Salil Vadhan (he/they)

Announcements

Start recording

My OH tomorrow 9am-1pm, sign up at salil.seas.harvard.edu

PS0 will be posted by Monday

Tentative new room: SEC 1.413

Agenda

1) Content overview
2) Syllabus
3) Q & A

Spectral Graph Theory in TCS and Beyond

theoretical

eigenvalues + eigenvectors of graphs and related linear algebra

understand graph structure

- how well-connected

TCS:
- solving linear systems
- derandomization
- sampling + counting
- web search
- max flow and other graph algs

⋮

Beyond:
- electric networks
- multi-robot systems
- computer vision
- epidemiology
- network economics

⋮

- clustering + coloring
- mixing of random walks

⋮

Matrices Associated w/ Graphs G (assume undirected)

adjacency matrix $M_G(i,j)$ = # edges from j to i

random-walk matrix $W_G(i,j) = \dfrac{M_G(i,j)}{\deg(j)}$

$$= M_G D_G^{-1}$$

where $D_G = \begin{pmatrix} \deg(1) & & & 0 \\ & \deg(2) & & \\ & 0 & \ddots & \\ & & & \deg(n) \end{pmatrix}$

Laplacian $L_G = D_G - M_G$

Normalized Laplacian

$$N_G = I - D_G^{-1/2} M_G D_G^{-1/2}$$

Random-Walk Laplacian

$$I - W_G = D_G^{1/2} N_G D_G^{-1/2}$$

Eigenvalues + Eigenvectors

Assume G undirected, d-regular (so $D_G = d \cdot I$)

Then $N_G = I - M_G/d = I - W_G$

is symmetric

$\Rightarrow$ orthonormal basis of eigenvectors

$v_1, \ldots, v_n$

w/ eigenvalues $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$

One eigenvector: $\vec{u} = (1/n, 1/n, \ldots, 1/n)$

uniform distribution

- $W_G \vec{u} = \vec{u}$ (by regularity)

  $\uparrow$

  $\vec{u}$ stationary dist.

- $N_G \vec{u} = (I - W_G)\vec{u} = 0$

Facts

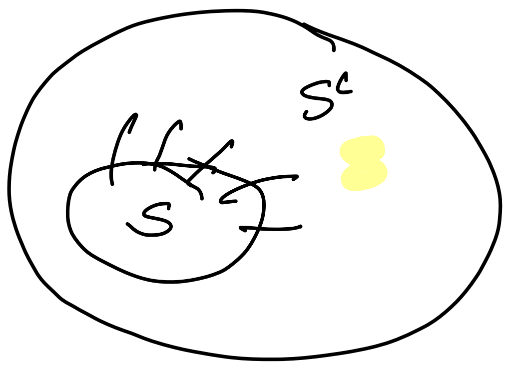
- $\nu_1 = 0$ and $v_1 = \vec{u}$.
- $1 \geq \nu_2 \geq 0$ gives a lot of info. about G

* Random Walks on G

converge to $\vec{u}$ in $O(\log n)/\nu_2$

* For every set S of at most $n/2$ vertices,

$$\#\text{ edges leaving } S = \Omega\left(\nu_2 \cdot \underbrace{|S| \cdot d}\right)$$



Cheeger's Inequality: A converse!

$\exists\, S$ of at most $n/2$ vertices such that

$$\#\text{ edges leaving } S \le O\left(\sqrt{\nu_2} \cdot |S| \cdot d\right)$$

Higher-Order Cheeger Inequalities (2011+)

ν_k "small" $\Longleftrightarrow$ can partition vertices into k sets cutting "few" edges

Other Topics

Markov Chains

1/2 lase $\Rightarrow$ random walks mix fast

$\Rightarrow$ fast algs (+ space-efficient algs)

for sampling from the stationary dist.

(even on an exponentially large G)

$\Rightarrow$ fast approx. counting algs

PageRank (Google ranking of webpages)

= stationary distribution of

$\approx$ web-link graph

Expander Graphs

Graphs that are very well connected (w.h. large)

but have few edges (e.g. $d = O(1)$)

Many applications in TCS incl.

derandomization

$1 \log_2 n + O(1)$ random bits

Intuition: random edge (i, j) in an expander

$\approx$ "indistinguishable"

two independent uniform random vertices

$2 \log_2 n$ random bits

High-dimensional expanders

generalization to hypergraphs — triples of vertices

e.g. vertices, edges, 2-d faces,

3-d faces, . . .

Spectral Sparsification (2004 + ...)

Expander graph = sparse graph that "approximates" the complete graph

Now: approximate an arbitrary graph by a sparse graph?

↳ solve problems on sparse approx.

Solving Laplacian Systems

Given G and $\vec{b} \in \mathbb{R}^n$, solve

$$L_G \vec{x} = \vec{b}$$

We'll see:

- randomized algs in time $\tilde{O}(m)$ ← # edges
- deterministic algs

in space $\tilde{O}(\log n)$ bits

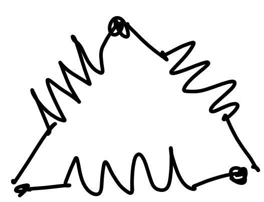
cf. best algs for general linear systems

$\approx O(n^{2.37\ldots})$ runtime

$O(\log^2 n)$ space

Electrical Flows

Given a resistor network G



and a vector $\vec{i}$ of external current

Then the vector of voltages $\vec{v}$

$$\vec{v} = L_G \vec{i}$$

Kirchoff's Matrix-Tree Theorem

Coeff of x

$$\det(xI - L_G) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

$$= \pm \frac{1}{n} \cdot \#(\text{spanning trees in } G)$$

$\Rightarrow$ poly-time alg to count + sample spanning trees!

Q: what spanning forests?

2019: solved using MCMC + High-dimensional expanders