where E is the evaporation rate ($\mathrm{kg\,m^{-2}\,s^{-1}}$) and z_m is the top of the moist layer ($z_m \approx 2$ km over much of the equatorial oceans). Substituting into (11.24) from the approximate continuity equation for q,

$$\nabla\bullet(\rho q\mathbf{V}) + \partial(\rho q w)/\partial z \approx 0 \tag{11.25}$$

we obtain

$$P = (\rho w q)_{z_m} + E \tag{11.26}$$

Using (11.26) we can relate the vertically averaged heating rate to the synoptic-scale variables $w(z_m)$ and $q(z_m)$.

We still, however, need to determine distribution of the heating in the vertical. The most common approach is to use an empirically determined vertical distribution based on observations. In that case, (11.18) can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\bullet\nabla\right)\ln\theta + w\frac{\partial\ln\theta}{\partial z} = \frac{L_c}{\rho c_p T}\eta(z)\left[(\rho w q)_{z_m} + E\right] \tag{11.27}$$

where $\eta(z) = 0$ for $z < z_c$ and $z > z_T$ and $\eta(z)$ for $z_c \le z \le z_T$ is a weighting function that must satisfy

$$\int_{z_c}^{z_T}\eta(z)dz = 1$$

Recalling that the diabatic heating must be approximately balanced by adiabatic cooling as indicated in (11.22), we see from (11.27) that $\eta(z)$ will have a vertical structure similar to that of the large-scale vertical mass flux, ρw . Observations indicate that for many tropical synoptic-scale disturbances, $\eta(z)$ reaches its maximum at about the 400-hPa level, consistent with the divergence pattern shown in Fig. 11.5.

The above formulation is designed to model average tropical conditions. In reality the vertical distribution of diabatic heating is determined by the local distribution of cloud heights. Thus, the cloud height distribution is apparently a key parameter in cumulus parameterization. A cumulus parameterization scheme in which this distribution is determined in terms of the large-scale variables was developed by Arakawa and Schubert (1974). A number of other schemes have been suggested in the past decade. Discussion of these is beyond the scope of this text.

11.4 EQUATORIAL WAVE THEORY

Equatorial waves are an important class of eastward and westward propagating disturbances in the atmosphere and in the ocean that are trapped about the equator (i.e., they decay away from the equatorial region). Diabatic heating by organized tropical convection can excite atmospheric equatorial waves, whereas wind stresses can excite oceanic equatorial waves. Atmospheric equatorial wave propagation can cause the effects of convective storms to be communicated over large longitudinal distances, thus producing remote responses to localized heat sources. Furthermore, by influencing the pattern of low-level moisture convergence, atmospheric equatorial waves can partly control the spatial and temporal distribution of convective heating. Oceanic equatorial wave propagation, however, can cause local wind stress anomalies to remotely influence the thermocline depth and the SST, as was discussed in Section 11.1.6.

11.4.1 Equatorial Rossby and Rossby–Gravity Modes

A complete development of equatorial wave theory would be rather complicated. In order to introduce equatorial waves in the simplest possible context we here use a shallow water model, analogous to that introduced in Section 7.3.2, and concentrate on the *horizontal* structure. Vertical propagation in a stratified atmosphere is discussed in Chapter 12. For simplicity, we consider the linearized momentum and continuity equations for a fluid system of mean depth h_e in a motionless basic state. Because we are interested only in the tropics, we utilize Cartesian geometry on an *equatorial β-plane*. In this approximation, terms proportional to $\cos\phi$ are replaced by unity, and terms proportional to $\sin\phi$ are replaced by y/a, where y is the distance from the equator and a is the radius of the earth. The Coriolis parameter in this approximation is given by

$$f \approx \beta y \tag{11.28}$$

where $\beta \equiv 2\Omega/a$, and Ω is angular velocity of the earth. The resulting linearized shallow water equations for perturbations on a motionless basic state of mean depth h_e may be written as

$$\partial u'/\partial t - \beta y v' = -\partial \Phi' \partial x \tag{11.29}$$

$$\partial v'/\partial t + \beta y u' = -\partial \Phi' \partial y \tag{11.30}$$

$$\partial \Phi'/\partial t + g h_e \left(\partial u'/\partial x + \partial v'/\partial y\right) = 0 \tag{11.31}$$

where $\Phi' = gh'$ is the geopotential disturbance, and primed variables designate perturbation fields.

The x and t dependence may be separated by specifying solutions in the form of zonally propagating waves:

$$\begin{pmatrix} u' \\ v' \\ \Phi' \end{pmatrix} = \begin{bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{\Phi}(y) \end{bmatrix} \exp\left[i\left(kx - \nu t\right)\right] \tag{11.32}$$

Substitution of (11.32) into (11.29)–(11.31) then yields a set of ordinary differential equations in y for the meridional structure functions $\hat{u}$, $\hat{v}$, $\hat{\Phi}$:

$$-i\nu\hat{u} - \beta y\hat{v} = -ik\hat{\Phi} \tag{11.33}$$

$$-i\nu\hat{v} + \beta y\hat{u} = -\partial\hat{\Phi}/\partial y \tag{11.34}$$

$$-i\nu\hat{\Phi} + gh_e\left(ik\hat{u} + \partial\hat{v}/\partial y\right) = 0 \tag{11.35}$$

If (11.33) is solved for $\hat{u}$ and the result substituted into (11.34) and (11.35), we obtain

$$\left(\beta^2 y^2 - \nu^2\right)\hat{v} = ik\beta y\hat{\Phi} + i\nu\partial\hat{\Phi}/\partial y \tag{11.36}$$

$$\left(\nu^2 - gh_e k^2\right)\hat{\Phi} + i\nu gh_e\left(\frac{\partial\hat{v}}{\partial y} - \frac{k}{\nu}\beta y\hat{v}\right) = 0 \tag{11.37}$$

Finally, (11.37) can be substituted into (11.36) to eliminate $\hat{\Phi}$, yielding a second-order differential equation in the single unknown, $\hat{v}$:

$$\frac{\partial^2\hat{v}}{\partial y^2} + \left[\left(\frac{\nu^2}{gh_e} - k^2 - \frac{k}{\nu}\beta\right) - \frac{\beta^2 y^2}{gh_e}\right]\hat{v} = 0 \tag{11.38}$$

Because (11.38) is homogeneous, we expect that nontrivial solutions satisfying the requirement of decay at large $|y|$ will exist only for certain values of ν, corresponding to frequencies of the normal mode disturbances.

Before discussing this equation in detail, it is worth considering the asymptotic limits that occur when either $h_e \to \infty$ or $\beta = 0$. In the former case, which is equivalent to assuming that the motion is nondivergent, (11.38) reduces to

$$\frac{\partial^2\hat{v}}{\partial y^2} + \left[-k^2 - \frac{k}{\nu}\beta\right]\hat{v} = 0$$

Solutions exist of the form $\hat{v} \sim \exp(ily)$, provided that ν satisfies the Rossby wave dispersion relationship, $\nu = -\beta k/(k^2 + l^2)$. This illustrates that for nondivergent barotropic flow, equatorial dynamics is in no way special. The rotation of the earth enters only in the form of the β effect; it is not dependent on f. However, if $\beta = 0$, all influence of rotation is eliminated and (11.38) reduces to the shallow water gravity wave model, which has nontrivial solutions for

$$\nu = \pm\left[gh_e\left(k^2 + l^2\right)\right]^{1/2}$$

Returning to (11.38), we seek solutions for the meridional distribution of $\hat{v}$, subject to the boundary condition that the disturbance fields vanish for $|y| \to \infty$.

This boundary condition is necessary because the approximation $f \approx \beta y$ is not valid for latitudes much beyond $\pm 30°$, so that solutions must be trapped equatorially if they are to be good approximations to the exact solutions on the sphere. Equation (11.38) differs from the classic equation for a harmonic oscillator in y because the coefficient in square brackets is not a constant, but is a function of y. For sufficiently small y this coefficient is positive and solutions oscillate in y, whereas for large y, solutions either grow or decay in y. Only the decaying solutions, however, can satisfy the boundary conditions.

It turns out[5] that solutions to (11.38), which satisfy the condition of decay far from the equator, exist only when the constant part of the coefficient in square brackets satisfies the relationship

$$\frac{\sqrt{gh_e}}{\beta}\left(-\frac{k}{\nu}\beta - k^2 + \frac{\nu^2}{gh_e}\right) = 2n+1; \quad n = 0,1,2,\ldots \tag{11.39}$$

which is a cubic dispersion equation determining the frequencies of permitted equatorially trapped free oscillations for zonal wave number k and meridional mode number n. These solutions can be expressed most conveniently if y is replaced by the nondimensional meridional coordinate

$$\xi \equiv \left(\beta \Big/ \sqrt{gh_e}\right)^{1/2} y$$

Then the solution has the form

$$\hat{v}(\xi) = v_0 \mathrm{H_n}(\xi)\exp\left(-\xi^2/2\right) \tag{11.40}$$

where v_o is a constant with velocity units, and $\mathrm{H}_n(\xi)$ designates the nth *Hermite polynomial*. The first few of these polynomials have the values

$$\mathrm{H}_0 = 1, \quad \mathrm{H}_1(\xi) = 2\xi, \quad \mathrm{H}_2(\xi) = 4\xi^2 - 2$$

Thus, the index n corresponds to the number of nodes in the meridional velocity profile in the domain $|y| < \infty$.

In general the three solutions of (11.39) can be interpreted as eastward and westward moving equatorially trapped gravity waves, and westward moving equatorial Rossby waves. The case $n = 0$ (for which the meridional velocity perturbation has a Gaussian distribution centered at the equator) must be treated separately. In this case the dispersion relationship (11.39) factors as

$$\left(\frac{\nu}{\sqrt{gh_e}} - \frac{\beta}{\nu} - k\right)\left(\frac{\nu}{\sqrt{gh_e}} + k\right) = 0 \tag{11.41}$$

[5] See Matsuno (1966).

The root $\nu/k = -\sqrt{gh_e}$, corresponding to a westward propagating gravity wave, is not permitted, as the second term in parentheses in (11.41) was implicitly assumed not to vanish when (11.36) and (11.37) were combined to eliminate $\hat{\Phi}$. The roots given by the first term in parentheses in (11.41) are

$$\nu = k\sqrt{gh_e}\left[\frac{1}{2} \pm \frac{1}{2}\left(1 + \frac{4\beta}{k^2\sqrt{gh_e}}\right)^{1/2}\right] \tag{11.42}$$

The positive root corresponds to an eastward propagating equatorial inertia–gravity wave, whereas the negative root corresponds to a westward propagating wave, which resembles an inertia–gravity wave for long zonal scale ($k \to 0$) and resembles a Rossby wave for zonal scales characteristic of synoptic-scale disturbances. This mode is generally referred to as a *Rossby–gravity* wave. The horizontal structure of the westward propagating $n = 0$ solution is shown in Fig. 11.13, whereas the relationship between frequency and zonal wave number for this and several other equatorial wave modes is shown in Fig. 11.14.

11.4.2 Equatorial Kelvin Waves

In addition to the modes discussed in the previous section, there is another equatorial wave that is of great practical importance. For this mode, which is called the equatorial *Kelvin wave*, the meridional velocity perturbation vanishes and (11.33)–(11.35) are reduced to the simpler set

$$-i\nu\hat{u} = -ik\hat{\Phi} \tag{11.43}$$

$$\beta y\hat{u} = -\partial\hat{\Phi}/\partial y \tag{11.44}$$

$$-i\nu\hat{\Phi} + gh_e\left(ik\hat{u}\right) = 0 \tag{11.45}$$

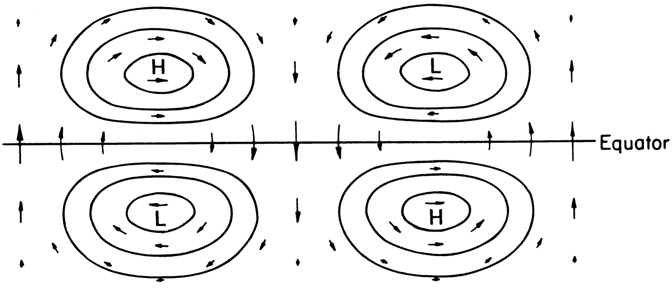

Fig. 11.13 Plan view of horizontal velocity and height perturbations associated with an equatorial Rossby–gravity wave. (Adapted from Matsuno, 1966.)

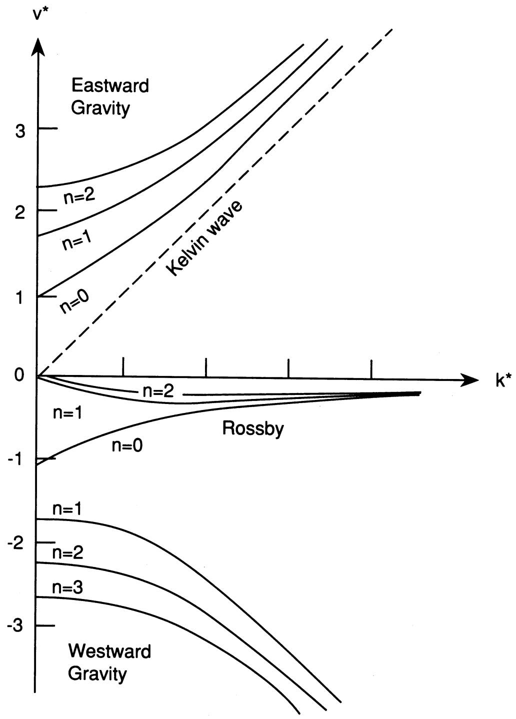

Fig. 11.14 Dispersion diagram for free equatorial waves. Frequency and zonal wave numbers have been nondimensionalized by defining

$$\nu^* \equiv \nu / \left(\beta\sqrt{gh_e}\right)^{1/2} \qquad k^* \equiv k\left(\sqrt{gh_e}/\beta\right)^{1/2}$$

Curves show dependence of frequency on zonal wave number for eastward and westward gravity modes and for Rossby and Kelvin modes. (k^* axis tic marks at unit interval with 0 on left.)

Combining (11.43) and (11.45) to eliminate $\hat{\Phi}$, we see that the Kelvin wave dispersion equation is identical to that for ordinary shallow water gravity waves:

$$c^2 \equiv (\nu/k)^2 = gh_e \tag{11.46}$$

According to (11.46), the phase speed c can be either positive or negative. However, if (11.43) and (11.44) are combined to eliminate $\hat{\Phi}$, we obtain a first-order equation for determining the meridional structure:

$$\beta y\hat{u} = -c\partial\hat{u}/\partial y \tag{11.47}$$

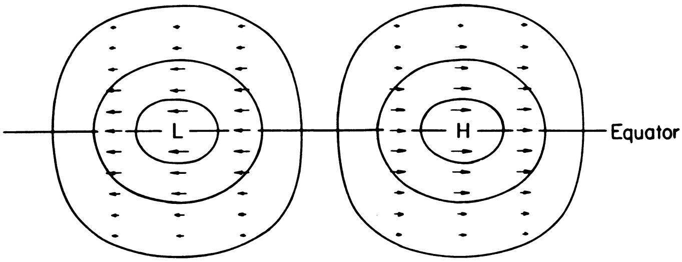

Fig. 11.15 Plan view of horizontal velocity and height perturbations associated with an equatorial Kelvin wave. (Adapted from Matsuno, 1966.)

which may be integrated immediately to yield

$$\hat{u} = u_o \exp\left(-\beta y^2/2c\right) \tag{11.48}$$

where u_0 is the amplitude of the perturbation zonal velocity at the equator. Equation (11.48) shows that if solutions decaying away from the equator are to exist, the phase speed must be positive ($c > 0$). Thus, Kelvin waves are eastward propagating and have zonal velocity and geopotential perturbations that vary in latitude as Gaussian functions centered on the equator. The e-folding decay width is given by

$$Y_K = |2c/\beta|^{1/2}$$

which for a phase speed $c = 30\,\text{m}\,\text{s}^{-1}$ gives $Y_K \approx 1600$ km.

The perturbation wind and geopotential structure for the Kelvin wave are shown in plan view in Fig. 11.15. In the zonal direction the force balance is exactly that of an eastward propagating shallow water gravity wave. A vertical section along the equator would thus be the same as that shown in Fig. 7.9. The meridional force balance for the Kelvin mode is an exact geostrophic balance between the zonal velocity and the meridional pressure gradient. It is the change in sign of the Coriolis parameter at the equator that permits this special type of equatorial mode to exist.

11.5 STEADY FORCED EQUATORIAL MOTIONS

Not all zonally asymmetric circulations in the tropics can be explained on the basis of inviscid equatorial wave theory. For quasi-steady circulations the zonal pressure gradient force must be balanced by turbulent drag rather than by inertia. The Walker circulation may be regarded as a quasi-steady equatorially trapped circulation that is generated by diabatic heating. The simplest models of such circulations specify the diabatic heating and use the equations of equatorial wave