exactly the same steps, only with f redefined as $\hat{f}$. Thus, the model can be interpreted as one where alternative savings vehicles do exist, but their supplies are such that substantially most of the world's reserve asset needs must be satisfied by debt that is subject to

[7] The model is suggestive of further links between government debt and private bank debt. We would expect that the yields on non-government-backed private bank debt (e.g., large time deposits) would especially rise relative to government-backed bank debt (e.g., checking deposits). Government-backed bank debt would rise in safety with increases in US government debt, if such increases lead to an increase in the reserve status of US government debt. Plausibly, banks would respond by increasing the supply of government-backed debt relative to non-government-backed debt. Krishnamurthy and Vissing-Jorgensen (2013) confirm this prediction in the data. The model also suggests a motive for increasing the backing of bank deposits during periods of rising US government debt, to avoid a contraction of the banking system. To investigate these issues thoroughly our model needs to be extended to accommodate a banking system with many banks, which we intend to do in future work.

rollover risk.[8]

Denmark and Switzerland have recently restricted their supplies of safe bonds. The result has been that the prices of their bonds have risen, with interest rates in both countries falling below zero. We can also see this in our model. Reducing $\underline{s}$ causes $\hat{f}$ to rise, and hence the price of safe bonds rises.

2.5 Non-monotone Strategies and Joint Safety Equilibria

We have restricted the agents' strategy space to threshold strategies that take values at the corner, i.e., invest in either country 1 or country 2. We conjecture, but do not have a proof yet, that the equilibrium with threshold-corner strategies constructed above will survive as the unique symmetric equilibrium if we expand the strategy space to monotone strategies. Specifically, denote the probability (or fraction) of investment in country 1 by $\phi(\delta_i) \in [0,1]$; the agent's strategy is called monotone if $\phi(\delta_i)$ is monotonically increasing in his signal δ_i of the country 1's fundamental.

However, we do know that once we allow agents to choose among non-monotone strategies, then for a higher z, it is possible to construct equilibria where both countries are safe given certain realizations of the relative fundamental signal $\tilde{\delta}$ (while one country fails if $\tilde{\delta}$ is too low or too high). We construct this non-monotone equilibrium (using "oscillating strategies") in Appendix A.2.

Under oscillating strategies agents invest in country 2 for low δ_i, then invest in country 1 given better signals, but go back to investing in country 2 for even higher signals. This strategy profile is driven by the strategic substitutes effect in our model, as it serves to equalize returns from investing when both countries are safe. Indeed, in the constructed equilibria with non-monotone strategies, non-monotonicity occurs only in the region where both countries are safe given the realization of fundamental $\tilde{\delta}$ and equilibrium investment strategies. In this region, knowing that both countries

[8] The total government debt of Switzerland in early 2015 was \$127bn. Central bank liabilities were nearer \$500bn, having grown significantly with the Europe crisis and the Swiss decision to maintain their exchange rate vis-a-vis the Euro. Total government debt in Denmark was \$155bn. Total central bank holdings of gold are approximately \$1.2tn, although this amount is largely backing for government liabilities, rather than privately investable gold. It is difficult to get a clear sense of the quantity of gold held privately as an investment, but it is likely not larger than the central bank holdings of \$1.2tn. The most liquid gold investment are gold ETFs. Total capitalization of US gold ETFs was \$39bn in early 2015. As a comparison, the total supply of Treasury bonds plus central bank liabilities (reserves, cash, repos) in early 2015 was over \$16tn.

will be safe, investors who are indifferent oscillate between investing in country 1 and country 2 depending on their private signal realizations. Note that the fundamental $\tilde{\delta}$ is no longer payoff relevant for safe countries (recall that without default the payoff of bonds is capped at one). Hence, the private signal is no longer payoff relevant, and this oscillation can be viewed as a randomization scheme to equalize returns. This is because every agent (say with a signal δ_i) in this region knows that other agents whose private signals span an interval of 2σ are investing in both countries in the right proportion, leading to equalized returns in both countries.

Though seemingly exotic, it is interesting that equilibria with non-monotone strategies lead to the economically plausible situation that when funding conditions in the world are plentiful (high z), both countries will be safe and there is a joint reserve asset. This possibility cannot emerge in the case we have analyzed where agents play monotone threshold strategies.

Importantly, we stress that all key qualitative properties in Proposition 1 derived under the threshold strategy equilibria are robust to considering the non-monotone strategy equilibria, with small modifications. The next proposition summarizes the results, which are parallel to Proposition 1.

Proposition 4 *We have the following results for the equilibrium with oscillating strategies.*

1. *The survival probability of the larger country 1 (hence the probability of country 1 bonds being the reserve asset) increases with the aggregate funding conditions z. However, a higher z also increases the survival probability of the smaller country 2.*

2. *For sufficiently favorable aggregate funding conditions $z \geq \underline{z}$ where $\underline{z}$ is derived in the appendix, the equilibrium with oscillating strategies exists. In this class of equilibria, the bonds issued by the larger country 1 are more likely to be the reserve asset than the bonds issued by the smaller country 2.*

3. *All else equal, the larger country 1 has the highest likelihood of survival when the size of country 2 goes to zero, i.e. $s \to 0$.*

Regarding the first result, recall that in the monotone threshold equilibria studied in the main text, a higher z increases the survival probability of the larger country 1 and at the same time *decreases* the survival probability of the smaller country 2. This is because only one country survives in the monotone threshold equilibria. In contrast, in the oscillating non-monotone strategy equilibria, both counties may survive, and thus improved aggregate funding conditions makes both countries safer. The second result of Proposition 4 reinforces that in Proposition 1, i.e., under sufficiently favorable aggregate funding conditions so that the non-monotone strategy equilibrium exists, the bonds of the larger country are more likely to be the reserve asset than the bonds of the smaller country. The third result is identical to that of Proposition 1.

3 Endogenous Reserve Asset Size

The preceding analysis suggest that there are costs and benefits of debt size. We now investigate this issue formally, considering incentives to alter debt size.

3.1 Incentives and Costs of Altering Debt Size

We suppose that the countries have natural debt sizes of s_1^* and s_2^*, which we will think of as determined by local economic and fiscal conditions. For example, countries with a higher GDP will naturally have a larger stock of debt outstanding. A country can choose to alter the size of its debt, but incurs some adjustment cost. For example, if country 1 increases its debt size to S_1 it must increase the surplus θ proportionately, via increase in the tax base, to support the larger debt issue.

We assume that increasing the debt size to S_i raises the surplus from θ_i to $\theta_i \frac{S_i}{s_i^*}$. Suppose that countries face an adjustment cost $c(S_i - s_i^*)$ which is increasing and convex in $S_i - s_i^*$, with $c(0) = 0$, $c'(0) = 0$. Then country 1 chooses S_1 to maximize (focusing again on monotone threshold strategies),

$$\underbrace{-\delta^* (S_1, S_2)}_{\textbf{benefit of reserve asset status}} \quad - \quad \underbrace{c(S_1 - s_1^*)}_{\textbf{adjustment cost}} \quad . \tag{14}$$

This objective can be understood as follows. The second term is the cost of adjustment. The first term captures the benefit of adjustment, i.e., the country is able to increase the probability of placing its debt and avoid default. Suppose that there is a constant loss associated with default, then the benefit is proportional to the survival probability, $\Pr\left(\tilde{\delta} > \delta^*\right)$. Given our uniform distribution assumption of $\tilde{\delta}$, $\Pr\left(\tilde{\delta} > \delta^*\right)$ is linear in $-\delta^*(S_1, S_2)$. Thus, equation (14) is country 1's objective function with an appropriate renormalization of the cost function. Likewise for country 2, the objective is

$$\delta^*(S_1, S_2) - c(S_2 - s_2^*)$$

where we note that the probability of placing country 2's debt is increasing in $\delta^*(S_1, S_2)$ (and linear in $\delta^*(S_1, S_2)$ given the uniform distribution of $\tilde{\delta}$).

We solve for $\delta^*(S_1, S_2)$ following the same analysis as in Section 2.2. The marginal investor with signal δ^* receives the following return from investing in country 1:

$$\Pi_1 = \int_{\frac{S_1(1-\theta)e^{-\delta^*}}{1+f}}^{1} \frac{S_1}{(1+f)\,x} dx = \frac{S_1}{1+f}\left(-\ln S_1 + \ln\frac{1+f}{1-\theta} + \delta^*\right),$$

while the return from investing in country 2 is

$$\Pi_2 = \int_0^{\frac{1+f-S_2(1-\theta)e^{\delta^*}}{1+f}} \frac{S_2}{(1+f)(1-x)} dx = \frac{S_2}{1+f}\left(-\ln S_2 + \ln\frac{1+f}{1-\theta} - \delta^*\right).$$

Setting $\Pi_1 = \Pi_2$ and solving, we find that the equilibrium threshold as a function of sizes of the two countries is

$$\delta^*(S_1, S_2) = \frac{S_2 - S_1}{S_1 + S_2} z + \frac{S_1 \ln S_1 - S_2 \ln S_2}{S_1 + S_2}. \tag{15}$$

where again $z \equiv \ln\frac{1+f}{1-\theta}$ measures the strength of savings/fundamentals. When $S_1 = 1$, we recover the equilibrium threshold for our base model in (11). When $S_1 = S_2$, the threshold is zero regardless of z, and each country is equally likely to be the reserve asset.

Next consider the benefit of increasing country size. Introduce the function $h(S_1, S_2; z)$, which is defined as the marginal impact of S_1 on $\delta^*(S_1, S_2)$:

$$h(S_1, S_2; z) \equiv \frac{\partial \delta^*(S_1, S_2)}{\partial S_1} = \frac{1}{(S_1 + S_2)^2}\left[S_1 + S_2(\ln S_1 + \ln S_2 + 1 - 2z)\right] \tag{16}$$

Due to symmetry, the (negative) derivative of δ^* with respect to S_2 is

$$-\frac{\partial \delta^*(S_1, S_2)}{\partial S_2} = h(S_2, S_1; z) = \frac{1}{(S_1 + S_2)^2}\left[S_2 + S_1(\ln S_1 + \ln S_2 + 1 - 2z)\right].$$

As a result, the first order conditions for endogenous debt sizes satisfy,

$$h(S_1, S_2; z) = c'(S_1 - s_1^*) \quad \text{and,} \quad h(S_2, S_1; z) = c'(S_2 - s_2^*).$$

Next we analyze the property of $h(S_1 = s_1^*, S_2 = s_2^*; z)$.

3.2 Equilibrium Characterization and "Phase Diagram"

We illustrate the solution in Figure 3. We first note that if there is no cost of changing size, then the equilibrium solves

$$h(S_1, S_2; z) = h(S_2, S_1; z) = 0.$$

The equilibrium is symmetric. Define the solution in this case as $\overline{S}(z) = S_1 = S_2$, and solving we find that:

$$1 + \ln \overline{S}(z) = z \Rightarrow \overline{S}(z) = e^{z-1}.$$

On Figure 3 we plot the phase diagram where $S_1 = s_1^*$ and $S_2 = s_2^*$ are below $\overline{S}(z)$. The solid black curve corresponds to the set of points where $h(S_1, S_2; z) = 0$; that is, these are points S_1 and S_2 where changing S_1 has no effect on δ^*. The dashed red curve corresponds to the set

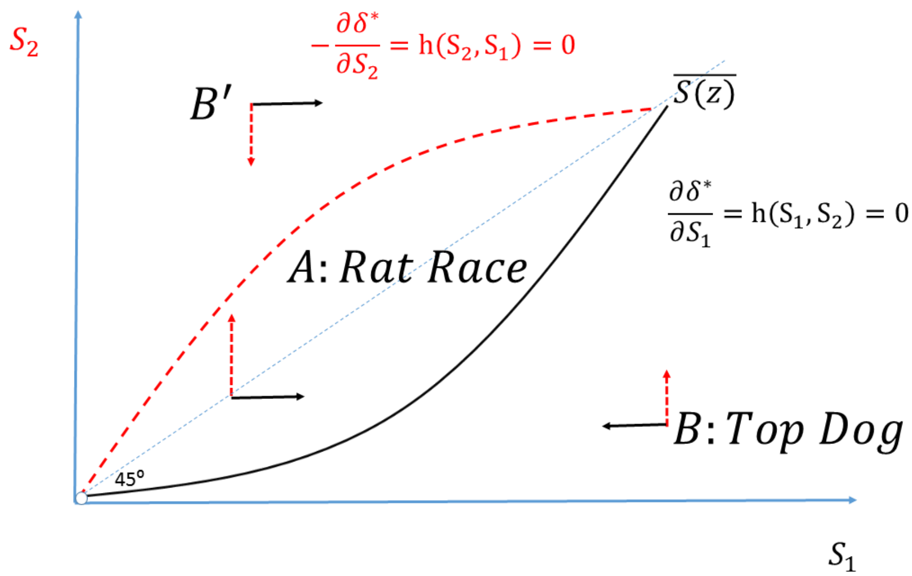

Figure 3: Diagram of incentives to alter size of debt

of points where $h(S_2, S_1; z) = 0$; that is, these are points where changing S_2 has no effect on δ^*. These two curves are symmetric around the 45-degree line and cross at the point $(\overline{S}(z), \overline{S}(z))$. The following lemma shows when country sizes are relatively small with $\max(S_1, S_2) < \overline{S}(z)$, the locus of $h(S_1, S_2; z) = 0$, which is the black curve in the figure, has $S_1 > S_2$. This implies that the black curve lies below the 45-degree line. The result is reversed for the red curve which is the locus of $-\frac{\partial \delta^*(S_1, S_2)}{\partial S_2} = h(S_2, S_1; z) = 0$ and is symmetric around the 45-degree line.

Lemma 1 *When* $\max(S_1, S_2) < \overline{S}(z)$*, the locus of* $h(S_1, S_2; z) = 0$ *has* $S_1 > S_2$..

Proof. We show that for $h(S_1 = S_2 + \epsilon, S_2; z) = 0$ to hold we must have $\epsilon > 0$. Using (16) we have:

$$h(S_1 = S_2 + \epsilon, S_2; z) = \epsilon + S_2(\ln(S_2 + \epsilon) - \ln S_2) + 2S_2(1 + \ln S_2 - z) = 0.$$

When $S_2 < \overline{S}(z)$, the last term $2S_2(1 + \ln S_2 - z)$ on the LHS is negative. To have $h = 0$, we must have that $\epsilon > 0$, since conditional on S_2 the sum of the first two terms is zero when $\epsilon = 0$ and increasing in ϵ. Q.E.D. ■

Intuitively, for any given level of S_2, starting from the 45-degree line country 1 has a strictly positive incentive to expand its debt size to enhance its reserve asset status, since $c'(0) = 0$. This is due to the size benefit, illustrated in the left panel of Figure 2. This explains why along the solid black curve with zero expansion incentive we must have $S_1 > S_2$. However, country 1 does not want its debts to become too large, because as illustrated by the right panel of Figure 2, becoming too large triggers rollover risk fears that may lead investors to coordinate on the smaller country's debt as the reserve asset. The black curve with $h(S_1, S_2; z) = 0$ balances these two effects.

Now suppose that the initial country sizes (s_1^*, s_2^*) lies to the right of the black curve, i.e., $h(S_1 = s_1^*, S_2 = s_2^*; z) > 0$. Then in equilibrium country 1 will choose an $S_1 < s_1^*$ (exactly how much less depends on the cost function). Likewise, for any S_1 if s_2^* is above the red curve, then country 2 will choose an $S_2 < s_2^*$. For points inside the solid black and dashed red curves, both countries want to expand. The arrows on Figure 3 indicate the direction, as in a phase diagram, in which countries will change debt size given a natural size of (s_1^*, s_2^*).

There are three regions of interest in Figure 3. Region A ("RAT RACE") corresponds to a case where countries' debts are similar in size – so a roughly symmetric case. We see that in this case both countries will increase the size of their debts. This is driven by the reserve asset effect: there is an externality whereby the larger country has a better reserve asset position, and countries compete to become the reserve asset by increasing their debt sizes. Of course, this competition to gain reserve asset status is ultimately self-defeating. Take the fully symmetric case where $s_1^* = s_2^*$ and thus $\delta^* = 0$ in equilibrium. The Nash equilibrium results in countries increasing debt sizes beyond their natural sizes in a rat race to become the reserve asset, while both would be better off and save adjustment costs if they coordinate not to expand. Region B and B' ("TOP DOG") are cases with asymmetric country sizes. Take region B (as B' follows the same logic). Here country 1 is large and country 2 is relatively small. In this case, country 1 is the top dog and not worried about losing its reserve asset status, and is primarily concerned about rollover risk. In this region country 1 chooses to contract its debt size, while country 2 expands, hoping to gain some reserve status.[9]

[9]For natural sizes beyond $\overline{S}(z)$, i.e. $\max(S_1, S_2) > \overline{S}(z)$, there is a region (not shown in Figure 3) outside A that

3.3 Funding Conditions and Incentives for Expansion

By using the fact that $\frac{\partial h(\cdot,\cdot;z)}{\partial z} < 0$ in (16), we can show that the region A—which is the area inside the solid black curve and the dashed green curve—expands unambiguously as the aggregate funding condition z rises. This result has interesting implications, as it suggests that countries, once facing an environment with better aggregate funding conditions, are drawn into the rate race where they compete to offer the reserve asset.

This comparative static offers a unique perspective on the expansion of relatively safe debt supplies in the period preceding the 2007 financial crisis. In the US, the government agencies, Fannie Mae and Freddie Mac, initiated a program ("Benchmark Notes") in 1998 whose purpose was to offer debt that could compete with US Treasury bonds as a large and liquid savings vehicle. Of course, the expansion of such agency debt stocks ultimately resulted in the bailouts of Fannie Mae and Freddie Mac by the US government, suggesting that welfare would have been improved without these programs, or if the US Treasury had coordinated the debt sizes of the Agencies along with that of the rest of the federal government. In Europe, the expansion of sovereign debt after the formation of the Euro can similarly be seen as a rat race to serve as the reserve asset within Europe. This rat race has also ended badly.

The figure offers a further prediction which may be comforting in today's world of high US government debt. Suppose that the world economy was in region A prior to the financial crisis, but has since transitioned to region B, where US government debt is the top dog. This transition is in keeping with the fact that the amount of US Treasury competitors has fallen since the crisis. The model suggests that the US then has an incentive to shrink its debt.

4 Coordination and Security Design

There can be gains from coordination. The results of the last section suggest that countries may be better off by coordinating to limit the size of their debt in some circumstances. In this section, we

covers the 45-degree line, in which both countries have incentives to shrink. There, the debt sizes are relative large compared to the aggregate funding conditions, and the rollover risk concerns drive both countries to issue less debt.

extend this analysis and characterize the benefits to coordinating through security design. We are motivated by the Eurobond proposals that have been floated over the last few years (see Claessens, Mody and Vallee, 2012, for a review of various proposals). A shared feature of the many proposals is to create a common Europe-wide reserve asset. Each country receives proceeds from the issuance of the "common bond" which is meant to serve as the reserve asset, in addition to proceeds from the sale of an individual country-specific bond. By issuing a common Euro-wide reserve asset, all countries benefit from investors' need for a reserve asset, as opposed to just the one country (Germany) which is the de-facto reserve asset in the absence of a coordinated security design. Our model, in which the determination of the reserve asset is endogenous, seems well-suited to analyze these issues formally. We are unaware of other similar models or formal analysis of this issue.

We return to the model with exogenous sizes, fixing $s_1 = 1$ and $s_2 = s$. The two countries issue a common bond of size $\alpha\,(1+s)$ as well as individual country bonds of size $(1-\alpha)s_i$. Suppose the equilibrium price for the common bonds is p_c. Let $\frac{s_i}{s_1+s_2}$ be the share of proceeds from the common bond issue that flows to country i, so that country i receives

$$\frac{s_i}{1+s} p_c \alpha\,(1+s) = s_i \alpha p_c$$

from the common bond auction. Country i issues individual bonds of size $(1-\alpha)\,s_i$ at endogenous price p_i so that total proceeds to country i are,

$$s_i(\alpha p_c + (1-\alpha)p_i).$$

We model the bond auction as a two-stage game. In the first-stage, countries auction the common bonds and investors pay a total of $f - \hat{f}$ to purchase these bonds, so that in equilibrium,

$$f - \hat{f} = (1+s)\alpha p_c \tag{17}$$

In the second stage, the investors use their remaining funds of $1 + \hat{f}$ to purchase individual country bonds. After both auctions, each country makes its default decision.

Country i avoids defaults whenever,

$$s_i(\alpha p_c + (1-\alpha)p_i) + s_i\theta_i > s_i, \tag{18}$$

which is a straightforward extension of the earlier default condition to include the common bond proceeds. We assume that default affects all of the country's obligations. The country defaults fully on its individual bonds and defaults on its portion of common bonds, so that investors in common bonds receive repayments only from any countries that do not default. We are currently investigating alternative structuring where common bonds are senior to individual country bonds, but the results are too preliminary to include in this draft.

We conjecture and verify the following solution.

Proposition 5 *There are two equilibria, a "maximum joint safety" equilibrium and a "minimum joint safety" equilibrium. In both equilibria, the determination of the reserve asset depends on α as follows:*

1. $\alpha \in [0, \underline{\alpha}]$*: There exists a threshold $\delta^*(\alpha)$. If $\tilde{\delta} > \delta^*(\alpha)$, then country 1 is the reserve asset, while if $\tilde{\delta} < \delta^*(\alpha)$ country 2 is the safe asset.*

2. $\alpha \in [\underline{\alpha}, 1]$*: Both countries satisfy (18) for $\tilde{\delta} \in [\delta_L(\alpha), \delta_H(\alpha)]$ so that both debts are safe, whereas for $\tilde{\delta}$ outside this interval, one country violates (18) and defaults and the other country's debt is the reserve asset.*

Furthermore, the two equilibria are defined as follows:

1. *In the "maximum joint safety" equilibrium, $\underline{\alpha} = \alpha_0$ which solves $\delta_L(\alpha_0) = \delta_H(\alpha_0)$;*

2. *In the "minimum joint safety" equilibrium, $\underline{\alpha} = \alpha_1$ which solves $\delta^*(\alpha_1) = 0$,*

with $\alpha_1 > \alpha_0$.

Figure 4 illustrates the statement of the proposition for a case of $s = 0.5$ (left panel) and $s = 0.25$ (right panel), both for $z = 1$. Focusing on the left panel, the black solid line plots δ^* for $\alpha \in [0, \alpha_1]$. We have that δ^* is less than zero for this case because $z = 1$ corresponding to the high savings cases we have illustrated in earlier graphs. At $\alpha = \alpha_1$ we have that δ^* equals zero. The dashed-lines in the figure indicate the upper/lower bounds of the realizations of $\tilde{\delta}$ in order for the economy to be in the joint safety region where both countries' debts are reserve assets. We see that this joint safety region begins at $\alpha = \alpha_0$, and expands as a function of α. Intuitively, increasing α shares some of the safety benefits of the large country with the small country and hence allows the small country to be safer given realizations of $\tilde{\delta}$. The two possible equilibria overlap between α_0 and α_1.

In the right panel we consider a case of $s = 0.25$. We note that the joint safety is a possibility even for $\alpha = 0$, provided that the $\tilde{\delta}$ realizations are sufficiently negative (i.e., country 2 is realized to have sufficiently strong fundamentals than country 1). It is still the case that increasing α makes country 2 safer, expanding the joint safety region and increasing δ^* towards zero. Increasing α makes country 1 less safe in the minimum joint safety region, while it makes country 1 more safe (i.e. δ_L falls) in the maximum joint safety region.

The main result that emerges from our analysis in this section, and which is evident in the figure, is that increases in α only create Pareto gains (when gains are thought of in terms of increasing country safety) when $\alpha > \alpha_1$. In this case, increases in α raise the safety of both country 1 and country 2. For $\alpha < \alpha_1$ and in the minimum safety equilibrium, increases in α reduce safety of one country while increasing safety of the other country. We illustrate this result further when we discuss welfare.

4.1 Case 1: Minimum joint safety

We first solve for an equilibrium where only one country is safe. This equilibrium is a small variant on the equilibrium we have derived so far involving threshold strategies. We will find the largest α so that this threshold equilibrium can exist, which then defines $\underline{\alpha} = \alpha_1$.

We work backwards from the second stage. In the second stage, investors have $1 + \hat{f}$ to purchase

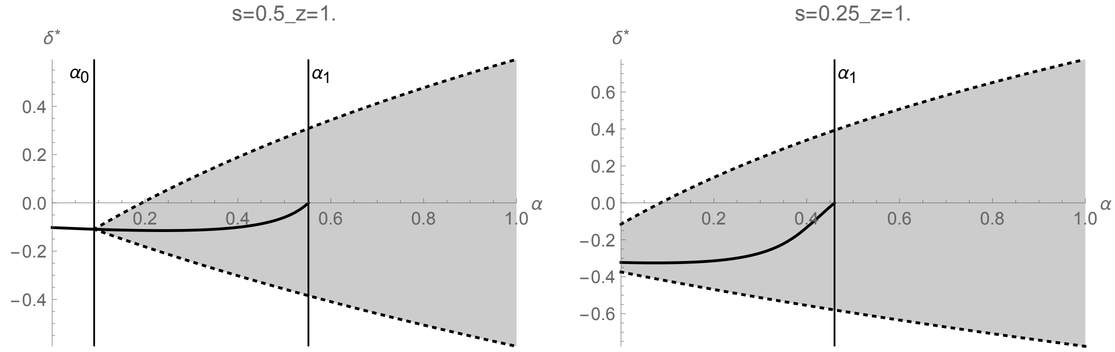

Figure 4: $\delta^*, \delta_H, \delta_L$ for the case of $s = 0.5$ and $s = 0.25$, as a function of α

individual country bonds. Consider the marginal investor with signal δ^* who considers that a fraction x of investors have signals exceeding his. Country 1 does not default if,

$$\alpha p_c + (1-\alpha)p_1 + \theta_1 > 1$$

Since, $f - \hat{f} = (1+s)\alpha p_c$ by (17)and $(1-\alpha)p_1 = x(1+\hat{f})$, we rewrite this condition as,

$$\frac{f-\hat{f}}{(1+s)} + x(1+\hat{f}) + \theta_1 > 1 \Rightarrow x \geq \frac{1-\theta_1 - \frac{1}{1+s}(f-\hat{f})}{1+\hat{f}}$$

We again take the limit as $\sigma \to 0$ and set $(1-\theta_1) = (1-\theta)e^{-\delta^*}$. Additionally, noting that the return to the marginal investor in investing in country 1 is $\frac{1-\alpha}{(1+\hat{f})x}$ if the country does not default (and zero recovery in default), we integrate over returns when the country does not default to find,

$$\Pi_1 = \frac{1-\alpha}{1+\hat{f}} \ln\left(\frac{1+\hat{f}}{(1-\theta)\,e^{-\delta^*} - \frac{1}{1+s}\left(f-\hat{f}\right)}\right).$$

When $\hat{f} = 0$, this equation is equal to the previously derived profit equation.

We repeat the same steps for the profits to investing in country 2 and find,

$$\Pi_2 = \frac{s\,(1-\alpha)}{1+\hat{f}} \ln\left(\frac{1+\hat{f}}{s\,(1-\theta)\,e^{\delta^*} - \frac{s}{1+s}\left(f-\hat{f}\right)}\right).$$

We solve for the threshold in the same way as before:

$$\Pi_1 = \Pi_2 \Rightarrow \delta^*(\hat{f}, \alpha) \tag{19}$$

Next we consider the stage 1 game and derive $\hat{f}$. Under the assumed equilibrium where only one country becomes the reserve asset and does not default, the return to investing in the common bond is,

$$\frac{1}{f - \hat{f}} \left[\int_{-\bar{\delta}}^{\delta^*} \alpha s \frac{d\delta}{2\bar{\delta}} + \int_{\delta^*}^{\bar{\delta}} \alpha \frac{d\delta}{2\bar{\delta}} \right].$$

The denominator in the front is the total amount of funds invested in the common bond, while the term in parentheses is the repayment on the common bonds in the cases of repayment only by country 2 and repayment only by country 1. The returns to keeping one dollar aside and investing in individual country bonds is,

$$\frac{1}{1 + \hat{f}} \left[\int_{-\bar{\delta}}^{\delta^*} (1 - \alpha) s \frac{d\delta}{2\bar{\delta}} + \int_{\delta^*}^{\bar{\delta}} (1 - \alpha) \frac{d\delta}{2\bar{\delta}} \right].$$

Again, the denominator in the front is the total amount of funds invested in individual bonds, while the term in parentheses is the repayment on individual bonds in the cases of repayment only by country 2 and repayment only by country 1. Note the similarity between these last two expressions. The similarity arises because along the nodes of country 2 defaulting or country 1 defaulting, the payoffs, state-by-state, to common bonds and individual bonds are αs_i and $(1 - \alpha) s_i$. Since the return from investing in common bonds in stage one of the game and waiting and investing in individual bonds in stage 2 must be equal, we have that:

$$\frac{\alpha}{f - \hat{f}} = \frac{1 - \alpha}{1 + \hat{f}}$$

which we can solve for $\hat{f}$ to give,

$$f - \hat{f} = \alpha\,(1 + f) \tag{20}$$

so that

$$p_c = \frac{f - \hat{f}}{\alpha\,(1 + s)} = \frac{1 + f}{1 + s}. \tag{21}$$

We combine equation (19) and (20) to solve for $\delta^*(\alpha)$.

We next consider the bound α_1 We have assumed that only one country becomes the reserve asset, while the other country does not default. However, inspecting (18) we see that as α rises, since $p_c > 0$, it may be that even a country that receives zero on its individual bonds will be able to avoid default. That is, the proceeds from the common bond, which is the reserve asset, provides a default cushion for the weaker country. In this case, it is possible that neither country defaults on its bonds, invalidating our equilibrium assumption that one country defaults for sure.

For each realization of δ, define

$$\theta_{def}\,(\delta) \equiv \min\left[\theta_1\,(\delta)\,, \theta_2\,(\delta)\right]$$

as the fundamental of the country with the worse realization of fundamentals. Take the best value of this worse realization of fundamentals, which occurs when $\delta = \delta^*$, since even at a slightly higher higher value than this realization one country narrowly beats the other country to become the reserve asset. Now we have constructed an equilibrium in which the beaten country defaults. But is this actually an equilibrium? For this realization, the beaten country defaults if,

$$\theta_{def}\,(\delta^*) + \alpha p_c < 1 \iff \alpha\frac{1+f}{1+s} < 1 - \theta_{def}\,(\delta^*) \iff \alpha \le \frac{1+s}{1+f}\left[1 - \theta_{def}\,(\delta^*\,(\alpha))\right].$$

Then α_1 solves the equation,

$$\alpha_1 = \frac{1+s}{1+f}\left[1 - \theta_{def}\,(\delta^*\,(\alpha_1))\right]$$

Let us investigate the slope $\delta^*_\alpha\,(0)$. We see from the graphs that there are situations in which

$\delta_\alpha^{*\prime}(0) < 0$ so that the large country gains when a small fraction of common bonds are issued. The intuition is the following. Equalizing returns, and realizing that $\hat{f} = f - \alpha(1+f)$, we see that increasing α decreases the available funding for individual bonds. When we look at the returns, we are comparing

$$\ln\left(\frac{1+\hat{f}}{(1-\theta)\,e^{-\delta^*} - \frac{1}{1+s}\left(f-\hat{f}\right)}\right) \quad \text{vs} \quad s\ln\left(\frac{1+\hat{f}}{s\,(1-\theta)\,e^{\delta^*} - \frac{s}{1+s}\left(f-\hat{f}\right)}\right)$$

We see there are two effects. In the denominator, a higher α leads to a lower $\hat{f}$, which means there is less money left over for individual bonds as common bonds divert some of the funding. Size doesn't matter for the numerator as we are in a "winner takes all" equilibrium for individual bonds. This is the *funding effect*. The denominator also changes, however, as $\alpha > 0$ introduces common bond proceeds $\frac{f-\hat{f}}{1+s}$ per common bond sold. We see that these common bonds are scaled by the size of the country as common bonds are issued proportional to country size. Common bond proceeds make a country safer, all else equal. Thus, this is the *safety effect.* For small s, most of the common bond proceeds accrue to the large country. This can lead to a situation in which the large country becomes even safer as δ^* adjusts downward.

4.2 Case 2: Maximum joint safety

We now construct an equilibrium in which both countries can be safe. Given this equilibrium, we will compute the minimum value of α for which this equilibrium exists. As we show, this minimum value, denoted α_0, will generally be less than α_1. This overlap implies that at least two equilibria exist for some parameters of the model, as described in the Proposition.

The possibility that both countries may be safe means that our equilibrium construction using threshold strategies is no longer possible. In a region where both countries are known to be safe (recall we consider the limit where $\sigma \to 0$), investors must be indifferent between investing in the two countries, choosing strategies so that in equilibrium the return on both countries' bonds are equal. This "joint safety" region prevails for realizations of $\tilde{\delta} \in [\delta_L, \delta_H]$. Outside this interval we are back to the case where the signals are so strong that only one country survives. We now focus on deriving the boundary between these regions given by functions $\delta_L(\alpha)$ and $\delta_H(\alpha)$. Given these functions we can solve for the boundary α_0 , which occurs when $\delta_L(\alpha_0)$=$\delta_H(\alpha_0)$ and the joint safety region vanishes.

We depart from monotone threshold strategies. Let us conjecture a non-monotone strategy whereby investment in country 1 and in country 2 alternate on discrete intervals of length $k\sigma$ and $(2-k)\,\sigma$, with $k \in (0,2)$. We will refer to this strategy as the oscillating strategy. The investor strategy is $\phi\,(y) \in \{0,1\}$ which is investment in country 1:

$$\phi\,(y) = \begin{cases} 0, & y < \delta_L \\ 1, & y \in [\delta_L, \delta_L + k\sigma] \cup [\delta_L + 2\sigma, \delta_L + (2+k)\,\sigma] \cup [\delta_L + 4\sigma, \delta_L + (4+k)\,\sigma] \cup ... \\ 0, & y \in [\delta_L + k\sigma, \delta_L + 2\sigma] \cup [\delta_L + (2+k)\,\sigma, \delta_L + 4\sigma] \cup [\delta_L + (4+k)\,\sigma, \delta_L + 6\sigma] \cup ... \\ 1, & y > \delta_H \end{cases} \tag{22}$$

In the limit as σ approaches zero, these strategies resemble mixed strategies, so that over any interval strictly inside $[\delta_L, \delta_H]$ the probability of investing in each country is such that bond returns are equalized.

4.2.1 Fraction of agents in investing in country 1

Consider a region on which all investors know that both countries are safe. In this case, the total investment in country 1 and 2 has to be $\left(1+\hat{f}\right)\frac{1}{1+s}$ and $\left(1+\hat{f}\right)\frac{s}{1+s}$, respectively, as otherwise one country will offer strictly higher (expected) returns than the other country. Take an agent with signal δ. Let us introduce the function $\rho\,(\delta)$, which is the expected proportion of agents investing in country 1 given (own) signal δ. Then, given the assumed strategy for all agents and given that we are in the region where both countries are safe,

$$\rho(\delta) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi\,(y)}{2\sigma} dy = \frac{k\sigma}{2\sigma}$$

We choose k so that,

$$\rho(\delta) = \frac{1}{1+s} \iff k = \frac{2}{1+s} \tag{23}$$

The last equality arises because in equilibrium the proportion must be constant and equal to $\frac{1}{1+s}$ in order to have returns equalized across the two safe bonds.

For any value of δ and x, the proportion of agents investing in country 1 is given by

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \begin{cases} 0, & \delta + 2\sigma x < \delta_L \\ \frac{\delta+2\sigma x-\delta_L}{2\sigma} & \delta + 2\sigma x \in (\delta_L, \delta_L + k\sigma) \\ \frac{1}{1+s}, & \delta_H - (2-k)\sigma > \delta > \delta_L + k\sigma \end{cases} \tag{24}$$

We see that $\rho(\delta, x)$ is strictly increasing in x in the vicinity of δ_L, and flat everywhere else. Note further that

$$\rho(\delta_L, x) = \begin{cases} x, & x \in \left[0, \frac{1}{1+s}\right] \\ \frac{1}{1+s}, & x \in \left(\frac{1}{1+s}, 1\right] \end{cases} \tag{25}$$

where we observe that $\rho(\delta_L, x)$ is less than or equal to $\frac{1}{1+s}$.

4.2.2 Lower boundary δ_L

Let $\Pi_i(\delta)$ be the expected payoff of investing in country i for a given agent with signal δ. In the completely safe case, investors were indifferent between both strategies because both paid exactly the same in all states of the world. This is not the case at δ_L, as the countries (at least in the eyes of the agent) are not always safe. Thus, the country with perceived default risk (at δ_L this is country 1) will need to offer a higher return conditional on survival to be attractive to the agent. Consider now the agent with signal δ_L. In his eyes, the return from investing only in country 2 (i.e. $\phi = 0$) is given by

$$\Pi_2(\delta_L) = \int_0^1 \frac{s}{\left(1+\hat{f}\right)(1-\rho(\delta_L, x))} dx \tag{26}$$

where we integrate over all x as country 1 is safe regardless of x. Thus, substituting in for $\rho\left(\delta_L, x\right)$, we have

$$\Pi_2\left(\delta_L\right) = \frac{s}{1+\hat{f}}\left[\int_0^{\frac{1}{1+s}} \frac{1}{1-x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{\frac{s}{1+s}} dx\right] = \frac{s}{1+\hat{f}}\left[\ln\frac{1+s}{s} + 1\right] < \frac{1+s}{1+\hat{f}} \tag{27}$$

where we used $s\ln\frac{1+s}{s} < 1$. Here, we see that the payoff to investing in country 2 is lower than the expected payoff that would have realized if both countries were safe. This comes about because more people (in expectation) invest in country 2, and less people invest in country 1. This, of course, means that the expected payoff (conditional on survival) of investing in country 1 bonds is higher.

The minimum proportion of agents investing in country 1 that are needed to make country 1 safe if the true state of the world is δ solves,

$$\theta_1\left(\delta\right) + \alpha p_c + \left(1+\hat{f}\right)\rho_1^{min}\left(\delta\right) = 1 \iff \rho_1^{min}\left(\delta\right) = \frac{1-\theta_1\left(\delta\right) - \alpha p_c}{1+\hat{f}}$$

Define $x_{min}\left(\delta_L\right)$ as the solution to $\rho\left(\delta_L, x\right) = \rho_1^{min}\left(\delta_L\right)$. Given equation (25), we have that,

$$x_{min}\left(\delta_L\right) = \frac{1-\theta_1\left(\delta_L\right) - \alpha p_c}{1+\hat{f}} \tag{28}$$

The expected return of investing in country 1 given one's own signal δ_L and the strategies $\phi\left(y\right)$ of other agents is given by,

$$\begin{aligned}
\Pi_1\left(\delta_L\right) &= \int_{x_{min}\left(\delta_L\right)}^1 \frac{1}{\left(1+\hat{f}\right)\rho(\delta_L, x)} dx \\
&= \frac{1}{1+\hat{f}}\left[\int_{x_{min}\left(\delta_L\right)}^{\frac{1}{1+s}} \frac{1}{x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{1/\left(1+s\right)} dx\right] \\
&= \frac{1}{1+\hat{f}}\left[\ln\frac{1}{1+s} - \ln x_{min}\left(\delta_L\right) + s\right]
\end{aligned}$$

Indifference requires that

$$\begin{aligned}
\Pi_2(\delta_L) &= \Pi_1(\delta_L) \\
\Longleftrightarrow \frac{s}{1+\hat{f}}\left[\ln\frac{1+s}{s}+1\right] &= \frac{1}{1+\hat{f}}\left[\ln\frac{1}{1+s}-\ln x_{min}+s\right] \\
\Longleftrightarrow s\ln\frac{1+s}{s} &= \ln\frac{1}{1+s}-\ln x_{min}(\delta_L) \\
\Longleftrightarrow x_{min}(\delta_L) &= \exp\left[s\ln s-(1+s)\ln(1+s)\right] \qquad (29)
\end{aligned}$$

We combine our two equations for $x_{min}(\delta_L)$, (28) and (29), to find an equation that determines δ_L:

$$\exp\{s\ln s-(1+s)\ln(1+s)\} = \frac{1-\theta_1(\delta_L)-\alpha p_c}{1+\hat{f}} \qquad (30)$$

We see that σ does not appear, and thus this solution is independent of signal noise. Since, $1-\theta_1(\delta)=(1-\theta)\exp(-\delta)$, we further have,

$$\frac{s^s}{(1+s)^{(1+s)}} = \frac{(1-\theta)e^{-\delta_L}-\alpha p_c}{1+\hat{f}}$$

which gives

$$\delta_L = -\ln\left\{\frac{1}{1-\theta}\left[\left(1+\hat{f}\right)\frac{s^s}{(1+s)^{(1+s)}}+\alpha p_c\right]\right\} \qquad (31)$$

For $\tilde{\delta}<\delta_L$, only country 2 is safe.

4.2.3 Upper boundary δ_H

The derivation of the upper boundary δ_H is analogous. We have

$$\rho(\delta,x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy = \begin{cases} \frac{1}{1+s}, & \delta-2\sigma(1-x)<\delta_H-(2-k)\sigma \\ \frac{\delta+2\sigma x-\delta_H}{2\sigma} & \delta-2\sigma(1-x)\in[\delta_H-(2-k)\sigma,\delta_H] \end{cases} \qquad (32)$$

so that

$$\rho\left(\delta_H, x\right)=\begin{cases}\frac{1}{1+s}, & x<\frac{1}{1+s}\\ x, & x\in\left[\frac{1}{1+s},1\right]\end{cases} \tag{33}$$

Consider now the agent with signal δ_H who is considering investing in country 1. In his eyes, the return from investing only in country 1 (i.e. $\phi=1$) is given by

$$\Pi_1\left(\delta_H\right)=\int_0^1 \frac{1}{\left(1+\hat{f}\right)\rho(\delta_H,x)dy}dx$$

where we integrated over all x as country 2 is always safe in the vicinity of δ_H. Thus, plugging in, we have

$$\Pi_1\left(\delta_H\right)=\frac{1}{1+\hat{f}}\left[\int_0^{\frac{s}{1+s}}\frac{1}{1-x}dx+\int_{\frac{s}{1+s}}^1\frac{1}{\frac{1}{1+s}}dx\right]=\frac{1}{1+\hat{f}}\left[\ln\left(1+s\right)+1\right]<\frac{1+s}{1+\hat{f}} \tag{34}$$

where we used $\ln\left(1+s\right)<s$ for $s>0$.

The default condition for country 2 is

$$s\theta_2\left(\delta\right)+s\alpha p_c+\left(1+\hat{f}\right)\left[1-\rho_2^{max}\left(\delta\right)\right]=s \iff \left[1-\rho_2^{max}\left(\delta\right)\right]=s\frac{1-\theta_2\left(\delta\right)-\alpha p_c}{1+\hat{f}}$$

where $\rho_2^{max}\left(\delta\right)$ is the maximum amount of people investing in country 1 so that country 2 does not default. Let us assume for the moment that at δ_H, we have $\left[1-\rho_2^{max}\left(\delta_H\right)\right]<\frac{s}{1+s}$, that is country 2 would survive even if less than $\frac{s}{1+s}$ of investors invest in it. Define $x_{max}\left(\delta_H\right)$ as the solution to $\rho\left(\delta_H, x_{max}\right)=\rho_2^{max}\left(\delta_H\right)$. Given equation (33), we have that,

$$1-x_{max}\left(\delta_H\right)=s\frac{1-\theta_2\left(\delta_H\right)-\alpha p_c}{1+\hat{f}} \tag{35}$$

The return to investing in country 2 is,

$$\begin{aligned}
\Pi_2(\delta_H) &= \int_0^{x_{max}(\delta_H)} \frac{s}{\left(1+\hat{f}\right)(1-\rho(\delta_H,x))dy}dx \\
&= \frac{s}{1+\hat{f}}\left[\int_0^{\frac{1}{1+s}} \frac{1}{1-\frac{1}{1+s}}dx + \int_{\frac{1}{1+s}}^{x_{max}(\delta_H)} \frac{1}{1-x}dx\right] \\
&= \frac{s}{1+\hat{f}}\left[\frac{1}{s} + \ln\frac{s}{1+s} - \ln(1-x_{max}(\delta_H))\right]
\end{aligned}$$

Indifference requires

$$\begin{aligned}
\Pi_1(\delta_H) &= \Pi_2(\delta_H) \\
\iff \frac{1}{1+\hat{f}}[\ln(1+s)+1] &= \frac{s}{1+\hat{f}}\left[\frac{1}{s} + \ln\frac{s}{1+s} - \ln(1-x_{max}(\delta_H))\right] \\
\iff \ln(1-x_{max}(\delta_H)) &= \ln s - \frac{1+s}{s}\ln(1+s) \\
\iff 1-x_{max}(\delta_H) &= \frac{s}{(1+s)^{\frac{1+s}{s}}} \qquad (36)
\end{aligned}$$

Combining the expressions for $x_{max}(\delta_H)$ from (35) and (36), we solve,

$$\delta_H = \ln\left\{\frac{1}{1-\theta}\left[\frac{1+\hat{f}}{(1+s)^{\frac{1+s}{s}}} + \alpha p_c\right]\right\} \qquad (37)$$

For $\tilde{\delta} > \delta_H$, only country 1 is safe.

4.2.4 $\alpha > 0$ and stage 1 asset allocation.

Let us now consider the stage 1 asset allocation, that is, the determination of $\hat{f}$ and $p_c = \frac{f-\hat{f}}{\alpha(1+s)}$ (there are $\alpha(1+s)$ units of common bonds, and there is $f-\hat{f}$ money invested in them). Consider an $\alpha > 0$. Then, we know that the expected returns from investing in common bonds in stage 1 and investing in individual country bonds in stage 2 have to be equalized. The return to investing in individual bounds is

$$\begin{aligned}
R_i &= (\text{only country } 2) + (\text{joint survival}) + (\text{only country } 1) && (38)\\
&= \int_{-\underline{\delta}}^{\delta_L} \frac{(1-\alpha)\,s}{1+\hat{f}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_L}^{\delta_H} \frac{(1-\alpha)\,(1+s)}{1+\hat{f}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_H}^{\overline{\delta}} \frac{(1-\alpha)}{1+\hat{f}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} && (39)\\
&= \frac{1-\alpha}{1+\hat{f}} \left[\int_{-\underline{\delta}}^{\delta_L} s\frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_L}^{\delta_H} (1+s)\frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_H}^{\overline{\delta}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} \right] && (40)
\end{aligned}$$

and the expected return for common bonds is given by

$$R_c = \frac{\alpha}{f-\hat{f}} \left[\int_{-\underline{\delta}}^{\delta_L} s\frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_L}^{\delta_H} (1+s)\frac{d\delta}{\overline{\delta}-\underline{\delta}} + \int_{\delta_H}^{\overline{\delta}} \frac{d\delta}{\overline{\delta}-\underline{\delta}} \right] \tag{41}$$

Note the similarity between these last two expressions. The similarity arises because along the nodes of country 2 defaulting or country 1 defaulting, the payoffs, state-by-state, to common bonds and individual bonds are αs_i and $(1-\alpha)s_i$. Thus,

$$\frac{\alpha}{f-\hat{f}} = \frac{1-\alpha}{1+\hat{f}} \iff f-\hat{f} = \alpha\,(1+f) \iff 1+\hat{f} = (1-\alpha)\,(1+f) \tag{42}$$

and,

$$\alpha p_c = \frac{f-\hat{f}}{1+s} = \alpha\frac{1+f}{1+s} \tag{43}$$

We can now solve out for the thresholds. Plugging the expression for αp_c into (31) and (37), we find,

$$\begin{aligned}
\delta_H &= z + \ln\left\{ \frac{1}{1+s} \left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}} (1-\alpha) + \alpha \right] \right\} && (44)\\
\delta_L &= -z - \ln\left\{ \frac{1}{1+s} \left[\left(\frac{s}{1+s}\right)^{s} (1-\alpha) + \alpha \right] \right\} && (45)
\end{aligned}$$

We now establish results concerning the existence of the oscillating equilibrium.

Proposition 6 *For a given z, define α_0 as the solution $z_{HL}(\alpha_0) = z$ where*

$$z_{HL}(\alpha) = \ln(1+s) - \frac{1}{2}\left\{\ln\left[\left(\frac{s}{1+s}\right)^{s}(1-\alpha)+\alpha\right] + \ln\left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}}(1-\alpha)+\alpha\right]\right\}$$

Then, the oscillating equilibrium exists for $\alpha > \alpha_0$. Furthermore, we have $\delta^*(\alpha_0) = \delta_H(\alpha_0) = \delta_L(\alpha_0)$ as well as $\delta^*_\alpha(\alpha_0) < 0$.

There is a smooth transition from δ^* to the oscillating equilibrium as α crosses α_0 in that the thresholds $\delta^*, \delta_H, \delta_L$ all coincide at $\alpha = \alpha_0$. Furthermore, this transition occurs at a point at which $\delta^*(\alpha)$ is decreasing in α, implying that it occurs at a point at which the large country gains safety from increasing α.

4.3 Welfare

Suppose that default results in a deadweight cost equal to the debt size of the defaulting country. That is, default, by country 1 results in deadweight cost of one, while default of country 2 results in deadweight cost of s. For each value of α define q^1, q^2 and q^{12} as the probabilities that country 1, country 2 and both countries are safe. Then aggregate welfare, summed across both countries and bondholders is,

$$W(\alpha) = \hat{W} - (1 - q^1 - q^{12}) - s(1 - q^2 - q^{12})$$

where $\hat{W}$ is a constant independent of α.

Welfare is only affected by the incidence of default given that default is a deadweight cost and all other transactions involve transfers between parties. The deadweight cost of default is equal to size, and hence the last two terms on the right of this expression are the deadweight costs due to country 1 default and country 2 default, respectively.

Figure 5 plots welfare as a function of α when $z = 1$. The left panel plots the case of $s = 0.5$ while the right panel plots that of $s = 0.25$. In both cases we see that welfare is increasing in α along maximum joint safety equilibrium, when the equilibrium exists. This is because in the joint safety

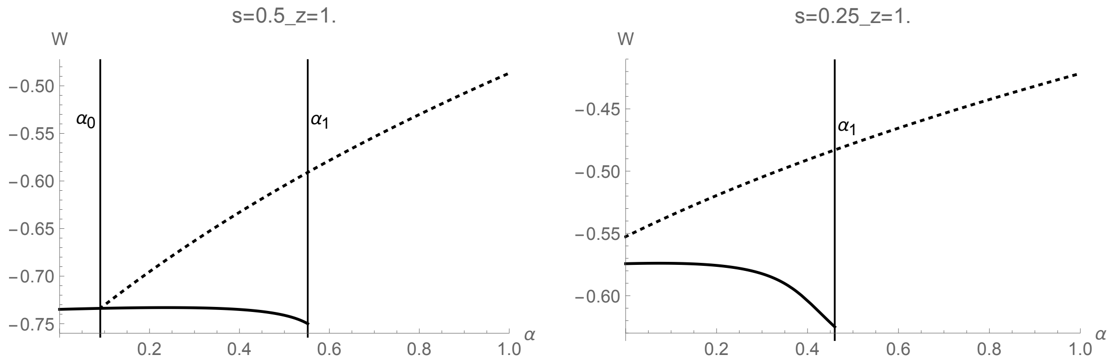

Figure 5: Welfare for the case of $s = 0.5$ and $s = 0.25$, as a function of α

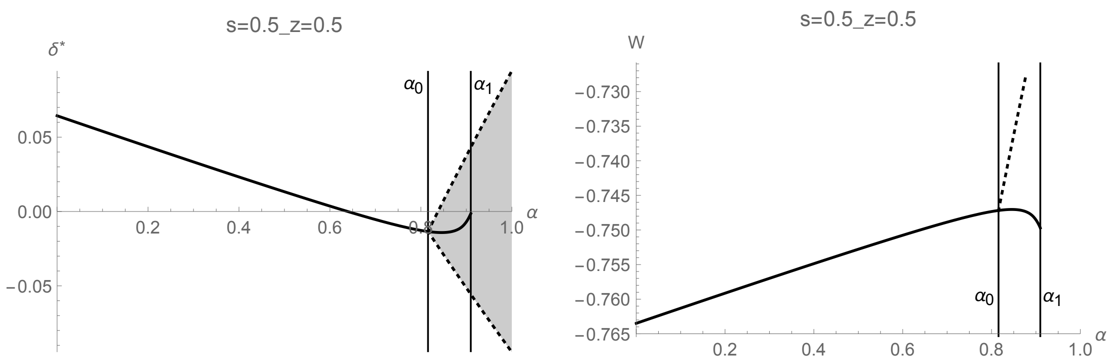

Figure 6: δ and welfare for the case of $s = 0.5$, $z = 0.5$, as a function of α

equilibrium, increasing α increases the safety of the common bonds and the safety of individual bonds. In the region where $\alpha > \alpha_0$, if equilibrium is in the minimum safety equilibrium, we see that welfare is decreasing in α. This reflects an effect that if equilibrium is threshold where only one country survives, the increase in α transfers some reserve asset status from country 1 to country 2 and hence decreases the safety of country 1 (the initial reserve asset), thereby reducing welfare.

Figure 6 plots the equilibrium determination of δ^* and the joint safety region as well as welfare for a case where $z = 0.5$. In this small z case, the small country is the reserve asset when $\alpha = 0$. Increasing α undercuts the small country's reserve asset status, until above α_0, there is the possibility of the joint safety region. We see that welfare is increasing in α even when $\alpha < \alpha_0$. This is because the large country gains some reserve asset status as α rises (i.e., δ^* falls), thus reducing the rollover risk of the large country's debt, and since default of the large country is assumed to involve larger deadweight costs, welfare is improved.

Common bonds unambiguously increase welfare only when they are large enough in size. A small step towards a fiscal union can be worse than no step. Earlier in this section we showed that only in the case where $\alpha > \alpha_1$ do increases in α increase both countries' safety. We thus conclude that for Euro bonds to work they need to be large enough.

5 Conclusion

TO BE WRITTEN

A Appendix

A.1 Additive Fundamental Structure

We have considered the specification of $1-\theta_i = (1-\theta)\exp\left((-1)^i \tilde{\delta}\right)$ for country i's fundamental. We now show that results are qualitatively similar with the alternative additive specification

$$\theta_1 = \theta + \tilde{\delta}, \text{ and } \theta_2 = \theta - \tilde{\delta}.$$

As $x = \Pr\left(\tilde{\delta} + \epsilon_j > \delta^*\right) = \frac{\tilde{\delta}+\sigma-\delta^*}{2\sigma} \Rightarrow \tilde{\delta} = \delta^* + (2x-1)\sigma$, we know that

$$\begin{aligned} \theta_1 &= \theta + \tilde{\delta} = \theta + \delta^* + (2x-1)\sigma \\ \theta_2 &= \theta - \tilde{\delta} = \theta - \delta^* - (2x-1)\sigma \end{aligned}$$

Given x, the large country 1 survives if and only if

$$p_1 - 1 + \theta_1 = (1+f)x - 1 + \theta + \delta^* + (2x-1)\sigma \geq 0 \Leftrightarrow x \geq \frac{1-\theta-\delta^*+\sigma}{1+f+2\sigma}$$

which implies the expected return from investing in country 1 is

$$\Pi_1 = \int_{\frac{1-\theta-\delta^*+\sigma}{1+f+2\sigma}}^{1} \frac{1}{(1+f)x}dx = \frac{1}{1+f}\ln\frac{1+f+2\sigma}{1-\theta-\delta^*+\sigma}.$$

For country 2, the bond is paid back if

$$\begin{aligned} (1+f)x' - s + s\theta_2 &= (1+f)x' - s + s\left[\theta - \delta^* - (2x-1)\sigma\right] \geq 0 \\ \Leftrightarrow x' &\geq \frac{s(1-\theta+\delta^*-\sigma)}{1+f+2s\sigma} \end{aligned}$$

which implies an expected return of

$$\Pi_2 = \int_{\frac{s(1-\theta+\delta^*-\sigma)}{1+f+2s\sigma}}^{1} \frac{s}{(1+f)x'}dx' = \frac{s}{1+f}\ln\frac{1+f+2s\sigma}{s(1-\theta+\delta^*+\sigma)}$$

As a result, the equilibrium threshold δ^* is pinned by by the indifference condition

$$\ln\frac{1+f+2\sigma}{1-\theta-\delta^*+\sigma} = s\ln\frac{1+f+2s\sigma}{s(1-\theta+\delta^*+\sigma)}.$$

Letting $\sigma \to 0$ we obtain

$$\ln\frac{1+f}{1-\theta-\delta^*} = s\ln\frac{1+f}{s(1-\theta+\delta^*)}. \tag{A.1}$$

We no longer have close-form solution for δ^* in (A.1), as δ^* shows up in both sides. However, the solution is unique because LHS (RHS) is increasing (decreasing) in δ^*. Finally, to ensure $\delta^* < 0$ so that the larger country 1 is relatively safer, we require the same sufficient condition of $z = \ln\frac{1+f}{1-\theta} > 1$ in this alternative specification.

A.2 Equilibrium with Non-monotone Strategies

We now construct an equilibrium in which 1) agents take non-monotone strategies, and 2) both countries could be safe when the ex post realizations of $\widetilde{\delta}$ fall into an interval $[\delta_L, \delta_H]$ where δ_L and δ_H are endogenously determined. Given this equilibrium, we will compute the minimum value of $z = \underline{z}$ for which this equilibrium exists.

The possibility that both countries may be safe means that our equilibrium construction using threshold strategies is no longer possible. In a region where both countries are known to be safe (recall we consider the limit where $\sigma \to 0$), investors must be indifferent between the two countries, choosing strategies so that in equilibrium the return on both countries' bonds are equal. This "joint safety" region prevails for realizations of $\tilde{\delta} \in [\delta_L, \delta_H]$. Outside this interval, i.e., $\tilde{\delta} \in [-\bar{\delta}, \delta_L) \cup (\delta_H, \bar{\delta}]$, we are back in the case where the signal is so strong that only one country is safe.

We will deriving the boundaries δ_L and δ_H as functions of z, i.e., $\delta_L(z)$ and $\delta_H(z)$. Given these functions we can solve for the minimum $\underline{z}$, which occurs when $\delta_L(\underline{z})$=$\delta_H(\underline{z})$ and the joint safety region vanishes.

We conjecture the following non-monotone strategy whereby investment in country 1 and in country 2 alternates on discrete intervals of length $k\sigma$ and $(2-k)\sigma$, with $k \in (0,2)$. The investor i's strategy given his private signal δ_i

is $\phi(\delta_i) \in \{0,1\}$:

$$\phi(\delta_i) = \begin{cases} 0, & \delta_i < \delta_L \\ 1, & \delta_i \in [\delta_L, \delta_L + k\sigma] \cup [\delta_L + 2\sigma, \delta_L + (2+k)\sigma] \cup [\delta_L + 4\sigma, \delta_L + (4+k)\sigma] \cup ... \\ 0, & \delta_i \in [\delta_L + k\sigma, \delta_L + 2\sigma] \cup [\delta_L + (2+k)\sigma, \delta_L + 4\sigma] \cup [\delta_L + (4+k)\sigma, \delta_L + 6\sigma] \cup ... \\ 1, & \delta_i > \delta_H \end{cases} \tag{A.2}$$

As we will show shortly, the non-monotone oscillation occurs only when both countries are safe, where the equilibrium requires proportional investment in each safe country to equalize returns across two safe bonds. Clearly, k determines the fraction of agents in investing in country 1 when oscillation occurs, to which we turn next.

A.2.1 Fraction of agents in investing in country 1

Consider the region where all investors know that both countries are safe. In this case, the total investment in country 1 and 2 has to be $\frac{1+f}{1+s}$ and $\frac{s(1+f)}{1+s}$, respectively, otherwise one country will offer strictly higher returns than the other. Take an agent with signal δ; introduce the function $\rho(\delta)$, which is the expected proportion of agents investing in country 1 given (own) signal δ. Then, given the assumed strategy for all agents and given that we are in the region where both countries are safe,

$$\rho(\delta) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \frac{k\sigma}{2\sigma}.$$

We choose k so that,

$$\rho(\delta) = \frac{1}{1+s} \iff k = \frac{2}{1+s} \tag{A.3}$$

The last equality arises because in equilibrium the proportion must be constant and equal to $\frac{1}{1+s}$ in order that returns are equalized across the two safe bonds.

Recall that x denotes the fraction of agents with signal realizations above the agent's private signal δ; and x follows a uniform distribution on $[0,1]$. For any value of δ and x,

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \begin{cases} 0, & \delta + 2\sigma x < \delta_L \\ \frac{\delta+2\sigma x-\delta_L}{2\sigma}, & \delta + 2\sigma x \in (\delta_L, \delta_L + k\sigma) \\ \frac{1}{1+s}, & \delta_H - (2-k)\sigma > \delta > \delta_L + k\sigma \end{cases} \tag{A.4}$$

We see that $\rho(\delta, x)$ is strictly increasing in x in the vicinity of δ_L, and flat everywhere else. When we evaluate δ at the marginal agent with signal $\delta = \delta_L$, we have

$$\rho(\delta_L, x) = \begin{cases} 0, & x = 0 \\ x, & x \in \left(0, \frac{1}{1+s}\right) \\ \frac{1}{1+s}, & x > \frac{1}{1+s} \end{cases} \tag{A.5}$$

where we observe that $\rho(\delta_L, x)$ is less than or equal to $\frac{1}{1+s}$.

A.2.2 Lower boundary δ_L

Let $V_\phi(\delta)$ be the expected payoff of strategy $\phi \in \{0,1\}$ when given signal δ. In the completely safe region discussed above (for δ exceeding δ_L sufficiently), investors were indifferent between both strategies because both paid exactly the same in all states of the world. This is not the case for agent with the threshold signal δ_L: as the agent knows investors with signal below is always investing in country 2, country 1 is perceived default risk. We now calculate the return of investing in either country, from the perspective of the boundary agent δ_L.

For the boundary agent δ_L, the return from investing only in country 2 (i.e. $\phi = 0$) is given by

$$\Pi_2(\delta_L) = \int_0^1 \frac{s}{(1+f)(1-\rho(\delta_L, x))} dx \tag{A.6}$$

where we integrate over all x as country 2 is safe regardless of x. Thus, plugging in, we have

$$\Pi_2(\delta_L) = \frac{s}{1+f}\left[\int_0^{\frac{1}{1+s}} \frac{1}{1-x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{\frac{s}{1+s}} dx\right] = \frac{s}{1+f}\left[\ln\frac{1+s}{s} + 1\right] < \frac{1+s}{1+f}. \tag{A.7}$$

where we used $s \ln \frac{1+s}{s} < 1$. Here, we see that payoff to investing in country 2 is lower than the expected payoff that would have realized if both countries were safe. This reflects the strategic substitution effect: because more people (in expectation) invest in the safe country 2, the return in country 2 is lower.

Now we turn to investment return for country 1. Since country 1 has default risk, we need to calculate the cutoff $x = x_{min}$ so that country 1 becomes safe if there are $x > x_{min}$ measure of agents receiving better signals. To derive x_{min}, we first solve for $\rho_1^{min}(\delta)$, which is the minimum proportion of agents investing in country 1 that are needed to make country 1 safe given fundamental δ. We have

$$\theta_1(\delta) + (1+f)\rho_1^{min}(\delta) = 1 \iff \rho_1^{min}(\delta) = \frac{1-\theta_1(\delta)}{1+f}$$

Define x_{min} as the solution to $\rho(\delta_L, x) = \rho_1^{min}(\delta_L)$. Given equation (A.5), we have that,

$$x_{min} = \frac{1-\theta_1(\delta_L)}{1+f}. \tag{A.8}$$

The expected return of investing in country 1 given one's own signal δ_L and the conjectured strategies $\phi(\cdot)$ of everyone else is given by,

$$\begin{aligned}
\Pi_1(\delta_L) &= \int_{x_{min}}^{1} \frac{1}{(1+f)\rho(\delta_L, x)} dx = \frac{1}{1+f}\left[\int_{x_{\min}}^{\frac{1}{1+s}} \frac{1}{x} dx + \int_{\frac{1}{1+s}}^{1} \frac{1}{1/(1+s)} dx\right] \\
&= \frac{1}{1+f}\left[\ln \frac{1}{1+s} - \ln x_{min} + s\right]. && \text{(A.9)}
\end{aligned}$$

The the boundary agentδ_L must be indifferent between investing in either country, i.e., $\Pi_2(\delta_L) = \Pi_1(\delta_L)$. Plugging in (A.6) and (A.9), we have

$$\frac{s}{1+f}\left[\ln \frac{1+s}{s} + 1\right] = \frac{1}{1+f}\left[\ln \frac{1}{1+s} - \ln x_{min} + s\right] \iff x_{min} = \frac{s^s}{(1+s)^{1+s}}. \tag{A.10}$$

We combine our two equations for x_{min}, (A.8) and (A.10), and use $1 - \theta_1(\delta_L) = (1-\theta)\exp(-\delta_L)$, to obtain:

$$\frac{s^s}{(1+s)^{1+s}} = \frac{(1-\theta)\exp(-\delta_L)}{1+f}.$$

Recall $z = \ln \frac{1+f}{1-\theta}$; we have

$$\delta_L(z) = -z + (1+s)\ln(1+s) - s\ln s \tag{A.11}$$

A.2.3 Upper boundary δ_H

The derivation is symmetric to the above. We have

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \begin{cases} \frac{1}{1+s}, & \delta - 2\sigma(1-x) < \delta_H - (2-k)\sigma \\ \frac{\delta+2\sigma x-\delta_H}{2\sigma} & \delta - 2\sigma(1-x) \in (\delta_H - (2-k)\sigma, \delta_H) \\ 1, & \delta - 2\sigma(1-x) > \delta_H \end{cases} \tag{A.12}$$

so that

$$\rho(\delta_H, x) = \begin{cases} \frac{1}{1+s}, & x < \frac{1}{1+s} \\ x, & x \in \left(\frac{1}{1+s}, 1\right) \\ 1, & x = 1 \end{cases} \tag{A.13}$$

Consider now the agent with signal δ_H who is considering investing in country 1. In his eyes, the return from investing only in country 1 (i.e. $\phi = 1$) is given by

$$\Pi_1(\delta_H) = \int_0^1 \frac{1}{\left(1+\hat{f}\right)\rho(\delta_H, x)dy} dx = \frac{1}{1+f}\left[\ln(1+s) + 1\right] < \frac{1+s}{1+f},$$

where we integrated over all x as country 1 is always safe in the vicinity of δ_H.

The default condition for country 2 is

$$s\theta_2(\delta_H) + (1+f)\left[1 - \rho_2^{max}(\delta_H)\right] = s \iff 1 - \rho_2^{max}(\delta_H) = s\frac{1-\theta_2(\delta_H)}{1+f}$$

where $\rho_2^{max}(\delta)$ is the maximum amount of agents investing in country 1 so that country 2 does not default. Assume, but later verify, that at δ_H we have $1-\rho_2^{max}(\delta_H) < \frac{s}{1+s}$, that is, country 2 would survive even if less than $\frac{s}{1+s}$ of investors invest in country 2. Define $x_{max}(\delta_H)$ as the solution to $\rho(\delta_H, x_{max}) = \rho_2^{max}(\delta_H)$; (A.13) implies that

$$1 - x_{max}(\delta_H) = s\frac{1-\theta_2(\delta_H)}{1+f}. \tag{A.14}$$

As a result, the return to country 2 is,

$$\begin{aligned}\Pi_2(\delta_H) &= \int_0^{x_{max}(\delta_H)} \frac{s}{(1+f)(1-\rho(\delta_H,x))dy}dx = \frac{s}{1+\hat{f}}\left[\int_0^{\frac{1}{1+s}} \frac{1}{1-\frac{1}{1+s}}dx + \int_{\frac{1}{1+s}}^{x_{max}(\delta_H)} \frac{1}{1-x}dx\right] \\ &= \frac{s}{1+f}\left[\frac{1}{s} + \ln\frac{s}{1+s} - \ln(1-x_{max}(\delta_H))\right]\end{aligned}$$

Indifference at the boundary agent δ_H requires $\Pi_1(\delta_H) = \Pi_2(\delta_H)$, which yields $1-x_{max}(\delta_H) = \frac{s}{(1+s)^{\frac{1+s}{s}}}$. Combining this result with (A.14) and $1-\theta_2(\delta_H) = (1-\theta)\exp(\delta_H)$, we solve,

$$\delta_H(z) = z - \frac{1+s}{s}\ln(1+s) \tag{A.15}$$

A.2.4 Verifying the equilibrium

We now verify the interior agents $\delta \in (\delta_L, \delta_H)$ have the appropriate incentives to play the conjectured strategy. We just showed that the investor with signal $\delta = \delta_L$ is indifferent; and similar to the argument in the threshold equilibrium, it is easy to show that agents with $\delta < \delta_L$ find it optimal to invest in country 2. Consider an investor with signal $\delta = \delta_L + k\sigma$. Regardless of his relative position (as measured by x) in the signal distribution, this agent knows that a proportion $\frac{1}{1+s}$ of investors invest in country 1, thus making it safe for sure. Further, he knows that a proportion $\frac{s}{1+s}$ of investors invest in country 2, also making it safe. Therefore, this agent knows that (i) both countries are completely safe and that (ii) investment flows give arbitrage free prices. He is thus indifferent, and so is every investor with $\delta_L + k\sigma < \delta < \delta_H - (2-k)\sigma$.

We now consider the investors with $\delta \in (\delta_L, \delta_L + k\sigma)$? We know that country 2 will always survive, and thus we have

$$\Pi_2(\delta) = \int_0^1 \frac{s}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{1-\phi(y)}{2\sigma}dy}dx.$$

Note that for any x with $x \geq -\frac{\delta-\delta_L-k\sigma}{2\sigma}$ we are in the oscillating region; for x below we are in the increasing part. Let $\varepsilon \equiv \frac{\delta-\delta_L}{2\sigma} \in \left(0, \frac{1}{1+s}\right)$ so that so that $\delta = \delta_L + 2\sigma\varepsilon$. Thus, we have

$$1-\rho(\delta,x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{1-\phi(y)}{2\sigma}dy = \begin{cases} 1-\varepsilon-x, & x\in\left(0,\frac{1}{1+s}-\varepsilon\right), \\ \frac{s}{1+s}, & x\in\left(\frac{1}{1+s}-\varepsilon, 1\right). \end{cases} \tag{A.16}$$

Then, we have

$$\Pi_2(\delta) = \frac{s}{1+f}\left[\int_0^{\frac{1}{1+s}-\varepsilon}\frac{1}{1-\varepsilon-x}dx + \int_{\frac{1}{1+s}-\varepsilon}^{1}\frac{1}{\frac{s}{1+s}}dx\right] = \Pi_2(\delta_L) + \frac{s\left(\ln(1-\varepsilon) + \frac{1+s}{s}\varepsilon\right)}{1+f}$$

For investment in country 1, we know that, since $\delta > \delta_L$, we have $\rho_1^{min}(\delta) < \rho_1^{min}(\delta_L)$. First, note that

$$\rho(\delta,x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy = \begin{cases} \varepsilon + x, & x\in\left(0,\frac{1}{1+s}-\varepsilon\right) \\ \frac{1}{1+s}, & x\in\left(\frac{1}{1+s}-\varepsilon, 1\right) \end{cases}$$

Let $x_{min}(\delta)$ be the measure of investors with higher signals than δ so that country 1 is safe. Since $\rho_1^{min}(\delta) = \frac{1-\theta_1(\delta)}{1+f}$, $x_{min}(\delta)$ is the lowest $x \in [0,1]$ such that

$$\rho(\delta,x) = \varepsilon + x \geq \rho_1^{min}(\delta).$$

Thus, we have

$$x_{min}(\delta) = x_{min}(\delta_L + 2\sigma\varepsilon) \quad = \quad \max\left\{\frac{1-\theta_1(\delta_L+2\sigma\varepsilon)}{1+f} - \varepsilon, 0\right\}. \tag{A.17}$$

The expected investment return from country 1 is

$$\begin{aligned}\Pi_1(\delta) &= \int_{x:\rho(\delta,x)\geq\rho_1^{min}(\delta)} \frac{1}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy}dx \\ &= \Pi_1(\delta_L) + \frac{1}{1+f}\left[\ln x_{min}(\delta_L) - \ln\left[\varepsilon + x_{min}(\delta_L+2\sigma\varepsilon)\right] + (1+s)\varepsilon\right]\end{aligned}$$

Thus, to show that $\Pi_1(\delta_L+2\sigma\varepsilon) \geq \Pi_2(\delta_L+2\sigma\varepsilon)$, we need to show that the following inequality holds for $\varepsilon \in \left(0, \frac{1}{1+s}\right)$:

$$g(\varepsilon) \equiv \ln x_{min}(\delta_L) - \ln\left[\varepsilon + x_{min}(\delta_L+2\sigma\varepsilon)\right] - s\ln(1-\varepsilon) \geq 0. \tag{A.18}$$

First, by using $\ln x_{min}(\delta_L) = s\ln s - (1+s)\ln(1+s)$ and $x_{min}\left(\delta_L + 2\sigma\frac{1}{1+s}\right) = 0$, we know the above inequality holds with equality at both end points $\varepsilon = 0$ and $\varepsilon = \frac{1}{1+s}$, i.e., $g(0) = g\left(\frac{1}{1+s}\right) = 0$. Second, it is easy to show that there exists a unique ε^* such that $\frac{1-\theta_1\left(\delta_L+2\sigma\varepsilon^*\right)}{1+f} = \varepsilon^*$, at which point (A.17) binds at zero. We further note that at $\varepsilon = 0$ we have $\frac{1-\theta_1(\delta_L)}{1+f} > 0$. Thus, in (A.17) we have $\varepsilon^* > 0$ and for $\varepsilon \in (0, \varepsilon^*)$ we have $x_{min}(\delta) = \frac{1-\theta_1(\delta_L+2\sigma\varepsilon)}{1+f} - \varepsilon > 0$, and for $\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right]$ we have $x_{min}(\delta) = 0$. Plugging in and taking derivative with respect to ε, we have

$$\frac{\partial}{\partial\varepsilon}\ln\left[\varepsilon + x_{min}(\delta_L+2\sigma\varepsilon)\right] = \begin{cases}\frac{-2\sigma\theta_1'(\delta_L+2\sigma\varepsilon)}{1-\theta_1(\delta_L+2\sigma\varepsilon)} & ,\varepsilon \in (0,\varepsilon^*) \\ \frac{1}{\varepsilon} & ,\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right]\end{cases}$$

Then, for (A.18), we have $g(\varepsilon)$ first rises and then drops:

$$g'(\varepsilon) = \begin{cases}\frac{2\sigma\theta_1'(\delta_L+2\sigma\varepsilon)}{1-\theta_1(\delta_L+2\sigma\varepsilon)} + \frac{s}{1-\varepsilon} > 0 & ,\varepsilon \in (0,\varepsilon^*), \\ -\frac{1}{\varepsilon} + \frac{s}{1-\varepsilon} = \frac{(1+s)\varepsilon-1}{1-\varepsilon} < 0 & ,\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right].\end{cases}$$

Combined with $g(0) = g\left(\frac{1}{1+s}\right) = 0$ we know that $g(\varepsilon) > 0, \forall\varepsilon \in \left(0, \frac{1}{1+s}\right)$, i.e., Thus, on $\varepsilon \in \left(0, \frac{1}{1+s}\right)$ the investors strictly want to invest in country 1.

Finally, we need to pick σ appropriately so that there exists some natural number $N > 1$ so that $2N\sigma = \delta_H - \delta_L$. For this particular choice of $\sigma = \hat{\sigma}$, the limiting case of zero signal noise can be achieved when we take the sequence of $\sigma_n = \hat{\sigma}/n$ for $n = 1, 2, ...$

A.2.5 Equilibrium properties

First, with joint safety, the probability of survival for country 1 (or the probability of its bonds being the reserve asset) is not longer one minus the probability of survival of country 2. Using $\tilde{\delta} \sim \mathbb{U}\left(-\overline{\delta}, \overline{\delta}\right)$, the probability of country 1 survival is

$$\Pr(\text{country 1 safe}) = \frac{\overline{\delta} - \delta_L}{2\overline{\delta}} = \frac{\overline{\delta} + z - (1+s)\ln(1+s) + s\ln s}{2\overline{\delta}}, \tag{A.19}$$

and the probability of country 2 survival is

$$\Pr(\text{country 2 safe}) = \frac{\delta_H + \overline{\delta}}{2\overline{\delta}} = \frac{\overline{\delta} + z - \frac{1+s}{s}\ln(1+s)}{2\overline{\delta}}.$$

As a result, the bonds issued by country 1 are more likely to be the reserve assets than that issued by country 2 if the following condition holds:

$$s\ln s - (1+s)\ln(1+s) + \frac{1+s}{s}\ln(1+s) = s\ln s + \left(\frac{1}{s} - s\right)\ln(1+s) > 0. \tag{A.20}$$

In the proof of Proposition 7 we show this condition always holds.[10]

Obviously, the above equilibrium construction requires that $\delta_L(z) < \delta_H(z)$. Since $\delta_L(z)$ in (A.11) is decreasing in z while $\delta_H(z)$ in (A.15) is increasing in z, this condition $\delta_L(z) < \delta_H(z)$ holds if $z > \underline{z}$ so that $\delta_L(\underline{z}) = \delta_H(\underline{z})$ which gives $\underline{z}$:

$$-\underline{z} + (1+s)\ln(1+s) - s\ln s = \underline{z} - \frac{1+s}{s}\ln(1+s) \Rightarrow \underline{z} = \frac{1}{2}\left[\left(2+s+\frac{1}{s}\right)\ln(1+s) - s\ln s\right]$$

Proposition 7 *We have the following results for the equilibrium with non-monotone strategies.*

1. *The lower equilibrium threshold $\delta_L(s,z)$ is decreasing in z, $\frac{\partial}{\partial z}\delta_L(z) < 0$. Hence, the probability that the larger country 1 is the reserve asset (given in (A.19)) is higher if the aggregate fundamental θ or aggregate saving f is higher. However, the probability of country 2's bonds being the reserve asset is also increasing with z.*

2. *When the aggregate funding conditions z is sufficiently high so that $z > \underline{z}$, the larger country 1 is more likely to be the reserve asset than the smaller country.*

3. *All else equal, country 1 has the highest likelihood of survival when country 2 size goes to zero $s \to 0$.*

Proof. The first result is obvious. For the second result we need to show that $F(s) \equiv s^2\ln s + \left(1-s^2\right)\ln(1+s) > 0$ holds for $s \in (0,1)$. It is clear that $F(0) = 0$ while $F(1) = 0$. Simple algebra shows that

$$F'(s) = 2s\ln s - 2s\ln(1+s) + 1, \frac{1}{2}F''(s) = \ln s - \ln(1+s) + 1 - \frac{s}{1+s} = \ln\left(\frac{s}{1+s}\right) + 1 - \frac{s}{1+s}.$$

Let $y = \frac{s}{1+s} <\in (0,1)$; then because it is easy to show $\ln y + 1 - y < 0$ (due to concavity of $\ln y$), we know that $F''(s) < 0$. As a result, $F(s)$ is concave but $F(0) = F(1) = 0$. This immediately implies that $F(s) > 0$, which is our desired result. The third claim follows because $-(1+s)\ln(1+s) + s\ln s$ is decreasing in s.

■

[10] We have considered survival, which can be either sole survival or joint survival. If we focus on sole survivals only, i.e., the bonds of country j are the only reserve asset, the condition is exactly the same. This is because

$$\begin{aligned} \Pr(\text{only country 1 survive}) &= \frac{-\delta_H + \overline{\delta}}{2\overline{\delta}} = \frac{\overline{\delta} - z + \frac{1+s}{s}\ln(1+s)}{2\overline{\delta}}, \\ \Pr(\text{only country 2 survive}) &= \frac{\delta_L + \overline{\delta}}{2\overline{\delta}} = \frac{\overline{\delta} - z + (1+s)\ln(1+s) - s\ln s}{2\overline{\delta}}. \end{aligned}$$

As a result, $\Pr(\text{only country 1 survive}) > \Pr(\text{only country 2 survive})$ if and only if

$$\frac{1+s}{s}\ln(1+s) - (1+s)\ln(1+s) + s\ln s > 0,$$

which is exactly the same as (A.20).