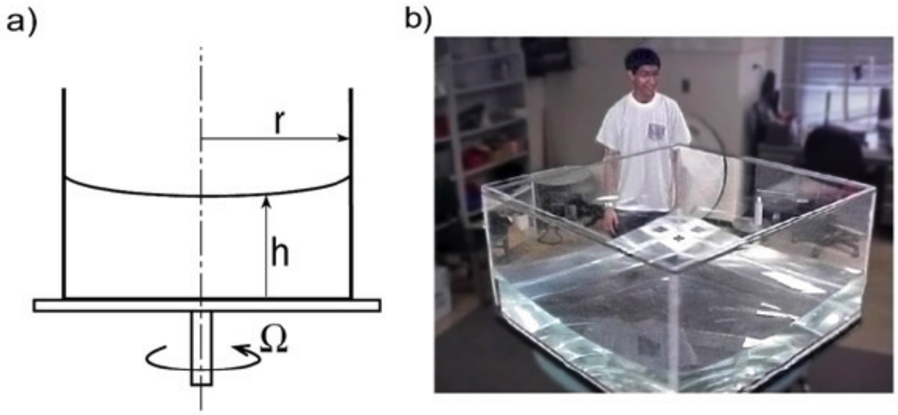
Effects of rotation

In a rotating reference frame, the momentum equation is:

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{z} - F = -2\vec{\Omega}\times\vec{u} + (-\vec{\Omega}\times\vec{\Omega}\times\vec{r})$$

The terms on the right hand side are the apparent accelerations: the Coriolis acceleration and the centrifugal acceleration.

Now we look at a couple of examples:
The surface of water in a tank is flat. When the tank is rotating, however, in equilibrium the water surface will have the shape of a parabola, as seen here:



This can be explained simply.

Because of friction, in equilibrium, water should rotate with the same angular speed as the tank. This is known as solid-body rotation. In an inertial reference frame, the acceleration is $\frac{D\vec{V}}{Dt} = \vec{\Omega}\times\frac{D\vec{r}}{Dt} = -\Omega^2\vec{r}$. This is provided by the radial pressure gradient.
In hydrostatic equilibrium, this is related to the height of the free surface so we have

$$\Omega^2 r = g\frac{\partial H}{\partial r}$$

In a reference frame that rotates with the tank, water is at rest. The centrifugal force is balancing the pressure gradient force. Integration w.r.t. r gives the parabola:

$$H = \frac{\Omega^2 r^2}{2g} \qquad (1.1)$$

If we modify the gravitational potential to be

$$\phi = gz - \frac{\Omega^2 r^2}{2}$$

Then the free surface is "flat" in the sense it has a constant ϕ. If we have a parabolic surface defined by (1.1) that is rotating with Ω, an object at rest (in the rotating frame) will be in balance everywhere.

Put a ball on this surface and set it in motion. In this case, one may neglect friction and pressure force so that

$$\frac{D\vec{u}}{Dt} = -2\vec{\Omega} \times \vec{u}$$

So the ball will experience a Coriolis acceleration "to the right" of u if the surface is rotating anticlockwise. You can feel it by pushing a dumbbell outward on a rotating chair. (it may make you dizzy). Note the Coriolis force is perpendicular to velocity and does no work.

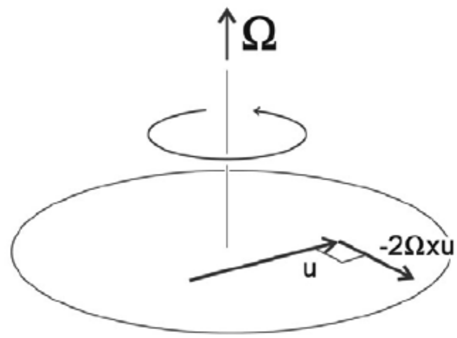


Figure 6.10: A fluid parcel moving with velocity u_{rot} in a rotating frame experiences a Coriolis acceleration $-2\mathbf{\Omega}\times u_{rot}$, directed 'to the right' of u_{rot} if, as here, $\mathbf{\Omega}$ is directed upwards, corresponding to anticlockwise rotation.

Now set the ball at the center and give it a kick. If we could neglect friction between the ball and the surface, then the ball doesn't even know the surface is rotating. In the inertial frame, it simply oscillates back and forth according to this equation:

$$\frac{d^2r}{dt^2} = -g\frac{dH}{dr} = -\Omega^2 r$$

and the frequency of the oscillation is Ω.

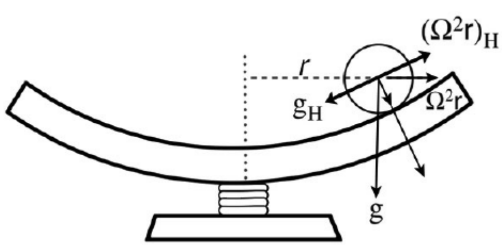


Figure 6.12: If a parabola of the form given by Eq.(6.33) is spun at rate Ω, then a ball carefully placed on it at rest does not fall in to the center but remains at rest: gravity resolved parallel to the surface, g_H, is exactly balanced by centrifugal accelerations resolved parallel to the surface, $(\Omega^2 r)_H$.

But, in the rotating reference frame, it follows the following equations and behaves quite differently.

$$\frac{du_{rot}}{dt} - 2\Omega v_{rot} = 0; \frac{dv_{rot}}{dt} + 2\Omega u_{rot} = 0$$

Since we start from the center, we pick the initial condition of 0 for u_{rot} (angular velocity) and v_0 for v_{rot} (radial velocity), the solution is:

$$u_{rot}(t) = v_o \sin 2\Omega t; \ v_{rot}(t) = v_o \cos 2\Omega t$$

From this, one can calculate the trajectory given the initial conditions for x and y.
The calculation result is

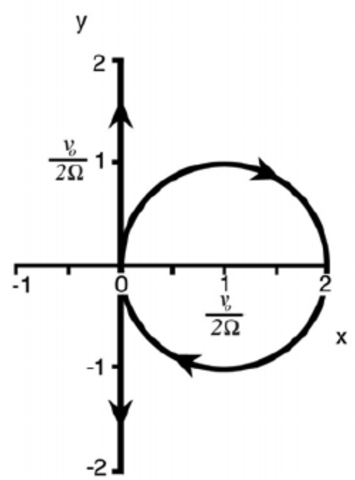


Figure 6.15: Theoretical trajectory of the puck during one complete rotation period of the table, $\frac{2\pi}{\Omega}$, in GFD Lab V in the inertial frame (straight line) and in the rotating frame (circle). We launch the puck from the origin of our coordinate system $x(0) = 0$; $y(0) = 0$ (chosen to be the center of the rotating parabola) with speed $u(0) = 0$; $v(0) = v_o$. The horizontal axes are in units of $\frac{v_o}{2\Omega}$. Observed trajectories are shown in Fig.6.14.

Observed trajectories in the lab:

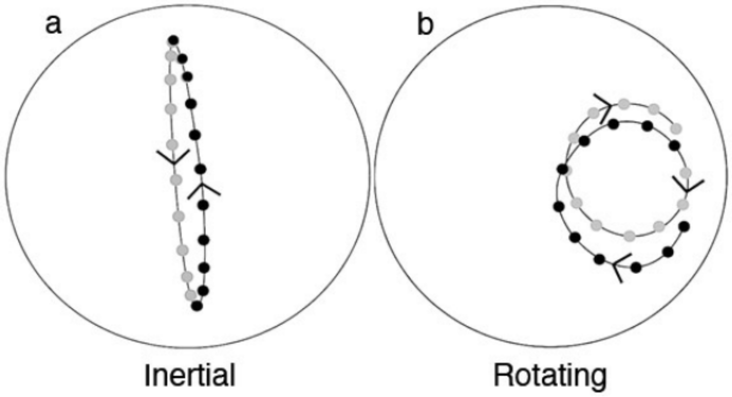


Figure 6.14: Trajectory of the puck on the rotating parabolic surface during one rotation period of the table $\frac{2\pi}{\Omega}$in (a) the inertial frame and (b) the rotating frame of reference. The parabola is rotating in an anticlockwise (cyclonic) sense.

Note that the different behavior is entirely due to the different reference frames.

The physics here is the same as that of the Foucault pendulum. Here is the wikipedia link on the Foucault pendulum http://en.wikipedia.org/wiki/Foucault_pendulum. These circles are called the inertial circles and have a period of π/Ω, half of the rotation period. These are well observed. Here is one example:

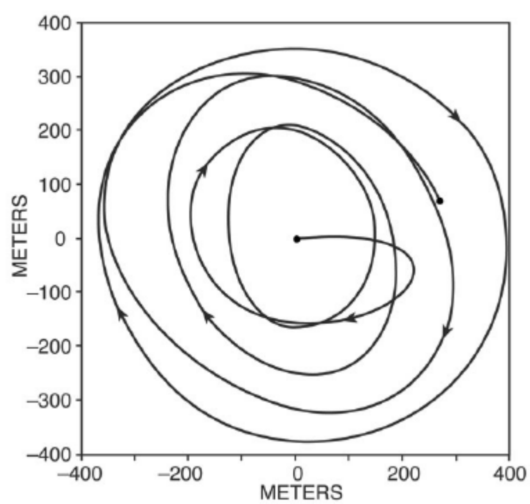


Figure 6.16: Inertial circles observed by a current meter in the main thermocline of the Atlantic Ocean at a depth of 500 m; 28°N, 54°W. Five inertial periods are shown. The inertial period at this latitude is 25.6 h and 5 inertial periods are shown. Courtesy of Carl Wunsch, MIT.

In fact, they are part of the explanation why cold wakes from hurricanes tend to be to the right of the storm (in the Northern Hemisphere).

Putting things on the sphere

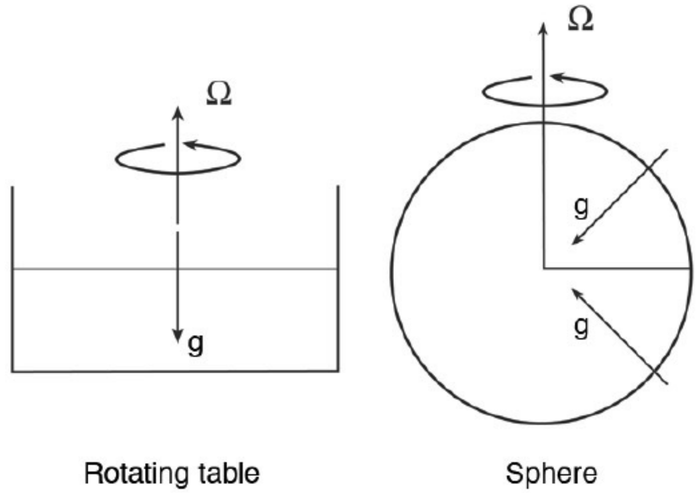


Figure 6.17: In the rotating table used in the laboratory Ω and g are always parallel or (as sketched here) anti-parallel to one another. This should be contrasted with the sphere.

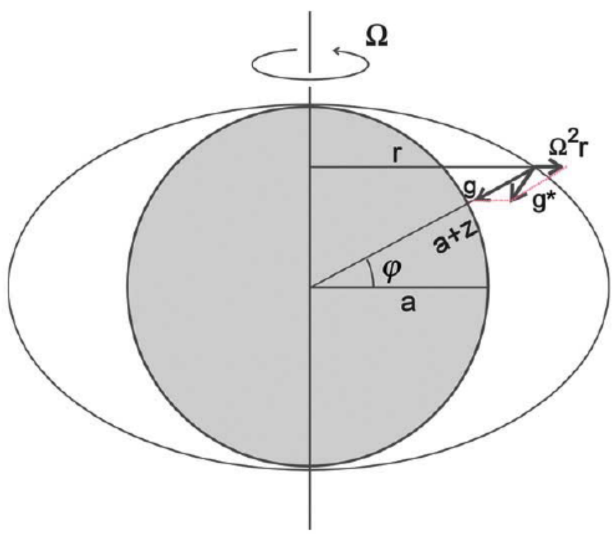


Figure 6.18: The centrifugal vector $\Omega \times \Omega \times r$ has magnitude $\Omega^2 r$, directed outward normal to the rotation axis. Gravity, g, points radially inwards to the center of the Earth. Over geological time the surface of the Earth adjusts to make itself an equipotential surface — close to a reference ellipsoid — which is always perpendicular the the vector sum of $\mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}$ and $\mathbf{g}$. This vector sum is 'measured' gravity: $\mathbf{g}^* = -g\widehat{\mathbf{z}} - \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}$.

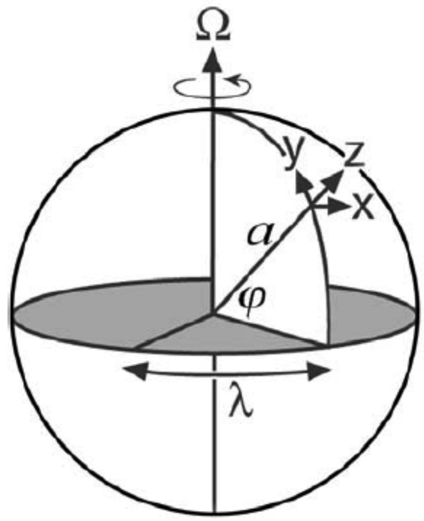


Figure 6.19: At latitude φ, longitude, λ, we define a local coordinate system such that the three coordinates in the (x, y, z) directions point (eastward, northward, upward): $dx = a\cos\varphi d\lambda$; $dy = ad\varphi$; $dz = dz$ where a is the radius of the earth. The velocity is $\mathbf{u} = (u, v, w)$ in the directions (x, y, z). See also Section 13.2.3.

At latitude φ, we define a local coordinate system such that the three coordinates in the (x, y, z) directions point (eastward, northward, upward), as shown. The components of $\mathbf{\Omega}$ in these coordinates are $(0, \Omega\cos\varphi, \Omega\sin\varphi)$. Therefore, expressed in these coordinates,

$$\begin{aligned}\mathbf{\Omega} \times \mathbf{u} &= (0, \Omega\cos\varphi, \Omega\sin\varphi) \times (u, v, w) \\ &= (\Omega\cos\varphi\, w - \Omega\sin\varphi\, v, \Omega\sin\varphi\, u, -\Omega\cos\varphi\, u)\ .\end{aligned}$$

The vertical component of the Coriolis acceleration is negligible compared to gravity. Also, the atmosphere and ocean are very thin for global scale motions. For these motions, horizontal velocities are much larger than the vertical velocity. And we are mostly concerned with the local vertical component of the rotation vector. Thus the Coriolis acceleration is

$$-f\hat{k} \times \vec{u}_h$$
$$f = 2\Omega\sin\varphi$$

where φ is the latitude. Typical values of f are shown below. The Coriolis force (for horizontal motions) is zero on the equator. That's why gravity wave adjustment without rotation approximately works near the equator, where we saw small temperature gradients.

latitude	f ($\times 10^{-4}\,\mathrm{s}^{-1}$)	β ($\times 10^{-11}\,\mathrm{s}^{-1}\,\mathrm{m}^{-1}$)
90°	1.46	0
60°	1.26	1.14
45°	1.03	1.61
30°	0.73	1.98
10°	0.25	2.25
0°	0	2.28

Table 6.1: Values of the Coriolis parameter, $f = 2\Omega \sin\varphi$ — Eq.(6.42) — and its meridional gradient, $\beta = \frac{df}{dy} = \frac{2\Omega}{a}\cos\varphi$ — Eq.(10.10) — tabulated as a function of latitude. Here Ω is the rotation rate of the Earth and a is the radius of the Earth.

While it is often convenient to express the equations of motion in local Cartesian coordinate, this brings some additional terms because the coordinates change with location (even without rotation).

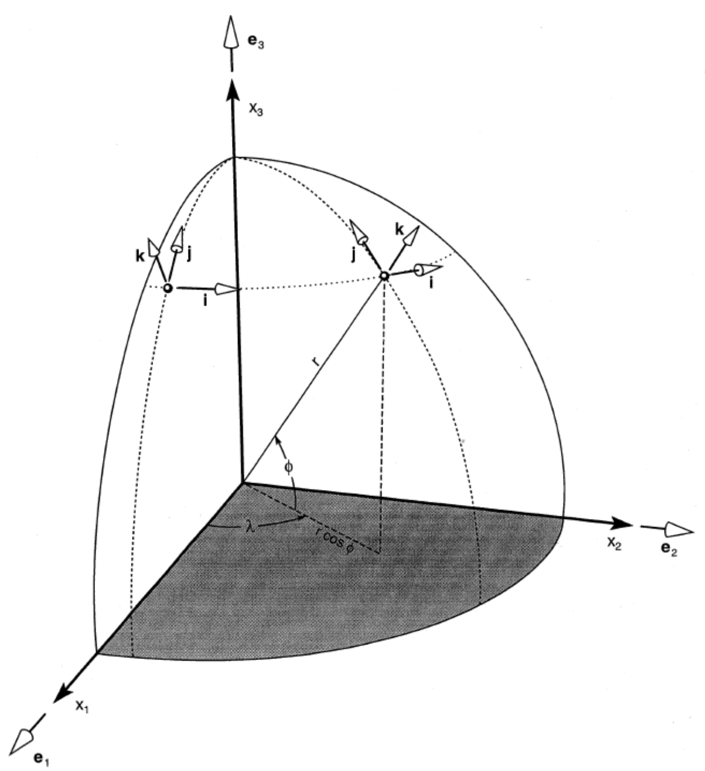


Figure 11.2 Spherical coordinates: longitude λ, latitude ϕ, and radial distance r. Coordinate vectors $\boldsymbol{e}_\lambda = \boldsymbol{i}$, $\boldsymbol{e}_\phi = \boldsymbol{j}$, and $\boldsymbol{e}_r = \boldsymbol{k}$ change with position (e.g., relative to fixed coordinate vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ of rectangular Cartesian coordinates).

Including these, the momentum equations become (Salby, section 11.2):

$$\frac{du}{dt} - 2\Omega(v \sin\phi - w\cos\phi) = -\frac{1}{\rho a \cos\phi}\frac{\partial p}{\partial \lambda} + uv\frac{\tan\phi}{a} - \frac{uw}{a} - D_\lambda,$$

$$\frac{dv}{dt} + 2\Omega u \sin\phi = -\frac{1}{\rho a}\frac{\partial p}{\partial \phi} - \frac{u^2 \tan\phi}{a} - \frac{uw}{a} - D_\phi,$$

$$\frac{dw}{dt} - 2\Omega u \cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{u^2 + v^2}{a} - D_z,$$

Neglect the curvature terms and with hydrostatic approximation is:

$$\begin{aligned} \frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv &= \mathcal{F}_x \, , \\ \frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu &= \mathcal{F}_y \, , \\ \frac{1}{\rho}\frac{\partial p}{\partial z} + g &= 0 \, , \end{aligned}$$

Adjustment under gravity with rotation

Now we consider the adjustment under gravity of a homogeneous shallow layer of fluid, a problem that we considered before, but now with rotation. First, we look at this program geost_adjust.m. We see that the surface no longer becomes flat (even when the centrifugal force is taken into account when defining "flat"). We can understand this behavior as described below.

Again use the hydrostatic equation, and linearize:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Taking ∂/∂y of the first equation and subtracting the ∂/∂x of the second equation, we have

$$\frac{\partial \varsigma}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\varsigma \equiv -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

We have defined the relative vorticity zeta. The above equation states angular momentum conservation, the same reason figure skates can rotate faster by pulling their arms close to their body. Together with the continuity equation, Lord Kelvin (1879) derived the following

$$\frac{\partial}{\partial t}\left(\frac{\varsigma}{f}-\frac{\eta}{H}\right)=0 \tag{1.2}$$

This is a very important result and is a linearized version of the potential vorticity (PV) conservation.

With Eq. (1.2), we can find the steady state solution:

In steady state, pressure gradient balances the Coriolis force. This is called the geostrophic balance.

$$-fv=-g\frac{\partial\eta}{\partial x}$$

$$+fu=-g\frac{\partial\eta}{\partial y}$$

Note that velocity is along constant pressure contours. In this case,

$$\varsigma=\frac{g}{f}\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)$$

and Eq. (1.2) becomes

$$\frac{g}{f^2}\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)-\frac{\eta}{H}=const$$

We see that spatial scale of the steady state response is $a=\sqrt{gH}\ /\ f$. This is the Rossby deformation radius. The adjustment process toward this steady state was studied by Rossby (1938) and known as the Rossby adjustment process.

A few points to note:
1. Not all potential energy can be converted to kinetic energy. This is because the final state must have the same PV as the initial state and must be geostrophic balance.
2. The scale of the final perturbation is that of the Rossby deformation radius a, which goes to infinity as f goes to 0. At midlatitude, a is roughly 1000km in the atmosphere and 30km in the ocean, although it depends on the vertical structure of the flow which affects the effective gravitational restoring force.
3. The time it takes to adjust toward equilibrium is a/sqrt(gH)=1/f.

These have important implications to the circulations in the atmosphere and ocean. If we are interested in motions with spatial scales larger than the Rossby deformation radius and time scales longer than the Rossby adjustment time, then we expect to see things in geostrophic balance.

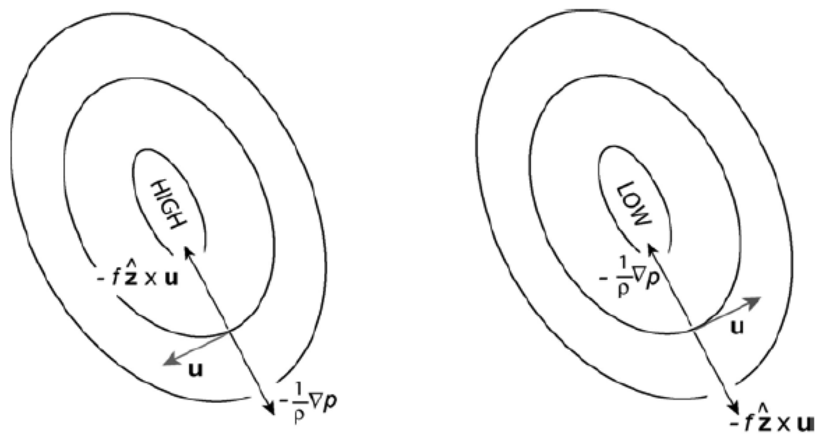


Figure 7.1: Geostrophic flow around (left) a high pressure center and (right) a low pressure center. (Northern hemisphere case, $f > 0$.) The effect of Coriolis deflecting flow 'to the right' — see Fig.6.10 — is balanced by the horizontal component of the pressure gradient force, $-\frac{1}{\rho}\nabla p$, directed from high to low pressure.

Another way of seeing this is through a scale analysis:

$$\frac{D\mathbf{u}}{Dt} + f\widehat{\mathbf{z}} \times \mathbf{u} = \underset{\frac{\mathcal{U}}{T}}{\frac{\partial \mathbf{u}}{\partial t}} + \underset{\frac{\mathcal{U}^2}{L}}{\mathbf{u}\cdot\nabla\mathbf{u}} + \underset{f\mathcal{U}}{f\widehat{\mathbf{z}} \times \mathbf{u}}$$

The ratio of the local rate of change in u and the advection term to that of the Coriolis term gives the Rossby number

$$R_o = \frac{U}{fL}$$

In the midlatitude atmosphere, f~10^{-4}/s and U~10m/s, and L~1000km so Ro~0.1. In the ocean, it is even smaller. The smallness of the Rossby number implies geostrophic balance.

$$f\hat{\mathbf{z}} \times \mathbf{u} + \frac{1}{\rho}\nabla p = 0$$

One can define the geostrophic wind/currents as:

$$\mathbf{u}_g = \frac{1}{f\rho}\hat{\mathbf{z}} \times \nabla p \ ,$$

or in component form:

$$(u_g, v_g) = \left(-\frac{1}{f\rho}\frac{\partial p}{\partial y}, \frac{1}{f\rho}\frac{\partial p}{\partial x}\right)$$

Geostrophic balance is well observed away from the boundary layer where friction may be neglected. To better see it in observations, we need to express the horizontal momentum equation using pressure as the vertical coordinate (as meteorologists like using pressure as the coordinate).

As discussed in Marshall and Plumb, Section 7.1, one can derive the following assuming hydrostatic balance:

$$\left(\frac{\partial p}{\partial x}\right)_z = g\rho\left(\frac{\partial z}{\partial x}\right)_p;$$
$$\left(\frac{\partial p}{\partial y}\right)_z = g\rho\left(\frac{\partial z}{\partial y}\right)_p.$$

so that

$$(u_g, v_g) = \left(-\frac{g}{f}\frac{\partial z}{\partial y}, \frac{g}{f}\frac{\partial z}{\partial x}\right)$$

In pressure coordinate, the geostrophic winds are nondivergent:

$$\nabla_p \cdot \mathbf{u}_g = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

This allows us to define a stream function that can often be useful:

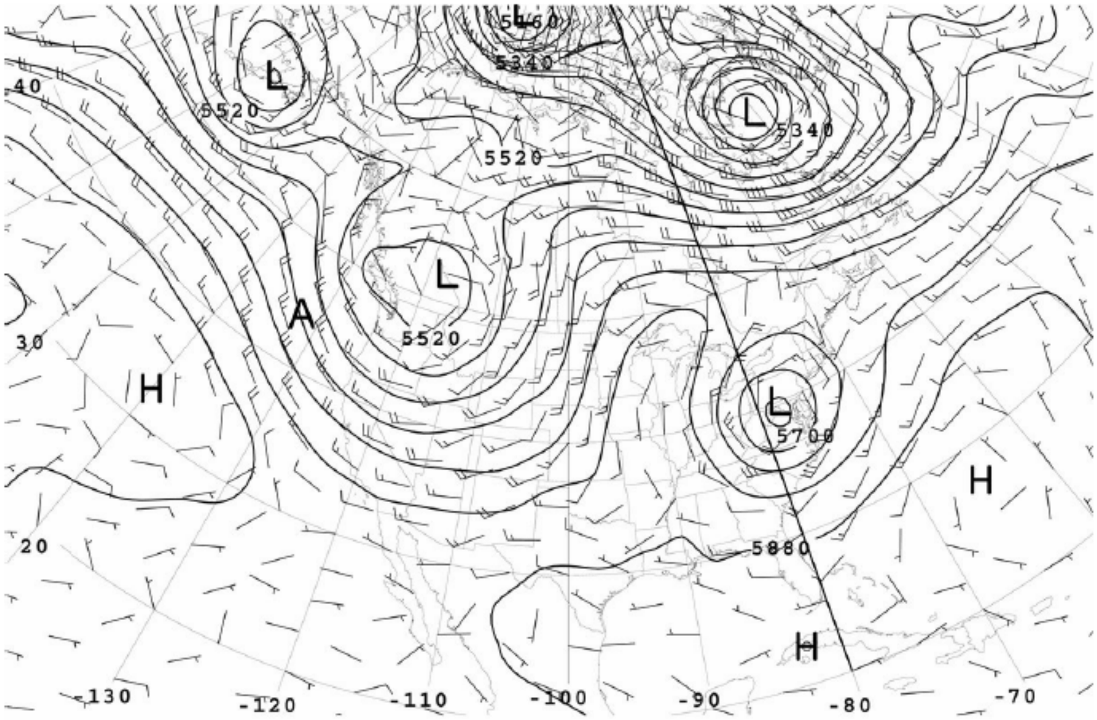
$$\psi_g = \frac{g}{f}z.$$

so that

$$u_g = -\frac{\partial \psi_g}{\partial y}; v_g = \frac{\partial \psi_g}{\partial x}$$

Note that all these fail at the equator where f=0.

Now let’s take a look at a synoptic chart:



(each full tick mark is 10m/s, half is 5m/s).

We see that when the height contours come close to each other, the winds are strong. Winds away from the boundary layer are very close to geostrophic balance. In regions of strong curvature, the Du/Dt term can be large, resulting in a three-way balance called the gradient wind balance.

Thermal wind balance

Geostrophic balance can be used together with hydrostatic balance to derive the thermal wind balance:

$$\rho g \partial z = -\partial p$$

$$\frac{\partial z}{\partial p} = \frac{-1}{g\rho} = \frac{-RT}{gp}$$

Take ∂/∂p of the geostrophic balance equation:

$$\frac{\partial u_g}{\partial p} = -\frac{g}{f}\frac{\partial^2 z}{\partial y \partial p} = \frac{R}{fp}\frac{\partial T}{\partial y}$$

$$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\frac{\partial T}{\partial y}$$

Similarly, we have $\frac{\partial v_g}{\partial \ln p} = \frac{-R}{f}\frac{\partial T}{\partial x}$

For the ocean, one can simply use the height coordinate:

$$\left(\frac{\partial u_g}{\partial z}, \frac{\partial v_g}{\partial z}\right) = \frac{g}{f\rho_{ref}}\left(\frac{\partial \rho}{\partial y}, -\frac{\partial \rho}{\partial x}\right)$$

This explains much of the wind structure in the atmosphere (and in the ocean).

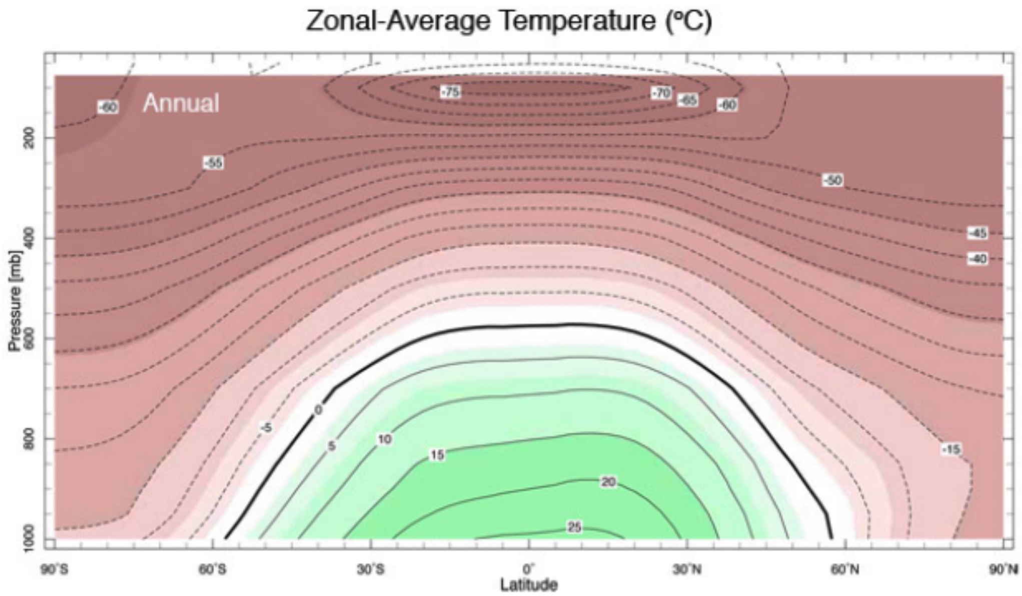


Figure 5.7: The zonally averaged annual mean temperature in °C.

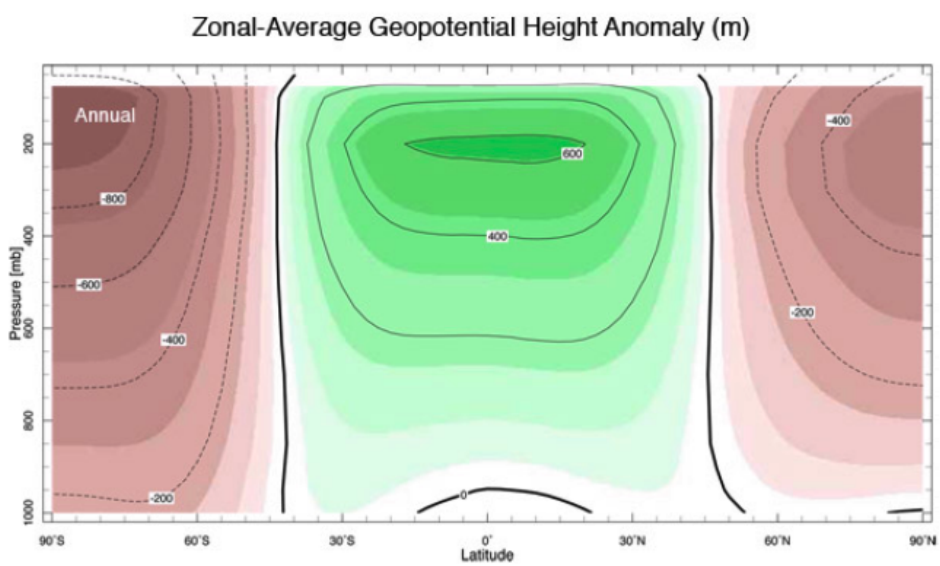
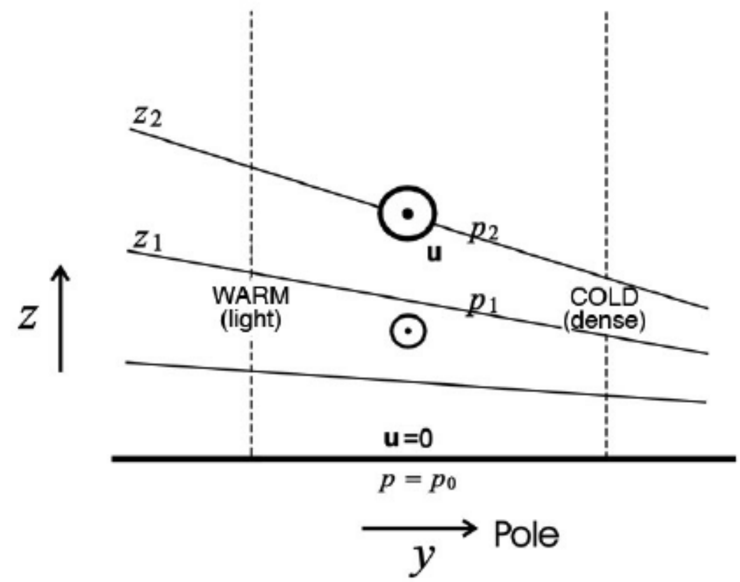
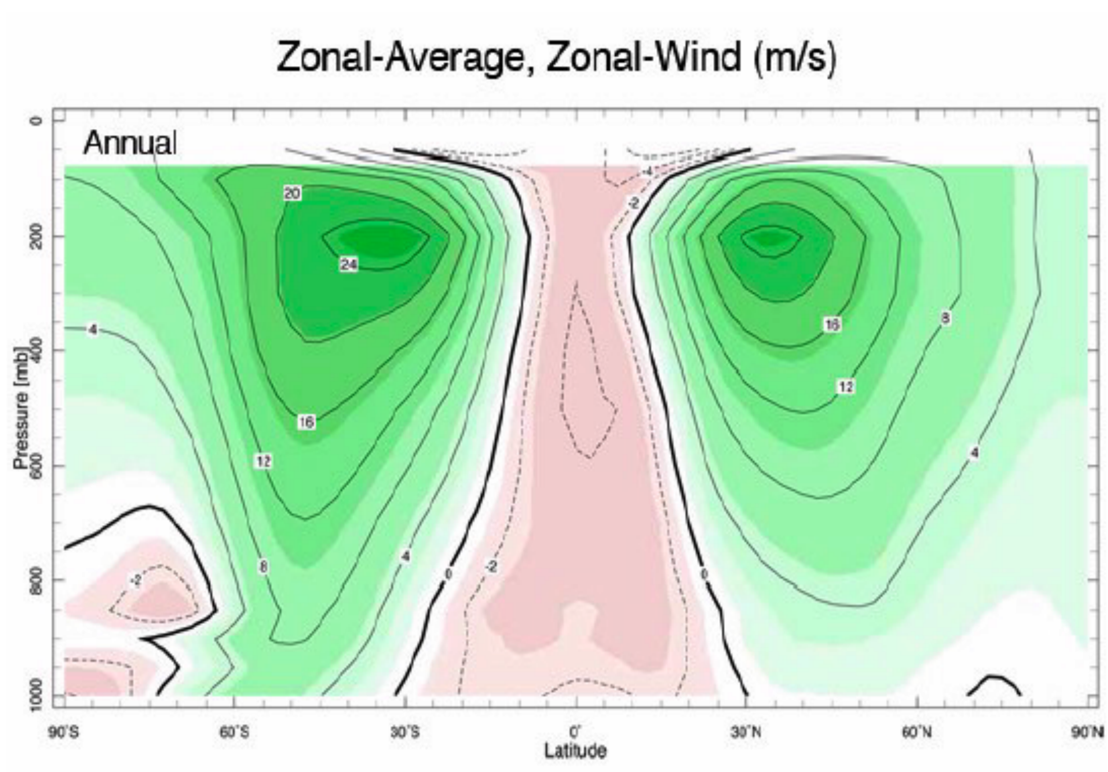


Figure 5.13: Zonal-mean geopotential height (m) for annual mean conditions. Values are departures from a horizontally uniform reference profile.





Here is a schematic of what's going on.

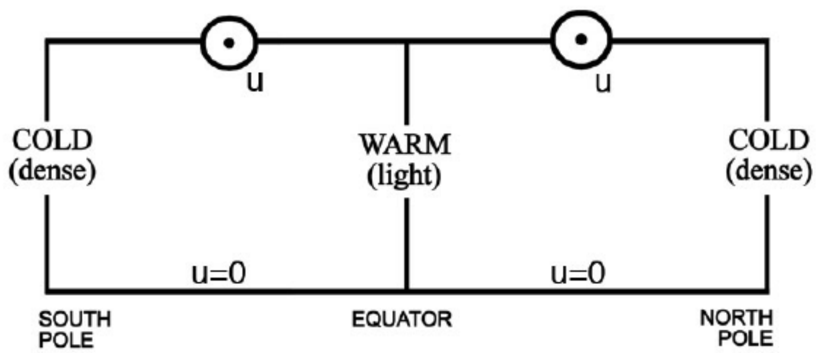


Figure 7.19: A schematic of westerly winds observed in both hemispheres in thermal wind balance with the equator-to-pole temperature gradient. (See Eq.(7.24) and the observations shown in Figs.5.7 and 5.20.)

The thermal wind balance works well even for instantaneous fields. This is not entirely surprising since it's a direct consequence of hydrostatic balance and geostrophic balance.

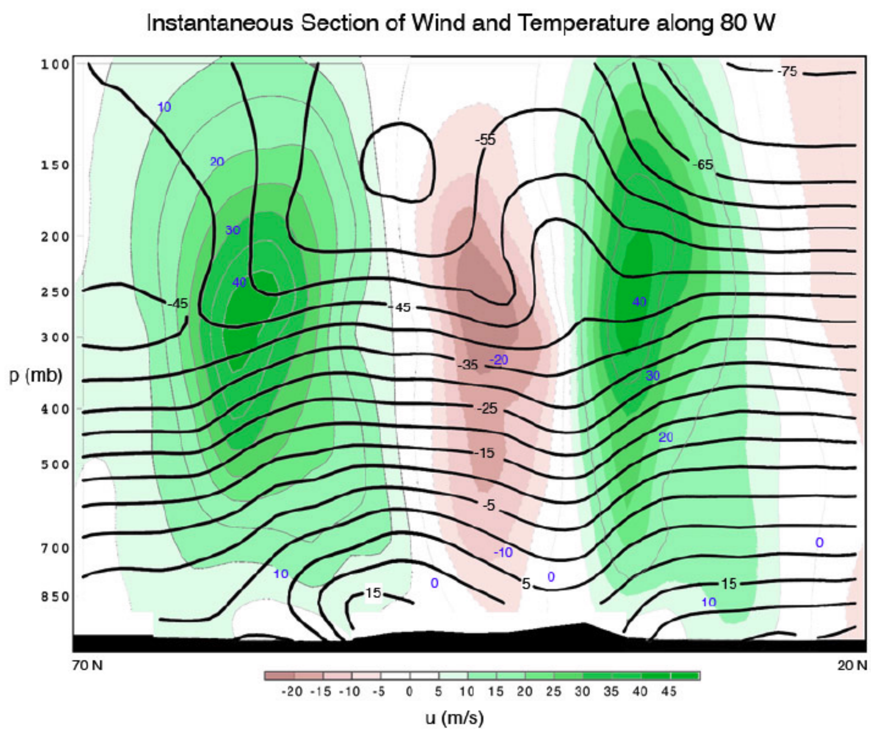


Figure 7.21: A cross section of zonal wind, u (color-scale, green indicating away from us and brown toward us) and thin contours every $5\,\mathrm{m\,s^{-1}}$), and potential temperature, T (thick contours every $5\,°\mathrm{C}$) through the atmosphere at 80°W extending from 20°N to 70°N on June 21st, 2003 on at 12GMT, as marked on Figs.7.20 and 7.4. Note that $\frac{\partial u}{\partial p} < 0$ in regions where $\frac{\partial T}{\partial y} < 0$ and visa-versa.

Taylor-Proudman Theorem

For homogeneous fluid (uniform density) in hydrostatic equilibrium, horizontal pressure gradient does not vary with height. If the flow is sufficiently slow and steady (measured by a small Ro) with negligible friction, the flow is in geostrophic balance. With the two statements above, we deduce that horizontal winds must be constant with height, i.e. the flow is two-dimensional. This is the essence of the Taylor-Proudman theorem.

The web site for Marshall and Plumb has interesting demonstrations of this effect (GFD0 and GFDVII). The actual atmosphere and ocean are not two-dimensional. Which assumption in deriving the theorem is violated?

When there is horizontal temperature gradient, the motion is close to be in thermal wind balance. And GFDVIII gives a good demo of this.

Also GFD IX is a demonstration similar (in spirit) to our matlab demo of gravity wave adjustment with rotation. The analogy to circulation in the atmosphere is shown here:

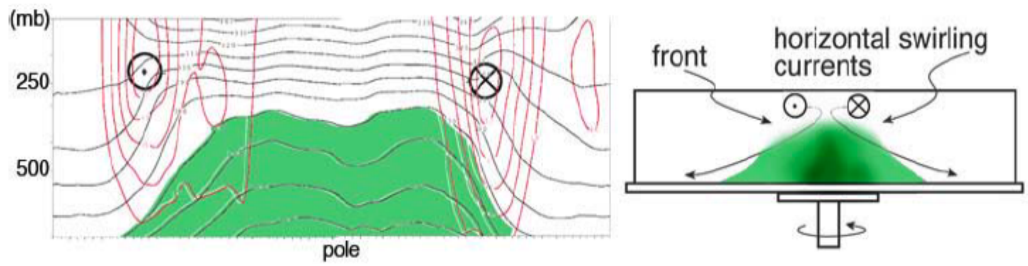


Figure 7.18: The dome of cold air that exists over the north pole shown in the instantaneous slice across the pole on the left (shaded green) is associated with strong upper-level winds marked $\otimes$ and $\odot$, and contoured in red. On the right we show a schematic diagram of the column of salty water studied in GFD Lab IX (cf. Figs.7.15 and 7.16. The column is prevented from slumping all the way to the bottom by the rotation of the tank. Differences in Coriolis forces acting on the spinning column provide a 'torque' which balances that of gravity acting on salty fluid trying to pull it down.

This also provides a way of thinking of the Rossby deformation radius. As the column of dense water is let to spread, water moves in at the top and moves out at the bottom. The Coriolis forcing will cause water to move in and out of the paper. This is the same as angular momentum conservation (from the perspective of the inertial frame)

$$\Omega r^2 + ur = const$$

$$u = -2\Omega \delta r$$

This will come to rest when thermal wind balance is established

$$2\Omega u = g' H / \delta r$$

The left hand is the Coriolis acceleration and the right hand side is the pressure gradient. This gives $\delta r = \sqrt{g' H} \,/\, 2\Omega$, the Rossby deformation radius.

Potential vorticity

First consider a shallow water system (uniform density and in hydrostatic equilibrium). In the absence of friction, we have, with full nonlinearity,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} (2)$$

Take $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$, we have

$$\frac{\partial \varsigma}{\partial t} + u\frac{\partial \varsigma}{\partial x} + v\frac{\partial \varsigma}{\partial y} + \frac{-\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{-\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial(\varsigma + f)}{\partial t} + u\frac{\partial(\varsigma + f)}{\partial x} + v\frac{\partial(\varsigma + f)}{\partial y} + \varsigma\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

With this, we have together with continuity:

$$\frac{D(\varsigma + f)}{Dt} + (\varsigma + f)\nabla \bullet \vec{V} = 0$$

$$\frac{DH}{Dt} + H\nabla \bullet \vec{V} = 0$$

Combine the two equations:

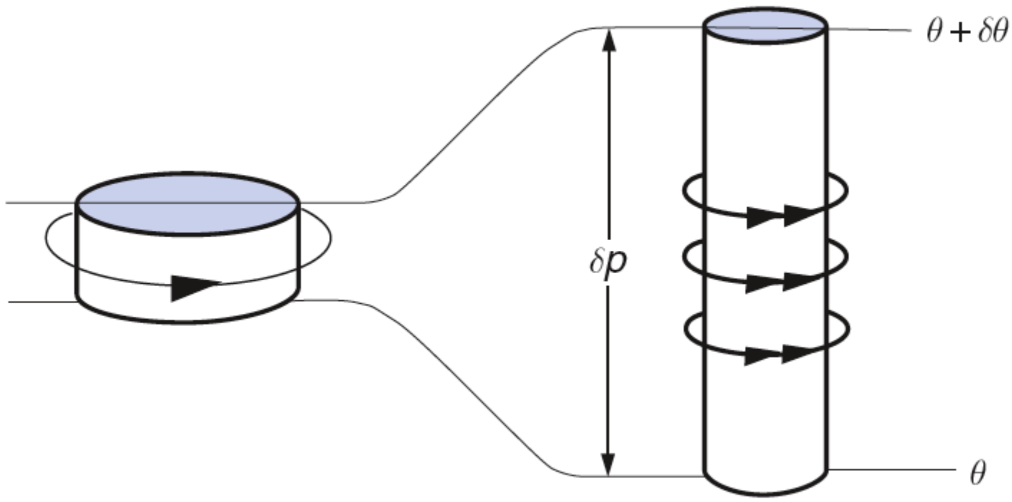
$$\frac{D(\varsigma + f)}{Dt} - \frac{(\varsigma + f)}{H}\frac{DH}{Dt} = 0$$

$$\Rightarrow \frac{D(\varsigma + f)}{Dt} + H(\varsigma + f)\frac{D(1/H)}{Dt} = 0$$

$$\Rightarrow \frac{D}{Dt}\left(\frac{\varsigma + f}{H}\right) = 0$$

This is a restatement of angular momentum conservation as the torque imparted by the pressure gradient force integrates to zero along a loop.

The quantity in parenthesis in the last line is the potential vorticity (PV) in shallow water systems and is conserved following fluid motion when there is no friction. It is the potential vorticity in the same sense as the potential temperature: it is the vorticity when the column is brought to a reference thickness.



In the atmosphere, density is not uniform. But one can define an analogous quantity:

$$P = \frac{\varsigma_\theta + f}{-\frac{1}{g}\frac{\partial p}{\partial \theta}}$$

The denominator is the thickness (or mass) between two isentropes. This is the Ertel PV or the isentropic PV and is conserved following adiabatic, frictionless motions. As such, they are useful tracers of atmospheric motion.

Do we expect the PV to be higher in the stratosphere or in the troposphere? PV is a key quantity in large-scale dynamics. As radiative adjustment is slow, PV is approximately conserved above the boundary layer on a day-to-day scale. For flows that are largely in hydrostatic and geostrophic balance, the entire flow structure (winds, temperature, pressure) can be deduced from the distribution of PV given the boundary conditions by inverting a Laplacian operator. As a simple example, consider an isolated positive PV anomaly. Part of this will be manifested in terms of positive vorticity and part of this will be in terms of enhanced stratification, as seen below. The depth-to-length ratio of the PV influence scales with f/N. The characteristic depth is called the Rossby height.

A schematic:

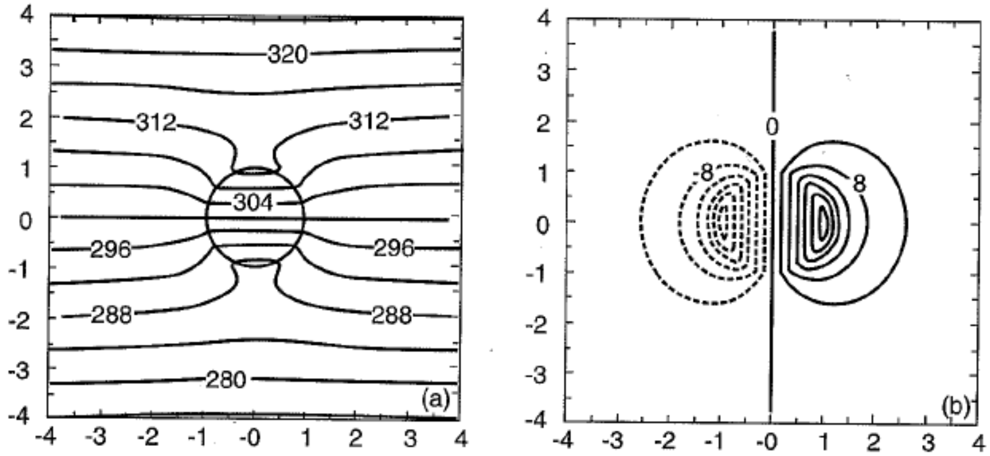


Fig. 6.10 Vertical cross sections showing potential temperature (left) and geostrophic wind (right) induced by a ball of constant potential vorticity of nondimensional radius unity. A constant standard atmosphere potential vorticity is added to the anomaly induced by the ball of potential vorticity. (Right) Solid contours indicate flow into the page and dashed contours indicate flow out of the page. The horizontal and scaled vertical distances are normalized by the radius of the potential vorticity ball. For a ball of 250-km radius the horizontal distance shown is 2000 km and the vertical distance is 20 km (assuming that the buoyancy frequency is 100 times the Coriolis frequency). (Adapted from Thorpe and Bishop, 1995)

A real world example: can you figure out the wind around the PV anomaly?

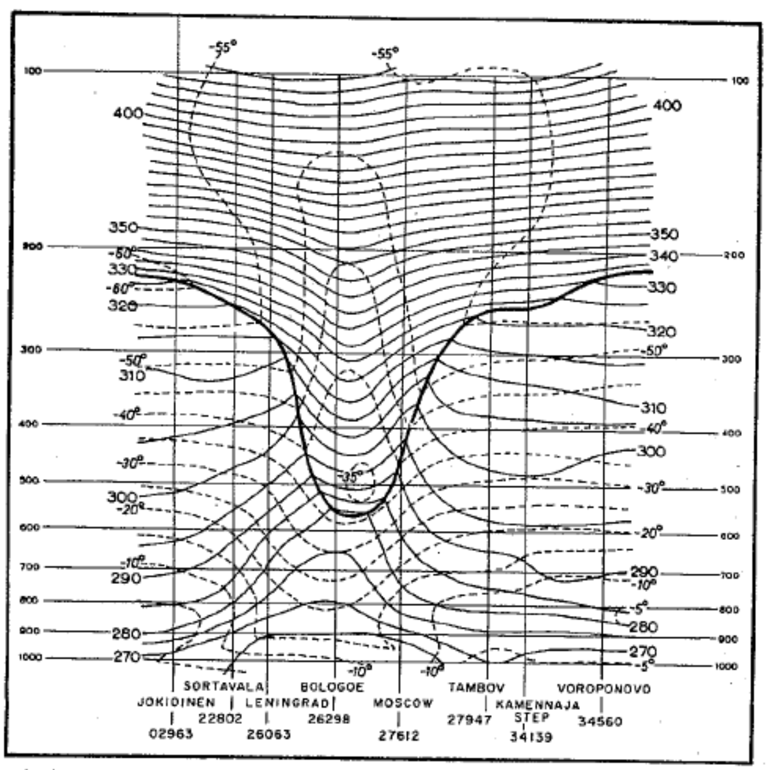


Figure 8. A vertical section through a cutoff cyclone at 12 GCT November 16 1959 produced by Peltonen (1963). The heavy line represents the tropopause; dashed lines are isotherms at 5°C intervals and solid lines isentropes every 5 K. The centre of the cyclone was at about 35°E, 58°N.

Effect of friction

In the boundary layer, friction from the surface cannot be neglected. Let's suppose a flow is in geostrophic balance around a low pressure so that the flow is along the isobars. Now "turn on" friction. Friction will slow down the wind so that the Coriolis force can no longer balance the pressure gradient; the latter wins. This causes flow into the low pressure. As the cross-isobar velocity develops, the Coriolis forcing associated with this also increases and in the end a three-way balance develops:

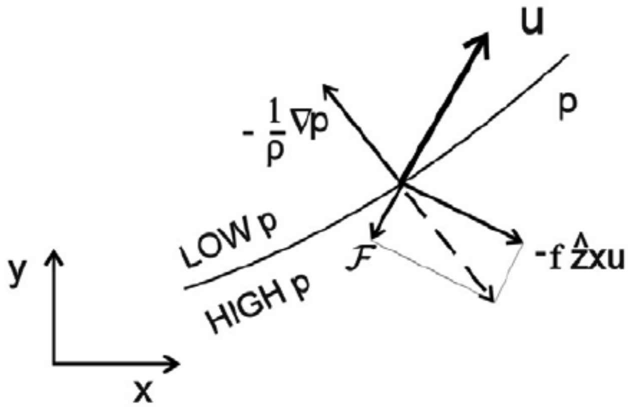


Figure 7.22: The balance of forces in Eq.(7.25): the dotted line is the vector sum $\mathcal{F} - f\widehat{\mathbf{z}} \times u$ and is balanced by $-\frac{1}{\rho}\nabla p$.

A schematic:

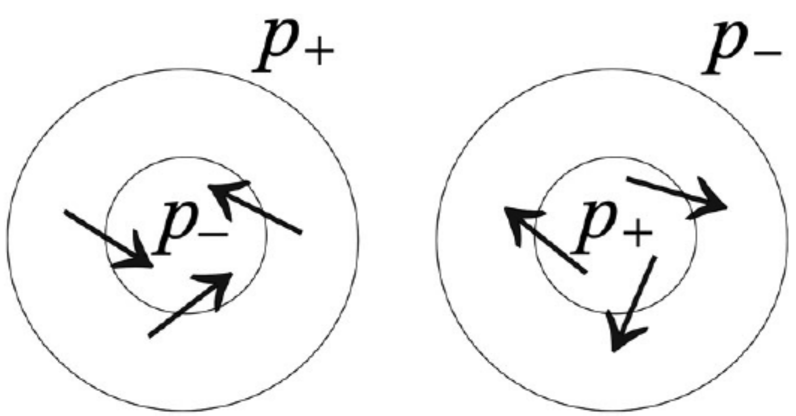


Figure 7.24: Flow spiralling in to a low pressure region (left) and out of a high pressure region (right) in a bottom Ekman layer. In both cases the ageostrophic flow is directed from high pressure to low pressure i.e. down the pressure gradient.

A lab demonstration:

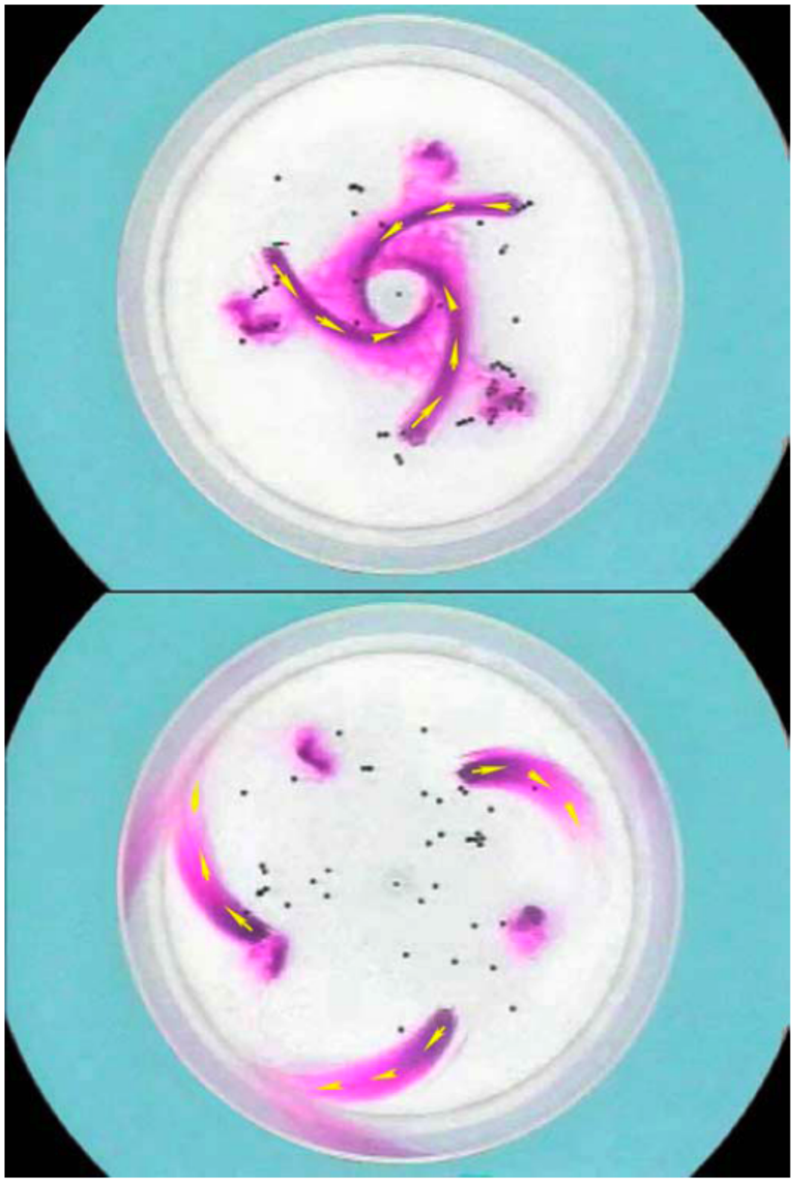
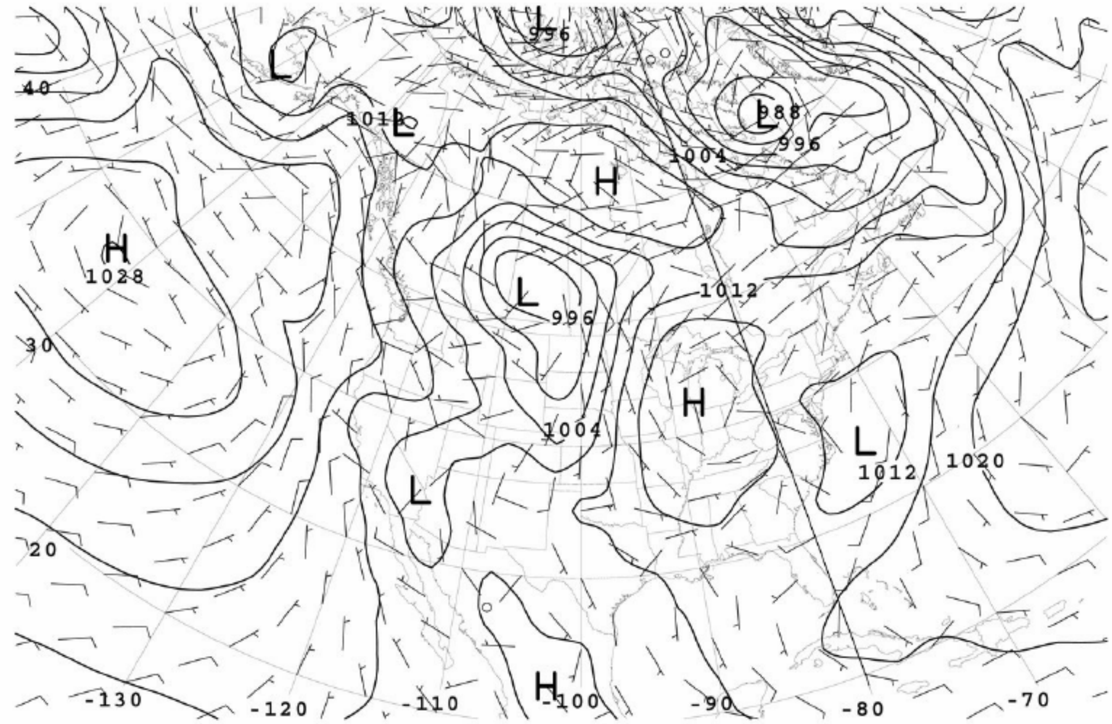


Figure 7.23: Ekman flow in a low pressure system (top) and a high pressure system (bottom) revealed by permanganate crystals on the bottom of a rotating tank. The black dots are floating on the free surface and mark out circular trajectories around the center of the tank directed anticlockwise (top) and clockwise (bottom).

An example from the atmosphere:



Note that departures from geostrophy (ageostrophic wind) are stronger over land. Why? As a simple illustrative example, let's suppose friction is proportional to wind (called Rayleigh drag) so that

$$-fv = -ku$$

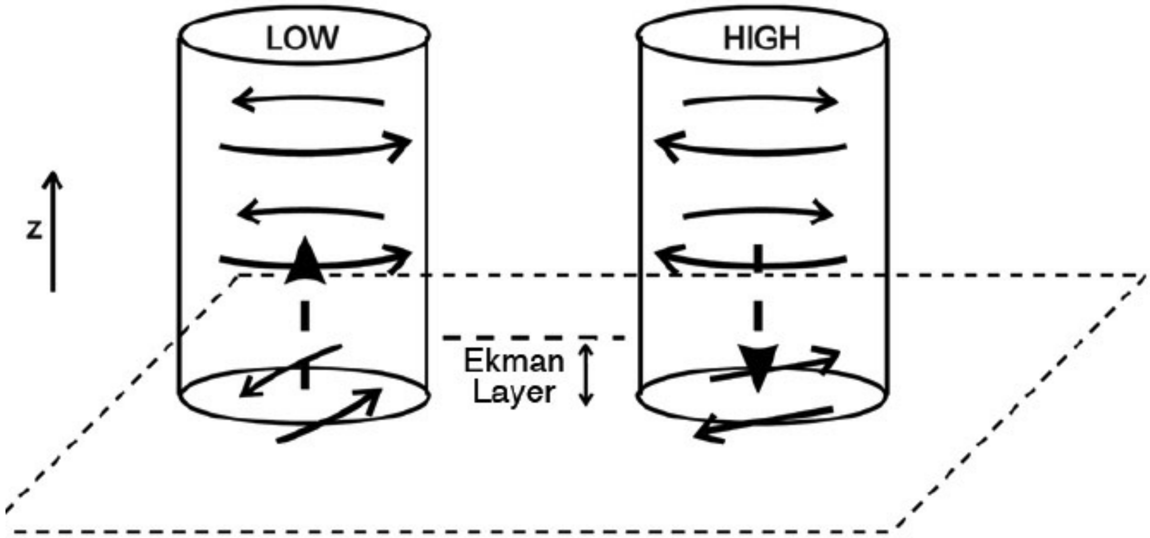
$$fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = -kv$$

The first equation already tells us that the ageostrophic winds are stronger for a stronger drag, but the two equations can be easily solved to give

$$u = \frac{1}{1 + k^2 / f^2}\frac{-1}{\rho f}\frac{\partial p}{\partial y}$$

$$\frac{v}{u} = \frac{k}{f}$$

Geostrophic winds are horizontally non-divergent. This is no longer true for ageostrophic flow. Going back to the schematic, what kind of vertical motion do it imply?



So friction causes convergence into a low pressure and a compensating upward motion (called Ekman pumping), and divergence out of a high pressure and a compensating downward motion (called Ekman suction). So high pressure systems tend to be associated with low precipitation and clear sky while low pressure systems are associated with rainy/cloudy conditions. This can be demonstrated with a cup of tea. Try spin up a cup of tea then stop. You will see the tealeaves at the bottom of the cup with move towards the center and form a pile. The same dynamics explain the formation of the eye in a developing hurricane.

The upper level divergence (convergence) from Ekman pumping (suction) is an efficient way for the upper level to feel the effect of bottom friction, and can spin down a midlatitude vortex in a few days, as compared to ~100days if it is done by vertical diffusion. In other words, material at the bottom doesn't have to be mixed to the upper level for it to feel the effect of bottom friction.

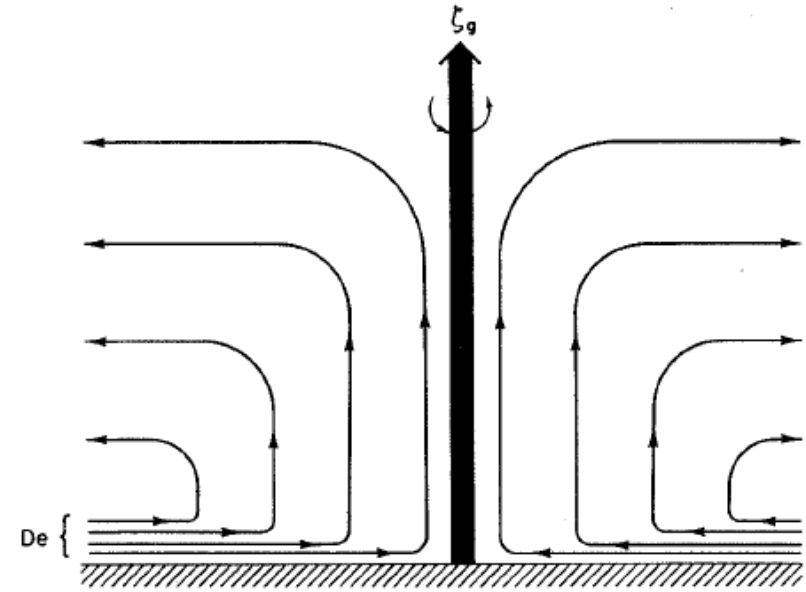


Fig. 5.7 Streamlines of the secondary circulation forced by frictional convergence in the planetary boundary layer for a cyclonic vortex in a barotropic atmosphere. The circulation extends throughout the full depth of the vortex.

When the atmosphere is stratified, things are more complicated but in general the outflow will be at a lower height:

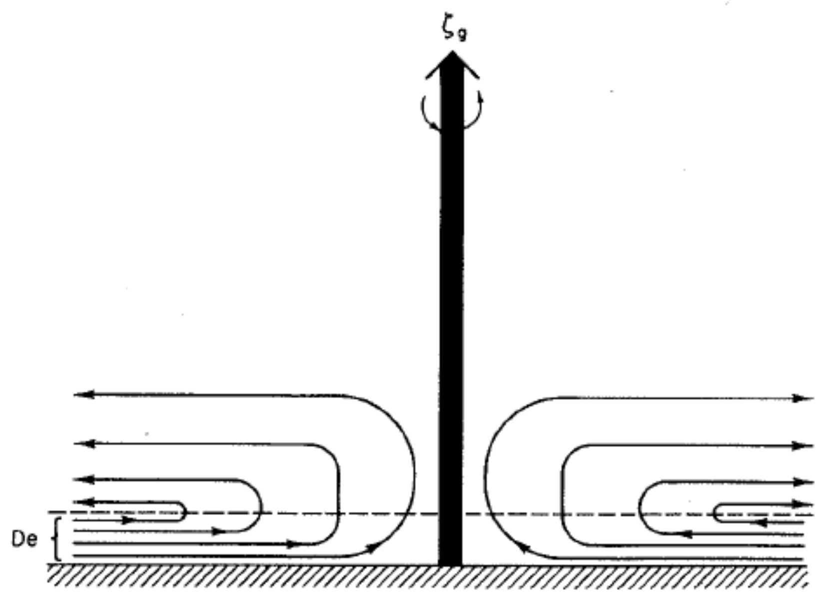


Fig. 5.8 Streamlines of the secondary circulation forced by frictional convergence in the planetary boundary layer for a cyclonic vortex in a stably stratified baroclinic atmosphere. The circulation decays with height in the interior.

We can use these ideas to connect the surface wind and pressure components of the general circulation.

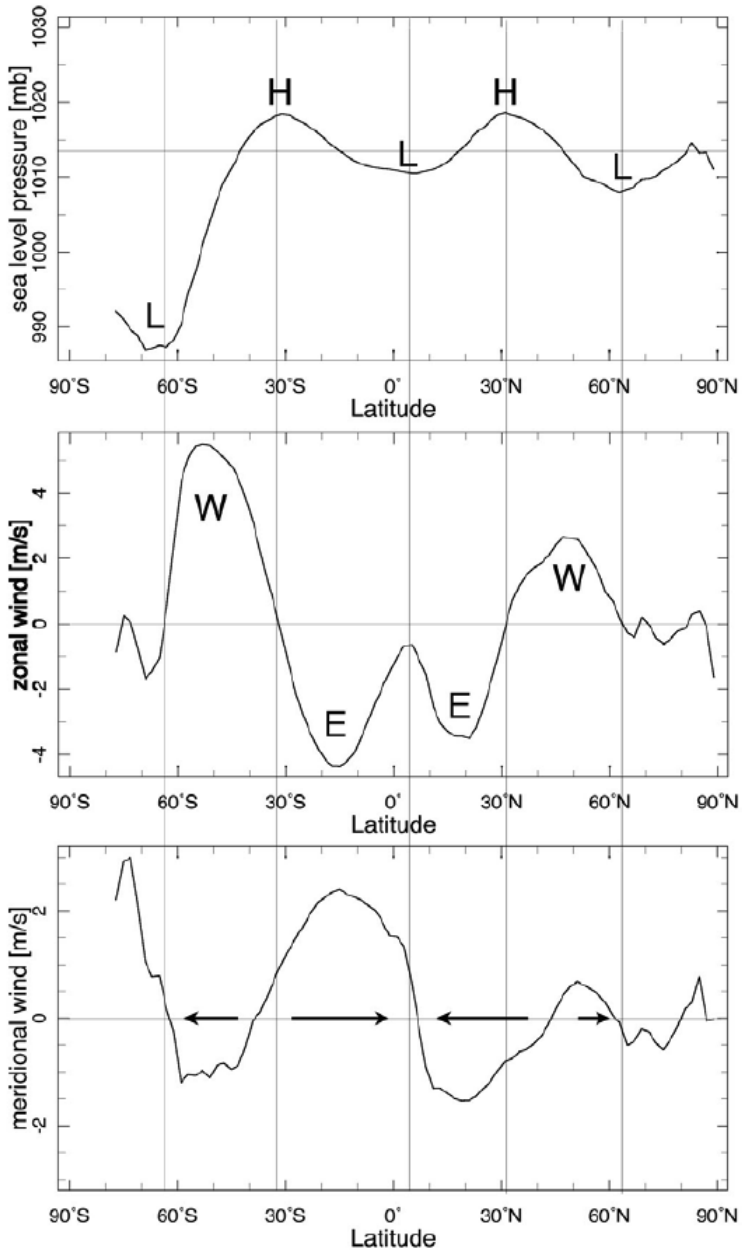


Figure 7.28: Anually and zonally averaged (top) sea level pressure in mbar, (middle) zonal wind in $\mathrm{m\,s^{-1}}$, and (bottom) meridional wind in $\mathrm{m\,s^{-1}}$. The horizontal arrows mark the sense of the meridional flow at the surface.