Chapter 6

The equations of fluid motion

In order to proceed further with our discussion of the circulation of the atmosphere, and later the ocean, we must develop some of the underlying theory governing the motion of a fluid on the spinning Earth. A differentially heated, stratified fluid on a rotating planet cannot move in arbitrary paths. Indeed, there are strong constraints on its motion imparted by the angular momentum of the spinning Earth. These constraints are profoundly important in shaping the pattern of atmosphere and ocean circulation and their ability to transport properties around the globe. The laws governing the evolution of both fluids are the same and so our theoretical discussion will not be specific to either atmosphere or ocean, but can and will be applied to both. Because the properties of rotating fluids are often counter-intuitive and sometimes difficult to grasp, alongside our theoretical development we will describe and carry out laboratory experiments with a tank of water on a rotating table (Fig.6.1). Many of the laboratory experiments we make use of are simplified versions of 'classics' that have become cornerstones of geophysical fluid dynamics. They are listed in Appendix 13.4. Furthermore we have chosen relatively simple experiments that, in the main, do nor require sophisticated apparatus. We encourage you to 'have a go' or view the attendant movie loops that record the experiments carried out in preparation of our text.

We now begin a more formal development of the equations that govern the evolution of a fluid. A brief summary of the associated mathematical concepts, definitions and notation we employ can be found in an Appendix 13.2.

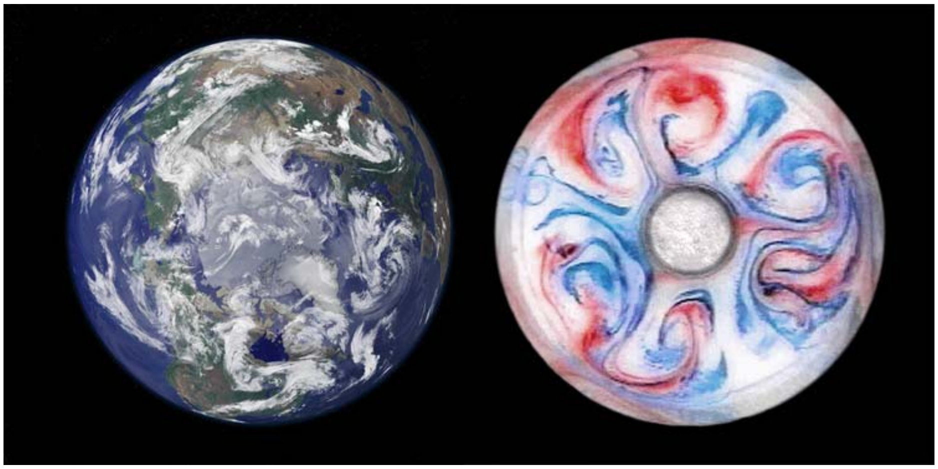

Figure 6.1: Throughout our text, running in parallel with a theoretical development of the subject, we study the constraints on a differentially heated, stratified fluid on a rotating planet (left), by making use of laboratory analogues designed to illustrate the fundamental processes at work (right). A complete list of the laboratory experiments can be found in Section 13.4.

6.1 Differentiation following the motion

When we apply the laws of motion and thermodynamics to a fluid to derive the equations that govern its motion, we must remember that these laws apply to material elements of fluid which are usually mobile. We must learn, therefore, how to express the rate of change of a property of a fluid element, *following that element as it moves along*, rather than at a fixed point in space. It is useful to consider the following simple example.

Consider again the situation sketched in Fig.4.13 in which a wind blows over a hill. The hill produces a pattern of waves in its lee. If the air is sufficiently saturated in water vapor, the vapor often condenses out to form cloud at the 'ridges' of the waves as described in Section 4.4 and seen in Figs.4.14 and 4.15.

Let us suppose that a steady state is set up so the pattern of cloud does not change in time. If $C = C(x, y, z, t)$ is the cloud amount, where (x, y) are horizontal coordinates, z is the vertical coordinate, t is time, then:

$$\left(\frac{\partial C}{\partial t}\right)_{\substack{\text{fixed point}\\ \text{in space}}} = 0,$$

where we keep at a fixed point in space, but at which, because the air is moving, there are constantly changing fluid parcels. The derivative $\left(\frac{\partial}{\partial t}\right)_{\text{fixed point}}$ is called the 'Eulerian derivative' after Euler[1].

But C is not constant *following along a particular parcel*; as the parcel moves upwards into the ridges of the wave, it cools, water condenses out, cloud forms, and so C increases (recall GFD Lab 1, Section 1.3.3); as the parcel moves down into the troughs it warms, the water goes back in to the gaseous phase, the cloud disappears and C decreases. Thus

$$\left(\frac{\partial C}{\partial t}\right)_{\substack{\text{fixed}\\ \text{particle}}} \neq 0$$

even though the wave-pattern is fixed in space and constant in time.

So, how do we mathematically express 'differentiation following the motion'? In order to follow particles in a continuum a special type of differentiation is required. Arbitrarily small variations of $C(x, y, z, t)$, a function of position and time, are given to the first order by:

$$\delta C = \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial x}\delta x + \frac{\partial C}{\partial y}\delta y + \frac{\partial C}{\partial z}\delta z$$

where the partial derivatives $\frac{\partial}{\partial t}$ etc. are understood to imply that the other variables are kept fixed during the differentiation. The fluid velocity is the

[1] Leonhard Euler (1707-1783). Euler made vast contributions to mathematics in the areas of analytic geometry, trigonometry, calculus and number theory. He also studied continuum mechanics, lunar theory, elasticity, acoustics, the wave theory of light, hydraulics and laid the foundation of analytical mechanics. In the 1750's Euler published a number of major pieces of work setting up the main formulas of fluid mechanics, the continuity equation and the Euler equations for the motion of an inviscid incompressible fluid.

rate of change of position of the fluid element, following that element along. The variation of a property C *following an element of fluid* is thus derived by setting $\delta x = u\delta t$, $\delta y = v\delta t$, $\delta z = w\delta t$, where u is the speed in the x-direction, v is the speed in the y-direction and w is the speed in the z-direction, thus:

$$(\delta C)_{\substack{\text{fixed}\\ \text{particle}}} = \left(\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z}\right)\delta t$$

where (u, v, w) is the velocity of the material element which by definition is the fluid velocity. Dividing by δt and in the limit of small variations we see that:

$$\left(\frac{\partial C}{\partial t}\right)_{\substack{\text{fixed}\\ \text{particle}}} = \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \frac{DC}{Dt}$$

in which we use the symbol $\frac{D}{Dt}$ to identify the rate of change following the motion:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla \ . \tag{6.1}$$

Here $\mathbf{u} = (u, v, w)$ is the velocity vector and $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the gradient operator. $\frac{D}{Dt}$ is called the Lagrangian derivative (after Lagrange; 1736 - 1813) [it is also called variously the 'substantial', the 'total' or the 'material' derivative]. Its physical meaning is 'time rate of change of some characteristic of a particular element of fluid' (which in general is changing its position). By contrast, as introduced above, the Eulerian derivative $\frac{\partial}{\partial t}$, expresses the rate of change of some characteristic at a *fixed point* in space (but with constantly changing fluid element because the fluid is moving).

Some writers use the symbol $\frac{d}{dt}$ for the Lagrangian derivative, but this is better reserved for the ordinary derivative of a function of one variable, the sense it is usually used in mathematics. Thus, for example, the rate of change of the radius of a rain drop would be written $\frac{dr}{dt}$ with the identity of the drop understood to be fixed. In the same context $\frac{D}{Dt}$ could refer to the motion of individual particles of water circulating within the drop itself. Another example is the vertical velocity, defined as $w = Dz/Dt$: if one sits in an air parcel and follow it around, w is the rate at which one's height changes[2].

[2]Meteorologists like working in pressure coordinates in which p is used as a vertical

The term $\mathbf{u}.\nabla$ in Eq.(6.1) represents *advection* and is the mathematical representation of the ability of a fluid to carry its properties with it as it moves. For example, the effects of advection are evident to us every day. In the northern hemisphere southerly winds (from the south) tend to be warm and moist because the air carries with it properties typical of tropical latitudes; northerly winds tend to be cold and dry because they advect properties typical of polar latitudes.

We will now use the Lagrangian derivative to help us apply the laws of mechanics and thermodynamics to a fluid.

6.2 Equation of motion for a non-rotating fluid

The state of the atmosphere or ocean at any time is defined by five key variables:

$$\mathbf{u} = (u, v, w);\ p \text{ and } T,$$

(six if we include specific humidity in the atmosphere, or salinity in the ocean). Note that by making use of the equation of state, Eq.(1.1), we can infer ρ from p and T. To 'tie' these variables down we need five independent equations. They are:

1. the laws of motion applied to a fluid parcel; this yields three independent equations in each of the three orthogonal directions
2. conservation of mass
3. the law of thermodynamics, a statement of the thermodynamic state in which the motion takes place.

These equations, five in all, together with appropriate boundary conditions, are sufficient to determine the evolution of the fluid.

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coordinate rather than z. In this coordinate an equivalent definition of "vertical velocity" is:

$$\omega = \frac{Dp}{Dt},$$

the rate at which pressure changes as the air parcel moves around. Since pressure varies much more quickly in the vertical than in the horizontal, this is still, for all practical purposes, a measure of vertical velocity, but expressed in units of hPa s^{-1}. Note also that upward motion has negative ω.

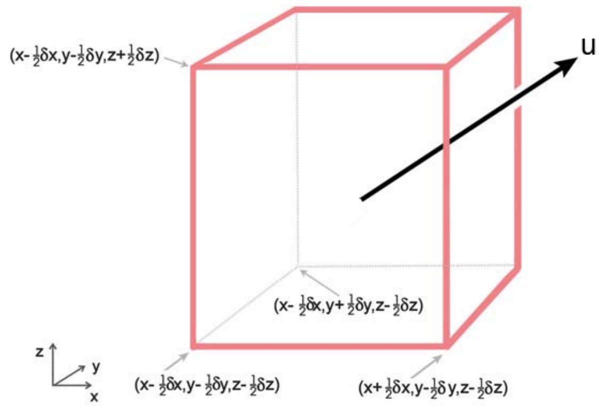

Figure 6.2: An elementary fluid parcel, conveniently chosen to be a cube of side δx, δy, δz, centered on (x, y, z). The parcel is moving with velocity $\mathbf{u}$.

6.2.1 Forces on a fluid parcel

We will now consider the forces on an elementary fluid parcel, of infinitesimal dimensions $(\delta x, \delta y, \delta z)$ in the three coordinate directions, centered on (x, y, z) (see Fig.6.2).

Since the mass of the parcel is $\delta M = \rho\ \delta x\ \delta y\ \delta z$, then, when subjected to a net force $\mathbf{F}$, Newton's Law of Motion for the parcel is

$$\rho\ \delta x\ \delta y\ \delta z \frac{D\mathbf{u}}{Dt} = \mathbf{F}\ , \tag{6.2}$$

where $\mathbf{u}$ is the parcel's velocity. As discussed earlier we must apply Eq.(6.2) to the same material mass of fluid, *i.e.*, we must follow the same parcel around. Therefore, the time derivative in Eq.(6.2) is the total derivative, defined in Eq.(6.1), which in this case is

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} &= \frac{\partial \mathbf{u}}{\partial t} + u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y} + w\frac{\partial \mathbf{u}}{\partial z} \\ &= \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u}\ . \end{aligned}$$

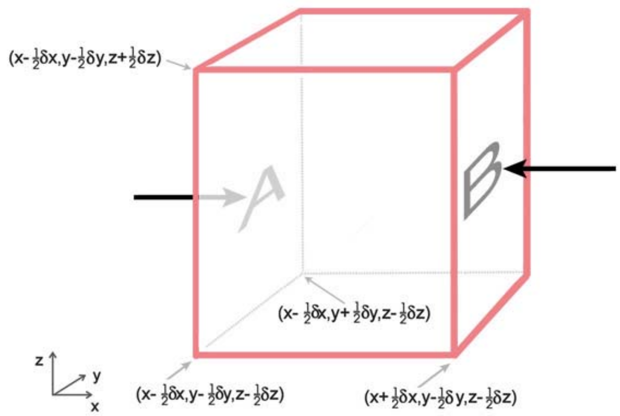

Figure 6.3: Pressure gradient forces acting on the fluid parcel. The pressure of the surrounding fluid applies a force to the right on face A and to the left on face B.

Gravity

The effect of gravity acting on the parcel in Fig.6.2 is straightforward: the gravitational force is $g\ \delta M$, and is directed downward,

$$\mathbf{F}_{gravity} = -g\rho\widehat{\mathbf{z}}\ \delta x\ \delta y\ \delta z, \tag{6.3}$$

where $\widehat{\mathbf{z}}$ is the unit vector in the upward direction and g is assumed constant.

Pressure gradient

Another kind of force acting on a fluid parcel is the pressure force within the fluid. Consider Fig.6.3. On each face of our parcel there is a force (directed inward) acting on the parcel equal to the pressure on that face multiplied by the area of the face. On face A, for example, the force is

$$F(A) = p(x - \frac{\delta x}{2}, y, z)\ \delta y\ \delta z\ ,$$

directed in the positive x-direction. Note that we have used the value of p at the mid-point of the face, which is valid for small δy, δz. On face B, there is an x-directed force

$$F(B) = -p(x + \frac{\delta x}{2}, y, z)\ \delta y\ \delta z\ ,$$

which is negative (toward the left). Since these are the only pressure forces acting in the x-direction, the net x-component of the pressure force is

$$F_x = \left[p(x - \frac{\delta x}{2}, y, z) - p(x + \frac{\delta x}{2}, y, z) \right]\ \delta y\ \delta z\ .$$

If we perform a Taylor expansion about the midpoint of the parcel, we have

$$\begin{aligned} p(x + \frac{\delta x}{2}, y, z) &= p(x, y, z) + \frac{\delta x}{2} \left(\frac{\partial p}{\partial x} \right)\ , \\ p(x - \frac{\delta x}{2}, y, z) &= p(x, y, z) - \frac{\delta x}{2} \left(\frac{\partial p}{\partial x} \right)\ , \end{aligned}$$

where the pressure gradient is evaluated at the midpoint of the parcel, and where we have neglected the small terms of $O(\delta x^2)$ and higher. Therefore the x-component of the pressure force is

$$F_x = -\frac{\partial p}{\partial x}\ \delta x\ \delta y\ \delta z\ .$$

It is straightforward to apply the same procedure to the faces perpendicular to the y- and z-directions, to show that these components are

$$\begin{aligned} F_y &= -\frac{\partial p}{\partial y}\ \delta x\ \delta y\ \delta z\ , \\ F_z &= -\frac{\partial p}{\partial z}\ \delta x\ \delta y\ \delta z\ . \end{aligned}$$

In total, therefore, the net pressure force is given by the vector

$$\begin{aligned} \mathbf{F}_{pressure} &= (F_x, F_y, F_z) \\ &= -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right)\ \delta x\ \delta y\ \delta z \qquad (6.4) \\ &= -\nabla p\ \delta x\ \delta y\ \delta z. \end{aligned}$$

Note that the net force depends only on the *gradient* of pressure, ∇p: clearly, a uniform pressure applied to all faces of the parcel would not introduce any net force.

Friction

For typical atmospheric and oceanic flows, frictional effects are negligible except close to boundaries where the fluid rubs over the Earth's surface. The atmospheric boundary layer — which is typically a few hundred meters to 1 km or so deep — is exceedingly complicated. For one thing, the surface is not smooth: there are mountains, trees, and other irregularities that increase the exchange of momentum between the air and the ground. (This is the main reason why frictional effects are greater over land than over ocean.) For another, the boundary layer is usually *turbulent*, containing many small-scale and often vigorous eddies; these eddies can act somewhat like mobile molecules, and diffuse momentum more effectively than molecular viscosity. The same can be said of oceanic boundary layers which are subject, for example, to the stirring by eddies generated by the action of the wind, as will be discussed in Section 10.1. At this stage, we will not attempt to describe such effects quantitatively but instead write the consequent frictional force on a fluid parcel as

$$\mathbf{F}_{fric} = \rho\, \mathcal{F}\, \delta x\, \delta y\, \delta z \tag{6.5}$$

where, for convenience, $\mathcal{F}$ is the frictional force *per unit mass*. For the moment we will not need a detailed theory of this term. Explicit forms for $\mathcal{F}$ will be discussed and employed in Sections 7.4.2 and 10.1.

6.2.2 The equation of motion

Putting all this together, Eq.(6.2) gives us

$$\rho\ \delta x\ \delta y\ \delta z \frac{D\mathbf{u}}{Dt} = \mathbf{F}_{gravity} + \mathbf{F}_{pressure} + \mathbf{F}_{fric}\ ,$$

Substituting from Eqs.(6.3), (6.4), and (6.5), and rearranging slightly, we obtain

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\widehat{\mathbf{z}} = \mathcal{F}\ . \tag{6.6}$$

This is our equation of motion for a fluid parcel.

Note that because of our use of vector notation, Eq.(6.6) seems rather simple. However, when written out in component form, as below, it becomes somewhat intimidating, even in Cartesian coordinates:

$$
\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x} &= \mathcal{F}_x && \text{(a)} \\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial y} &= \mathcal{F}_y && \text{(b)} \\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}+g &= \mathcal{F}_z\ . && \text{(c)}
\end{aligned}
\tag{6.7}
$$

Fortunately often we will be able to make a number of simplifications. One such simplification, for example, is that, as discussed in Section 3.2, large-scale flow in the atmosphere and ocean is almost always close to hydrostatic balance, allowing Eq.(6.7c) to be radically simplified as follows.

6.2.3 Hydrostatic balance

From the vertical equation of motion, Eq.(6.7c), we can see that if friction and the vertical acceleration Dw/Dt are negligible, we obtain:

$$\frac{\partial p}{\partial z} = -\rho g \tag{6.8}$$

thus recovering the equation of hydrostatic balance, Eq.(3.3). For large-scale atmospheric and oceanic systems in which the vertical motions are weak, the hydrostatic equation is almost always accurate, though it may break down in vigorous systems of smaller horizontal scale such as convection.[3]

6.3 Conservation of mass

In addition to Newton's laws there is a further constraint on the fluid motion: *conservation of mass*. Consider a fixed *fluid volume* as illustrated in Fig.6.4. The volume has dimensions $(\delta x, \delta y, \delta z)$. The mass of the fluid occupying this volume, $\rho\, \delta x\, \delta y\, \delta z$, may change with time if ρ does so. However, mass

[3]It might appear from Eq.(6.7c) that $|Dw/Dt| << g$ is a sufficient condition for the neglect of the acceleration term. This indeed is almost always satisfied. However, for hydrostatic balance to hold to sufficient accuracy to be useful, the condition is actually $|Dw/Dt| << g\Delta\rho/\rho$, where $\Delta\rho$ is a typical density variation on a pressure surface. Even in quite extreme conditions this more restrictive condition turns out to be very well satisfied.

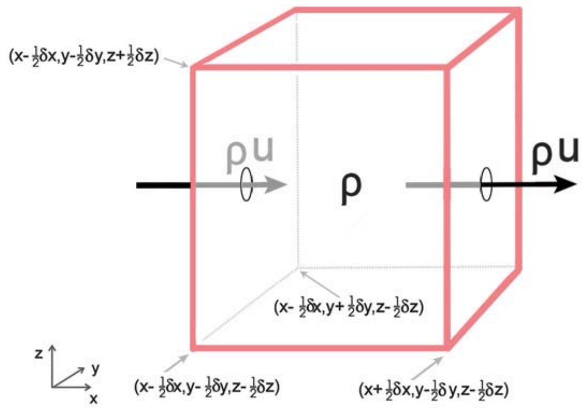

Figure 6.4: The mass of fluid contained in the fixed volume, $\rho\delta x\delta y\delta z$, can be changed by fluxes of mass out of and in to the volume, as marked by the arrows.

continuity tells us that this can only occur if there is a flux of mass into (or out of) the volume, *i.e.*,

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = \frac{\partial \rho}{\partial t}\,\delta x\;\delta y\,\delta z = (\text{net mass flux into the volume})\ .$$

Now the volume flux in the x-direction per unit time into the left face in Fig.6.4 is $u\left(x-\frac{1}{2}\delta x, y, z\right)\delta y\;\delta z$, so the corresponding mass flux is $[\rho u]\left(x-\frac{1}{2}\delta x, y, z\right)\delta y\;\delta z$ where $[\rho u]$ is evaluated at the left face. That out through the right face is $[\rho u]\left(x+\frac{1}{2}\delta x, y, z\right)\delta y\;\;\delta z$; therefore the net mass import in the x-direction into the volume is (again employing a Taylor expansion)

$$-\frac{\partial}{\partial x}(\rho u)\;\;\delta x\;\delta y\;\delta z\ .$$

Similarly the rate of net import of mass in the y-direction is

$$-\frac{\partial}{\partial y}(\rho v)\;\;\delta x\;\delta y\;\delta z$$

and in the z-direction is

$$-\frac{\partial}{\partial z}(\rho w)\;\;\delta x\;\delta y\;\delta z.$$

Therefore the net mass flux into the volume is $-\nabla \cdot (\rho\mathbf{u})\, \delta x\, \delta y\, \delta z$. Thus our *equation of continuity* becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \,. \tag{6.9}$$

This has the general form of a physical conservation law:

$$\frac{\partial \text{Concentration}}{\partial t} + \nabla \cdot (\text{flux}) = 0$$

in the absence of sources and sinks.

Using the *total derivative* D/Dt, Eq.(6.1), and noting that $\nabla \cdot (\rho\mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho$ (see the vector identities listed in Section 13.2) we may therefore rewrite Eq.(6.9) in the alternative, and often very useful, form:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \,. \tag{6.10}$$

6.3.1 Incompressible flow

For incompressible flow (*e.g.* for a liquid such as water in our laboratory tank or in the ocean), the following simplified approximate form of the continuity equation almost always suffices:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \,; \tag{6.11}$$

Indeed this is the definition of incompressible flow: it is *non-divergent* — no bubbles allowed! Note that in any real fluid, Eq.(6.11) is never *exactly* obeyed. Moreover, despite Eq.(6.10), use of the incompressibility condition should not be understood as implying that $\frac{D\rho}{Dt} = 0$; the density of a parcel of water can be changed by internal heating and/or conduction (see, for example, Section 11.1). While these density changes may be large enough to affect the buoyancy of the fluid parcel, they are too small to affect the mass budget. For example, the thermal expansion coefficient of water is typically $2 \times 10^{-4}\,\mathrm{K}^{-1}$ and so the volume of a parcel of water changes by only 0.02% per degree of temperature change.

6.3.2 Compressible flow

A compressible fluid such as air is nowhere close to being non-divergent — ρ changes markedly as fluid parcels expand and contract. This is inconvenient in the analysis of atmospheric dynamics. However it turns out that, provided the hydrostatic assumption is valid (as it nearly always is), one can get around this inconvenience by adopting pressure coordinates. In pressure coordinates, (x, y, p), the elemental fixed "volume" is $\delta x\ \delta y\ \delta p$. Since $z = z(x, y, p)$, the vertical dimension of the elemental volume (in geometric coordinates) is $\delta z = \frac{\partial z}{\partial p}\delta p$ and so its mass is δM given by:

$$\begin{aligned}\delta M &= \rho\, \delta x\, \delta y\, \delta z \\ &= \rho \left(\frac{\partial p}{\partial z}\right)^{-1} \delta x\, \delta y\, \delta p \\ &= -\frac{1}{g}\, \delta x\, \delta y\, \delta p\ ,\end{aligned}$$

where we have used hydrostatic balance, Eq.(3.3). So the mass of an elemental fixed volume *in pressure coordinates* cannot change! In effect, comparing the top and bottom line of the above, the equivalent of "density" in pressure coordinates — the mass per unit "volume" — is $1/g$, a constant. Hence, in the pressure-coordinate version of the continuity equation, there is no term representing rate of change of density; it is simply

$$\nabla_p \cdot \mathbf{u}_p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0. \tag{6.12}$$

where the subscript p reminds us that we are in pressure coordinates. The greater simplicity of this form of the continuity equation, as compared to Eqs.(6.9) or (6.10), is one of the reasons why pressure coordinates are favored in meteorology.

6.4 Thermodynamic equation

The equation governing the evolution of temperature can be derived from the first law of thermodynamics applied to a moving parcel of fluid. Dividing Eq.(4.12) by δt and letting $\delta t \longrightarrow 0$ we find:

$$\frac{DQ}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}. \tag{6.13}$$

$\frac{DQ}{Dt}$ is known as the 'diabatic heating rate' per unit mass. In the atmosphere, this is mostly due to latent heating and cooling (from condensation and evaporation of H_2O) and radiative heating and cooling (due to absorption and emission of radiation). If the heating rate is zero then $\frac{DT}{Dt} = \frac{1}{\rho c_p}\frac{Dp}{Dt}$: as discussed in Section 4.3.1, the temperature of a parcel will decrease in ascent (as it moves to lower pressure) and increase in descent (as it moves to higher pressure). Of course this is why we introduced potential temperature in Section 4.3.2: in adiabatic motion, θ is conserved. Written in terms of θ, Eq.(6.13) becomes

$$\frac{D\theta}{Dt} = \left(\frac{p}{p_0}\right)^{-\kappa} \frac{\dot{Q}}{c_p}. \tag{6.14}$$

where $\dot{Q}$ (with a dot over the top) is a shorthand for $\frac{DQ}{Dt}$. Here θ is given by Eq.(4.17), the factor $\left(\frac{p}{p_0}\right)^{-\kappa}$ converts from T to θ, and $\frac{\dot{Q}}{c_p}$ is the diabatic heating in units of $\mathrm{K\,s^{-1}}$. The analogous equations that govern the evolution of temperature and salinity in the ocean will be discussed in Chapter 11.

6.5 Integration, boundary conditions and restrictions in application

Eqs.(6.6), (6.11)/(6.12) and (6.14) are our five equations in five unknowns. Together with initial conditions and boundary conditions, they are sufficient to determine the evolution of the flow.

Before going on, we make some remarks about restrictions in the application of our governing equations. The equations themselves apply very accurately to the detailed motion. In practice, however, variables are always averages over large volumes. We can only tentatively suppose that the equations are applicable to the average motion, such as the wind integrated over a 100 km square box. Indeed, the assumption that the equations do apply to average motion is often incorrect. The treatment of turbulent motions remains one of the major challenges in dynamical meteorology and oceanography. Finally, our governing equations have been derived relative to a 'fixed' coordinate system. As we now go on to discuss, this is not really a restriction, but is usually inconvenient.

6.6 Equation of motion for a rotating fluid

Eq.(6.6) is an accurate representation of Newton's laws applied to a fluid observed from a fixed, inertial, frame of reference. However, we live on a rotating planet and observe winds and currents in its rotating frame. For example the winds shown in Fig.5.20 are not the winds that would be observed by someone looking back at the earth, as in Fig.1. Rather, they are the winds measured by observers on the planet rotating with it. In most applications it is easier and more desirable to work with the governing equations in a frame rotating with the earth. Moreover it turns out that rotating fluids have rather unusual properties and these properties are often most easily appreciated in the rotating frame. To proceed, then, we must write down our governing equations in a rotating frame. However, before going on to a formal 'frame of reference' transformation of the governing equations, we describe a laboratory experiment that vividly illustrates the influence of rotation on fluid motion and demonstrates the utility of viewing and thinking about fluid motion in a rotating frame.

6.6.1 GFD Lab III: Radial inflow

We are all familiar with the swirl and gurgling sound of water flowing down a drain. Here we set up a laboratory illustration of this phenomenon and study it in rotating and non-rotating conditions. We rotate a cylinder about its vertical axis: the cylinder has a circular drain hole in the center of its bottom, as shown in Fig.6.5. Water enters at a constant rate through a diffuser on its outer wall and exits through the drain. In so doing, the angular momentum imparted to the fluid by the rotating cylinder is conserved as it flows inwards, and paper dots floated on the surface acquire the swirling motion seen in Fig.6.6 as the distance of the dots from the axis of rotation decreases.

The swirling flow exhibits a number of important principles of rotating fluid dynamics — conservation of angular momentum, geostrophic (and cyclostrophic) balance (see Section 7.1) — all of which will be made use of in our subsequent discussions. The experiment also gives us an opportunity to think about frames of reference because it is viewed by a camera co-rotating with the cylinder.

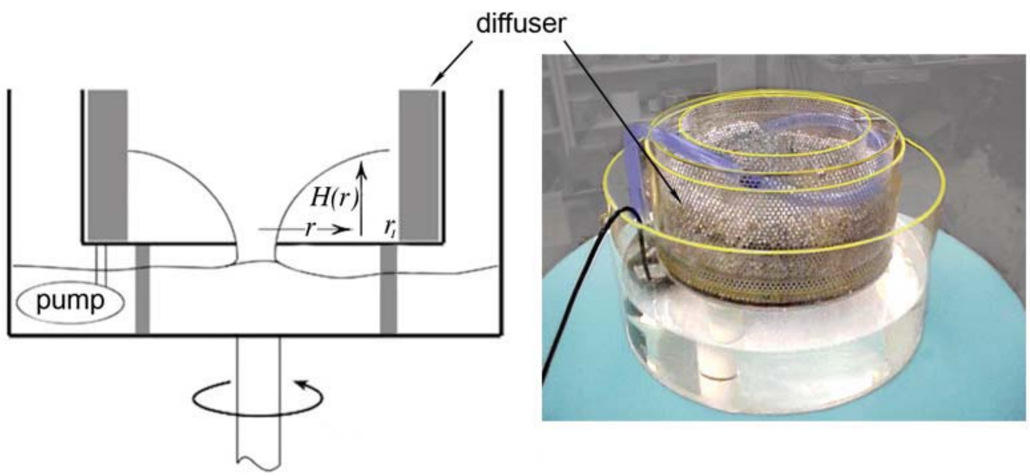

Figure 6.5: The radial inflow apparatus. A diffuser of 30 cm inside diameter is placed in a larger tank and used to produce an axially symmetric, inward flow of water toward a drain hole at the center. Below the tank there is a large catch basin, partially filled with water and containing a submersible pump whose purpose is to return water to the diffuser in the upper tank. The whole apparatus is then placed on a turntable and rotated in an anticlockwise direction. The path of fluid parcels is tracked by dropping paper dots on the free surface. See Whitehead and Potter (1977).

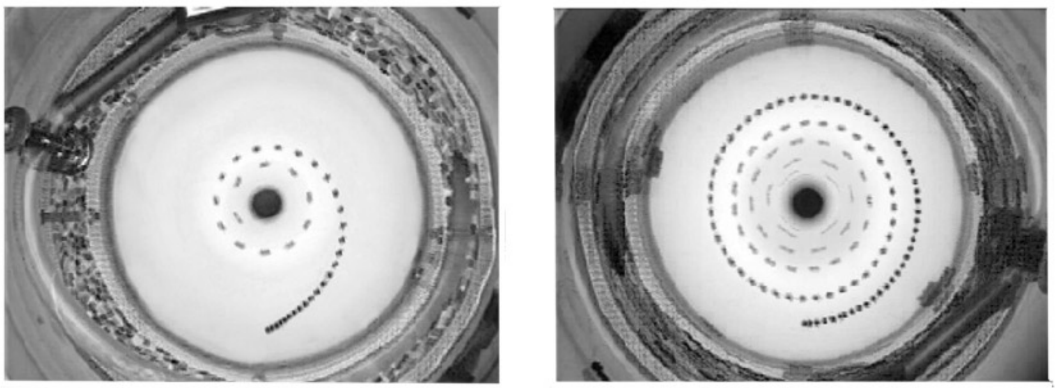

Figure 6.6: Trajectories of particles in the radial inflow experiment viewed in the rotating frame. The positions are plotted every $\frac{1}{30}$ s. On the left $\Omega = 5$rpm. On the right $\Omega = 10$rpm. Note how the pitch of the particle trajectory increases as Ω increases and how, in both cases, the speed of the particles increases as the radius decreases.

Observed flow patterns

When the apparatus is not rotating, water flows radially inward from the diffuser to the drain in the middle. The free surface is observed to be rather flat. When the apparatus is rotated, however, the water acquires a swirling motion: fluid parcels *spiral* inward as can be seen in Fig.6.6. Even at modest rotation rates of $\Omega = 10$rpm (corresponding to a rotation period of around 6 seconds)[4], the effect of rotation is marked and parcels complete many circuits before finally exiting through the drain hole. The azimuthal speed of the particle increases as it spirals inwards, as indicated by the increase in the spacing of the particle positions in the figure. In the presence of rotation the free surface becomes markedly curved, high at the periphery and plunging downwards toward the hole in the center, as shown in the photograph, Fig.6.7.

[4] An Ω of 10rpm (revolutions per minute) is equivalent to a rotation period $\tau = \frac{60}{10} = 6$ s. Various measures of table rotation rate are set out in Appendix 13.4.1.

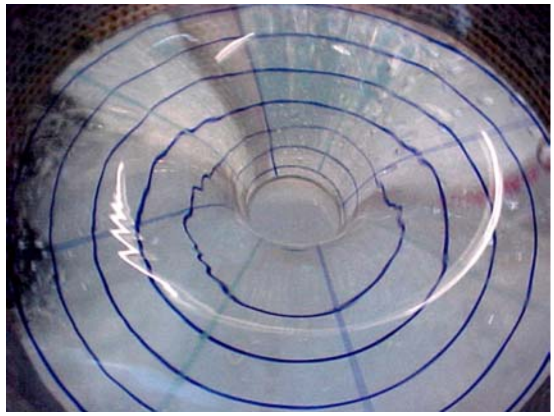

Figure 6.7: The free surface of the radial inflow experiment, in the case when the apparatus is rotated anticyclonically. The curved surface provides a pressure gradient force directed inwards that is balanced by an outward centrifugal force due to the anticlockwise circulation of the spiraling flow.

Dynamical balances

In the limit in which the tank is rotated rapidly, parcels of fluid circulate around many times before falling out through the drain hole (see the right hand frame of Fig.6.6); the pressure gradient force directed radially inwards (set up by the free surface tilt) is in large part balanced by a centrifugal force directed radially outwards.

If V_θ is the azimuthal velocity in the absolute frame (the frame of the laboratory) and v_θ is the azimuthal speed *relative* to the tank (measured using the camera co-rotating with the apparatus) then (see Fig.6.8):

$$V_\theta = v_\theta + \Omega r \tag{6.15}$$

where Ω is the rate of rotation of the tank in radians per second. Note that Ωr is the azimuthal speed of a particle stationary relative to the tank at radius r from the axis of rotation.

We now consider the balance of forces in the vertical and radial directions, expressed first in terms of the absolute velocity V_θ and then in terms of the relative velocity v_θ.

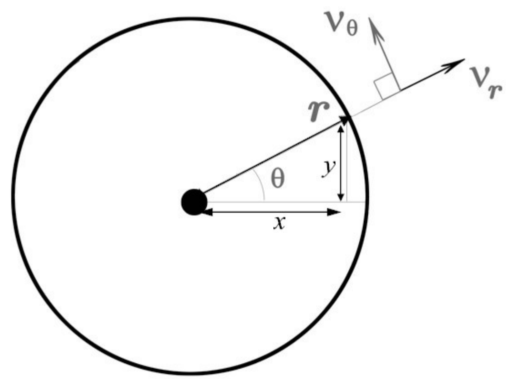

Figure 6.8: The velocity of a fluid parcel viewed in the rotating frame of reference: $v_{rot} = (v_\theta, v_r)$ in polar coordinates — see Section 13.2.3.

Vertical force balance We suppose that hydrostatic balance pertains in the vertical, Eq.(3.3). Integrating in the vertical and noting that the pressure vanishes at the free surface (actually $p =$ atmospheric pressure at the surface, which here can be taken as zero), and with ρ and g assumed constant, we find that:

$$p = \rho g \left(H - z\right) \tag{6.16}$$

where $H(r)$ is the height of the free surface (where $p = 0$) and we suppose that $z = 0$ (increasing upwards) on the base of the tank (see Fig.6.5, left).

Radial force balance in the non-rotating frame If the pitch of the spiral traced out by fluid particles is tight (*i.e.* in the limit that $\frac{v_r}{v_\theta} << 1$, appropriate when Ω is sufficiently large) then the centrifugal force directed radially outwards acting on a particle of fluid is balanced by the pressure gradient force directed inwards associated with the tilt of the free surface. This radial force balance can be written in the non-rotating frame thus:

$$\frac{V_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Using Eq.(6.16), the radial pressure gradient can be directly related to the gradient of free surface height enabling the force balance to be written[5]:

$$\frac{V_\theta^2}{r} = g\frac{\partial H}{\partial r} \tag{6.17}$$

Radial force balance in the rotating frame Using Eq.(6.15), we can express the centrifugal acceleration in Eq.(6.17) in terms of velocities in the rotating frame thus:

$$\frac{V_\theta^2}{r} = \frac{(v_\theta + \Omega r)^2}{r} = \frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r \tag{6.18}$$

Hence

$$\frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r = g\frac{\partial H}{\partial r} \tag{6.19}$$

The above can be simplified by measuring the height of the free surface relative to that of a reference parabolic surface[6], $\frac{\Omega^2 r^2}{2}$ as follows

$$\eta = H - \frac{\Omega^2 r^2}{2g}. \tag{6.20}$$

Then, since $\frac{\partial \eta}{\partial r} = \frac{\partial H}{\partial r} - \frac{\Omega^2 r}{g}$, Eq.(6.19) can be written in terms of η thus:

$$\frac{v_\theta^2}{r} = g\frac{\partial \eta}{\partial r} - 2\Omega v_\theta \tag{6.21}$$

Eq.(6.17) (non-rotating) and Eq.(6.21) (rotating) are completely equivalent statements of the balance of forces. The distinction between them is that the former is expressed in terms of V_θ, the latter in terms of v_θ. Note that Eq.(6.21) has the same form as Eq.(6.17) except (i) η (measured relative to the reference parabola) appears rather than H (measured relative to a flat surface) and (ii) an extra term, $-2\Omega v_\theta$, appears on the rhs of Eq.(6.21) — this is called the 'Coriolis acceleration'. It has appeared because we have

[5]Note that the balance Eq.(6.17) cannot hold *exactly* in our experiment because radial accelerations must be present associated with the flow of water inwards from the diffuser to the drain. But if these acceleration terms are small the balance (6.17) is a good approximation.

[6]By doing so, we thus eliminate from Eq.(6.19) the centrifugal term $\Omega^2 r$ associated with the background rotation. We will follow a similar procedure for the spherical earth in Section 6.6.3 (see also GFD Lab IV in Section 6.6.4).

chosen to express our force balance in terms of *relative*, rather than absolute velocities. We shall see that the Coriolis acceleration plays a central role in the dynamics of the atmosphere and ocean.

Angular momentum

Fluid entering the tank at the outer wall will have angular momentum because the apparatus is rotating. At r_1, the radius of the diffuser in Fig.6.5, fluid has velocity Ωr_1 and hence angular momentum Ωr_1^2. As parcels of fluid flow inwards they will conserve this angular momentum (provided that they are not rubbing against the bottom or the side). Thus conservation of angular momentum implies that:

$$V_\theta \, r = \text{constant} = \Omega r_1^2 \tag{6.22}$$

where V_θ is the azimuthal velocity at radius r in the laboratory (inertial) frame given by Eq.(6.15). Combining Eqs.(6.22) and (6.15) we find

$$v_\theta = \Omega \frac{(r_1^2 - r^2)}{r}. \tag{6.23}$$

We thus see that the fluid acquires a sense of rotation which is the same as that of the rotating table but which is greatly magnified at small r. If $\Omega > 0$ — i.e. the table rotates in an anticlockwise sense — then the fluid acquires an anticlockwise (cyclonic[7]) swirl. If $\Omega < 0$ the table rotates in a clockwise (anticyclonic) direction and the fluid acquires a clockwise (anticyclonic) swirl. This can be clearly seen in the trajectories plotted in Fig.6.6. Eq.(6.23) is, in fact, a rather good prediction for the azimuthal speed of the particles seen in Fig.6.6. We will return to this experiment later in Section 7.1.3 where we discuss the balance of terms in Eq.(6.21) and its relationship to atmospheric flows.

6.6.2 Transformation into rotating coordinates

In our radial inflow experiment we expressed the balance of forces in both the non-rotating and rotating frames. We have already written down the equations of motion of a fluid in a non-rotating frame, Eq.(6.6). Let us now

[7]The term cyclonic (anticyclonic) means that the swirl is in the same (opposite) sense as the background rotation.

formally transform it in to a rotating reference frame. The only tricky part is the acceleration term $D\mathbf{u}/Dt$, which requires manipulations analogous to Eq.(6.18) but in a general framework. We need to figure out how to transform the operator D/Dt (acting on a vector) into a rotating frame. Of course, D/Dt of a scalar is the same in both frames, since this means "the rate of change of the scalar following a fluid parcel". The same fluid parcel is followed from both frames, and so scalar quantities (e.g. temperature or pressure) do not change when viewed from the different frames. However, a vector is not invariant under such a transformation, since the coordinate directions relative to which the vector is expressed are different in the two frames.

A clue is given by noting that the velocity in the absolute (inertial) frame $\mathbf{u}_{in}$ and the velocity in the rotating frame $\mathbf{u}_{rot}$, are related (see Fig.6.9) through:

$$\mathbf{u}_{in} = \mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r} \ , \tag{6.24}$$

where $\mathbf{r}$ is the position vector of a parcel in the rotating frame, $\mathbf{\Omega}$ is the rotation vector of the rotating frame of reference, and $\mathbf{\Omega} \times \mathbf{r}$ is the vector product of $\mathbf{\Omega}$ and $\mathbf{r}$. This is just a generalization (to vectors) of the transformation used in Eq.(6.15) to express the absolute velocity in terms of the relative velocity in our radial inflow experiment. As we shall now go on to show, Eq.(6.24) is a special case of a general 'rule' for transforming the rate of change of vectors between frames, which we now derive.

Consider Fig.6.9. In the rotating frame, any vector $\mathbf{A}$ may be written

$$\mathbf{A} = \widehat{\mathbf{x}} A_x + \widehat{\mathbf{y}} A_y + \widehat{\mathbf{z}} A_z \tag{6.25}$$

where (A_x, A_y, A_z) are the components of $\mathbf{A}$ expressed instantaneously in terms of the three rotating coordinate directions, for which the unit vectors are $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$. In the rotating frame, these coordinate directions are fixed, and so

$$\left(\frac{D\mathbf{A}}{Dt}\right)_{rot} = \widehat{\mathbf{x}}\frac{DA_x}{Dt} + \widehat{\mathbf{y}}\frac{DA_y}{Dt} + \widehat{\mathbf{z}}\frac{DA_z}{Dt} \ .$$

However, *viewed from the inertial frame*, the coordinate directions in the

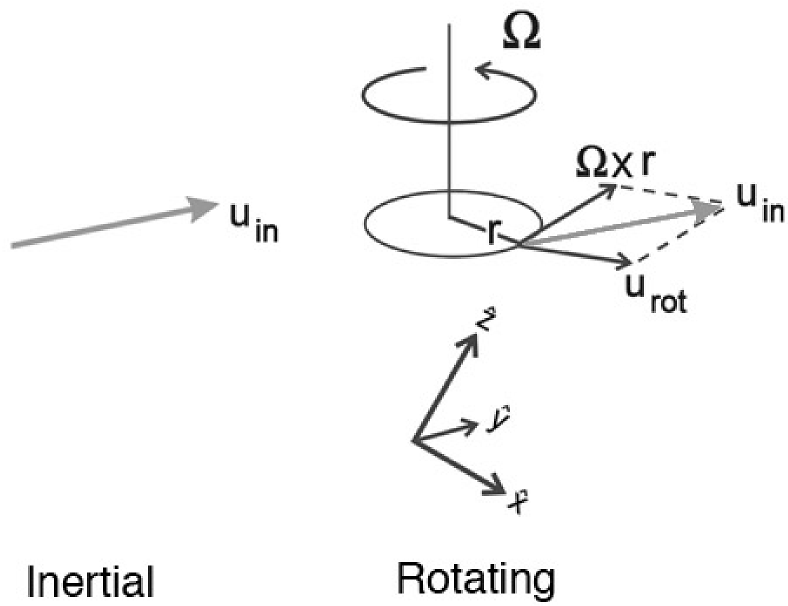

Figure 6.9: On the left is the velocity vector of a particle $\mathbf{u}_{in}$ in the inertial frame. On the right is the view from the rotating frame. The particle has velocity $\mathbf{u}_{rot}$ in the rotating frame. The relation between $\mathbf{u}_{in}$ and $\mathbf{u}_{rot}$ is $\mathbf{u}_{in} = \mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}$ where $\mathbf{\Omega} \times \mathbf{r}$ is the velocity of a particle fixed (not moving) in the rotating frame at position vector $\mathbf{r}$. The relationship between the rate of change of any vector $\mathbf{A}$ in the rotating frame and the change of $\mathbf{A}$ as seen in the inertial frame is given by: $\left(\frac{D\mathbf{A}}{Dt}\right)_{in} = \left(\frac{D\mathbf{A}}{Dt}\right)_{rot} + \mathbf{\Omega} \times \mathbf{A}$.

rotating frame are not fixed, but are rotating at rate $\mathbf{\Omega}$, and so

$$\begin{aligned}\left(\frac{D\widehat{\mathbf{x}}}{Dt}\right)_{in} &= \mathbf{\Omega}\times\widehat{\mathbf{x}}\ ,\\ \left(\frac{D\widehat{\mathbf{y}}}{Dt}\right)_{in} &= \mathbf{\Omega}\times\widehat{\mathbf{y}}\ ,\\ \left(\frac{D\widehat{\mathbf{z}}}{Dt}\right)_{in} &= \mathbf{\Omega}\times\widehat{\mathbf{z}}\ .\end{aligned}$$

Therefore, operating on Eq.(6.25),

$$\begin{aligned}\left(\frac{D\mathbf{A}}{Dt}\right)_{in} &= \widehat{\mathbf{x}}\frac{DA_x}{Dt}+\widehat{\mathbf{y}}\frac{DA_y}{Dt}+\widehat{\mathbf{z}}\frac{DA_z}{Dt}\\ &\quad+\left(\frac{D\widehat{\mathbf{x}}}{Dt}\right)_{in}A_x+\left(\frac{D\widehat{\mathbf{y}}}{Dt}\right)_{in}A_y+\left(\frac{D\widehat{\mathbf{z}}}{Dt}\right)_{in}A_z\\ &= \widehat{\mathbf{x}}\frac{DA_x}{Dt}+\widehat{\mathbf{y}}\frac{DA_y}{Dt}+\widehat{\mathbf{z}}\frac{DA_z}{Dt}\\ &\quad+\mathbf{\Omega}\times\left(\widehat{\mathbf{x}}A_x+\widehat{\mathbf{y}}A_y+\widehat{\mathbf{z}}A_z\right)\ ,\end{aligned}$$

whence

$$\left(\frac{D\mathbf{A}}{Dt}\right)_{in}=\left(\frac{D\mathbf{A}}{Dt}\right)_{rot}+\ \mathbf{\Omega}\times\mathbf{A}\ . \tag{6.26}$$

which yields our transformation rule for the operator $\frac{D}{Dt}$ acting on a vector.

Setting $\mathbf{A}=\mathbf{r}$, the position vector of the particle in the rotating frame, we arrive at Eq.(6.24). To write down the rate of change of velocity following a parcel of fluid in a rotating frame, $\left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{in}$, we set $\mathbf{A}\longrightarrow\mathbf{u}_{in}$ in Eq.(6.26) using Eq.(6.24) thus:

$$\begin{aligned}\left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{in} &= \left[\left(\frac{D}{Dt}\right)_{rot}+\mathbf{\Omega}\times\right]\left(\mathbf{u}_{rot}+\mathbf{\Omega}\times\mathbf{r}\right)\\ &= \left(\frac{D\mathbf{u}_{rot}}{Dt}\right)_{rot}+2\mathbf{\Omega}\times\mathbf{u}_{rot}+\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}\ ,\end{aligned} \tag{6.27}$$

since, by definition

$$\left(\frac{D\mathbf{r}}{Dt}\right)_{rot}=\mathbf{u}_{rot}\ .$$

Eq.(6.27) is a more general statement of Eq.(6.18): we see that there is a one-to-one correspondence between the terms.

6.6.3 The rotating equation of motion

We can now write down our equation of motion in the rotating frame. Substituting from Eq.(6.27) into the inertial-frame equation of motion (6.6), we have, in the rotating frame,

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\widehat{\mathbf{z}} \;=\; \underbrace{-2\boldsymbol{\Omega}\times\mathbf{u}}_{\substack{\text{Coriolis}\\ \text{accel}^n}} + \underbrace{-\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}}_{\substack{\text{Centrifugal}\\ \text{accel}^n}} + \mathcal{F} \tag{6.28}$$

where we have dropped the subscripts "$_{rot}$" and it is now understood that $D\mathbf{u}/Dt$ and $\mathbf{u}$ refer to the rotating frame.

Note that Eq.(6.28) is the same as Eq.(6.6) except that $\mathbf{u} = \mathbf{u}_{rot}$ and 'apparent' accelerations, introduced by the rotating reference frame, have been placed on the right-hand side of Eq.(6.28) [just as in Eq.(6.21)]. The apparent accelerations have been given names: the centrifugal acceleration $(-\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r})$ is directed radially outward (Fig.6.9) the Coriolis acceleration $(-2\boldsymbol{\Omega}\times\mathbf{u})$ is directed 'to the right' of the velocity vector if $\boldsymbol{\Omega}$ is anticlockwise, sketched in Fig.6.10. We now discuss these apparent accelerations in turn.

Centrifugal acceleration

As noted above, $-\,\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}$ is directed radially outwards. If no other forces were acting on a particle, the particle would accelerate outwards. Because centrifugal acceleration can be expressed as the gradient of a potential thus

$$-\,\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r} = \nabla\left(\frac{\Omega^2 r^2}{2}\right)$$

where r is the distance normal to the rotating axis (see Fig.6.9) it is convenient to combine $\nabla\left(\frac{\Omega^2 r^2}{2}\right)$ with $g\widehat{\mathbf{z}} = \nabla\,(gz)$, the gradient of the gravitational potential gz, and write Eq.(6.28) in the succinct form:

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\phi = -2\boldsymbol{\Omega}\times\mathbf{u} + \mathcal{F} \tag{6.29}$$

where

$$\phi = gz - \frac{\Omega^2 r^2}{2} \tag{6.30}$$

is a modified (by centrifugal accelerations) gravitational potential 'measured' in the rotating frame.[8] In this way gravitational and centrifugal accelerations can be conveniently combined in to a 'measured' gravity, $\nabla\phi$. This is discussed in Section 6.6.4 at some length in the context of experiments with a parabolic rotating surface GFD Lab IV.

Coriolis acceleration

The first term on the rhs of Eq.(6.28) is the "Coriolis acceleration"[9] — it describes a tendency for fluid parcels to turn, as shown in Fig.6.10 and investigated in GFD Lab V. (Note that in this figure, the rotation is anticlockwise, i.e. $\Omega > 0$, like that of the northern hemisphere viewed from above the north pole; for the southern hemisphere, the effective sign of rotation is reversed, see Fig.6.17.)

In the absence of any other forces acting on it, a fluid parcel would accelerate as

$$\frac{D\mathbf{u}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{u} \tag{6.31}$$

With the signs shown, the parcel would turn to the right in response to the Coriolis force (to the left in the southern hemisphere). Note that since, by definition, $(\boldsymbol{\Omega} \times \mathbf{u}) \cdot \mathbf{u} = \mathbf{0}$, the Coriolis force is *workless*: it does no work, but merely acts to change the flow direction.

To breath some life in to these acceleration terms we will now describe experiments with a parabolic rotating surface.

[8] Note that $\frac{\phi}{g} = 1 - \frac{\Omega^2 r^2}{2g}$ is directly analogous to Eq.(6.20) adopted in the analysis of the radial inflow experiment.

[9]  Gustave Gaspard Coriolis (1792-1843). French mathematician who discussed what we now refer to as the "Coriolis force" in addition to the already-known centrifugal force. The explanation of the effect sprang from problems of early 19th-century industry, i.e. rotating machines such as water-wheels.

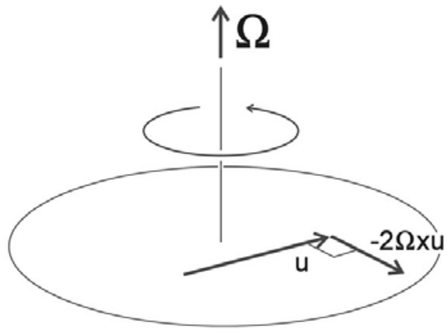

Figure 6.10: A fluid parcel moving with velocity u_{rot} in a rotating frame experiences a Coriolis acceleration $-2\mathbf{\Omega}\times\, u_{rot}$, directed 'to the right' of u_{rot} if, as here, $\mathbf{\Omega}$ is directed upwards, corresponding to anticlockwise rotation.

6.6.4 GFD Lab IV and V: Experiments with Coriolis forces on a parabolic rotating table

GFD Lab IV: studies of parabolic equipotential surfaces

We fill a tank with water, set it turning and leave it until it comes in to solid body rotation, i.e., the state in which fluid parcels have zero velocity in the rotating frame of reference. This is easily determined by viewing a paper dot floating on the free surface from a co-rotating camera. We note that the free-surface of the water is not flat; it is depressed in the middle and rises up to its highest point along the rim of the tank, as sketched in Fig.6.11. What's going on?

In solid-body rotation, $\mathbf{u} = \mathbf{0}$, $\mathcal{F} = 0$, and so Eq.(6.29) reduces to $\frac{1}{\rho}\nabla p + \nabla\phi = 0$ (a generalization of hydrostatic balance to the rotating frame). For this to be true:

$$\frac{p}{\rho} + \phi = \text{constant}$$

everywhere in the fluid (note that here we are assuming $\rho = $ constant). Thus on surfaces where $p = $ constant, ϕ must be constant too: i.e., p and ϕ surfaces must be coincident with one another.

At the free surface of the fluid, $p = 0$. Thus, from Eq.(6.30)

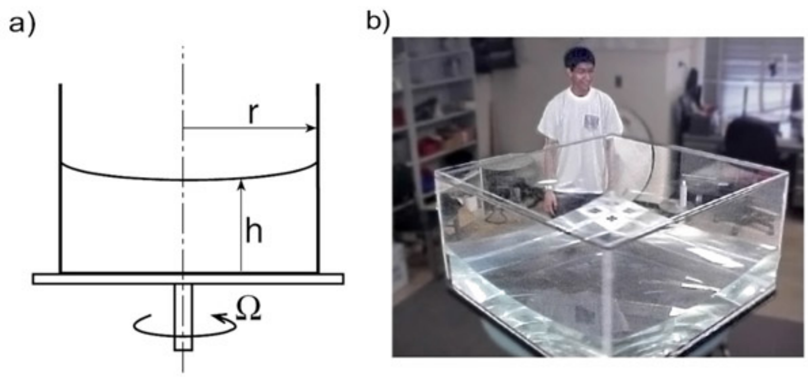

Figure 6.11: (a) Water placed in a rotating tank and insulated from external forces (both mechanical and thermodynamic) eventually comes in to solid body rotation in which the fluid does not move relative to the tank. In such a state the free surface of the water is not flat but takes on the shape of a parabola given by Eq.(6.33). (b) parabolic free surface of water in a tank of 1 m square rotating at $\Omega = 20$ rpm.

$$gz - \frac{\Omega^2 r^2}{2} = \text{constant} \tag{6.32}$$

the modified gravitational potential. We can determine the constant of proportionality by noting that at $r = 0$, $z = h(0)$, the height of the fluid in the middle of the tank (see Fig.6.11a). Hence the depth of the fluid h is given by:

$$h(r) = h(0) + \frac{\Omega^2 r^2}{2g} \tag{6.33}$$

where r is the distance from the axis or rotation. Thus the free surface takes on a parabolic shape: it tilts so that it is always perpendicular to the vector $\mathbf{g}^*$ (gravity modified by centrifugal forces) given by $\mathbf{g}^* = -g\widehat{\mathbf{z}} - \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r}$. If we hung a plumb line in the frame of the rotating table it would point in the direction of $\mathbf{g}^*$ i.e. slightly outwards rather than directly down. The surface given by Eq.(6.33) is the reference to which H is compared to define η in Eq.(6.20).

Let us estimate the tilt of the free surface of the fluid by inserting numbers into Eq.(6.33) typical of our tank. If the rotation rate is 10 rpm (so $\Omega \simeq 1$ s^{-1}) and the radius of the tank is 0.30 m, then with $g = 9.81\ \mathrm{m\,s^{-2}}$, we find $\frac{\Omega^2 r^2}{2g} \sim 5\,\mathrm{mm}$, a noticeable effect but a small fraction of the depth to which the tank is typically filled. If one uses a large tank at high rotation, however — see Fig.6.11(b) in which a 1 m square tank was rotated at a rate of 20 rpm — the distortion of the free surface can be very marked. In this case $\frac{\Omega^2 r^2}{2g} \sim 0.2\,\mathrm{m}$.

It is very instructive to construct a smooth parabolic surface on which one can roll objects. This can be done by filling a large flat-bottomed pan with resin on a turntable and letting the resin harden while the turntable is left running for several hours (this is how parabolic surfaces are made). The resulting parabolic surface can then be polished to create a low friction surface. The surface defined by Eq.(6.32) is an equipotential surface of the rotating frame and so a body carefully placed on it at rest (in the rotating frame) should remain at rest. Indeed if we place a ball-bearing on the rotating parabolic surface — and make sure that the table is rotating at the same speed as was used to create the parabola! — then we see that it does not fall in to the center, but instead finds a state of rest in which the component of gravitational force, g_H resolved along the parabolic surface is exactly balanced by the outward-directed horizontal component of the centrifugal force, $(\Omega^2 r)_H$, as sketched in Fig.6.12 and seen in action in Fig.6.13.

GFD Lab V: visualizing the Coriolis force We can use the parabolic surface discussed in Lab IV, in conjunction with a dry ice 'puck', to help us visualize the Coriolis force. On the surface of the parabola, $\phi =$ constant and so $\nabla\phi = 0$. We can also assume that there are no pressure gradients acting on the puck because the air is so thin. Furthermore, the gas sublimating off the bottom of the dry ice almost eliminates frictional coupling between the puck and the surface of the parabolic dish; thus we may also assume $\mathcal{F} = 0$. Hence the balance Eq.(6.31) applies.[10]

We can play games with the puck and study its trajectory on the parabolic turntable, both in the rotating and laboratory frames. It is useful to view the puck from the rotating frame using an overhead co-rotating camera. Fig.6.14 plots the trajectory of the puck in the inertial (left) and in the rotating

[10] A ball-bearing can also readily be used for demonstration purposes, but it is not quite as effective as a dry ice puck.

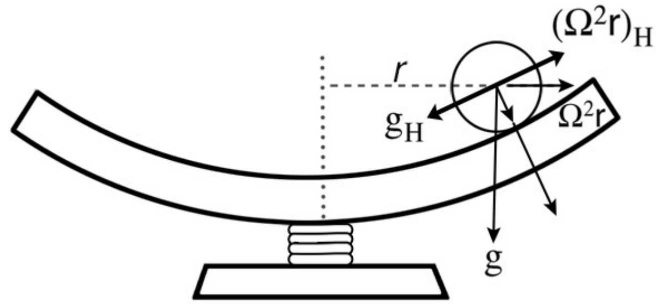

Figure 6.12: If a parabola of the form given by Eq.(6.33) is spun at rate Ω, then a ball carefully placed on it at rest does not fall in to the center but remains at rest: gravity resolved parallel to the surface, g_H, is exactly balanced by centrifugal accelerations resolved parallel to the surface, $(\Omega^2 r)_H$.

Figure 6.13: Studying the trajectories of ball bearings on a rotating parabola. A co-rotating camera views and records the scene from above.

(right) frame. Notice that the puck is 'deflected to the right' by the Coriolis force when viewed in the rotating frame if the table is turning anticlockwise (cyclonically). The following are useful reference experiments:

1. We place the puck so that it is motionless in the rotating frame of reference — it follows a circular orbit around the center of the dish in the laboratory frame.

2. We launch the puck on a trajectory that crosses the rotation axis. Viewed from the laboratory the puck moves backwards and forwards along a straight line (the straight line expands out in to an ellipse if the frictional coupling between the puck and the rotating disc is not negligible; see Fig.6.14a). When viewed in the rotating frame, however, the particle is continuously deflected to the right and its trajectory appears as a circle as seen in Fig.6.14b. This is the 'deflecting force' of Coriolis. These circles are called 'inertial circles'. (We will look at the theory of these circles below).

3. We place the puck on the parabolic surface again so that it appears stationary in the rotating frame, but is then slightly perturbed. In the rotating frame, the puck undergoes inertial oscillations consisting of small circular orbits passing through the initial position of the unperturbed puck.

Inertial circles It is straightforward to analyze the motion of the puck in GFD Lab V. We adopt a Cartesian (x, y) coordinate in the rotating frame of reference whose origin is at the center of the parabolic surface. The velocity of the puck on the surface is $\mathbf{u}_{rot} = (u, v)$ where $u_{rot} = dx/dt$ and $v_{rot} = dy/dt$ and we have reintroduced the subscript $_{rot}$ to make our frame of reference explicit. Further we assume that z increases upwards in the direction of Ω.

Rotating frame Let us write out Eq.(6.31) in component form (replacing $\frac{D}{Dt}$ by $\frac{d}{dt}$, the rate of change of a property of the puck). Noting that (see VI, Appendix 13.2):

$$2\mathbf{\Omega} \times \mathbf{u}_{rot} = (0, 0, 2\Omega) \times (u_{rot}, v_{rot}, 0) = (-2\Omega v_{rot}, 2\Omega u_{rot}, 0)$$

the two horizontal components of Eq.(6.31) are:

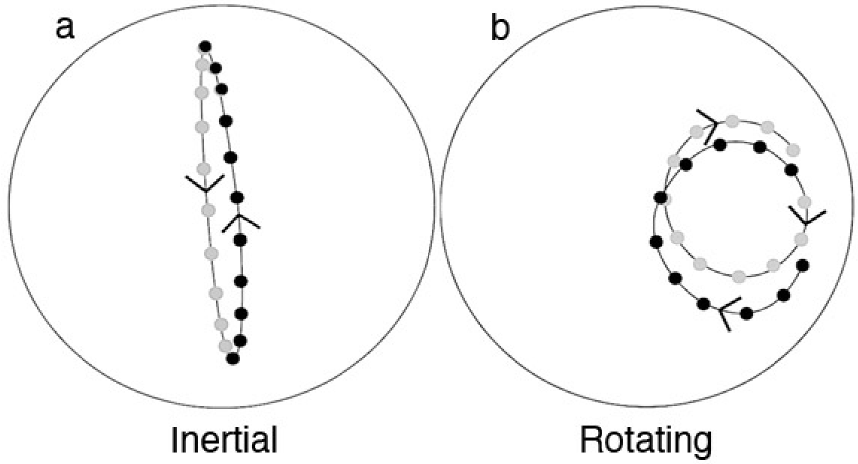

Figure 6.14: Trajectory of the puck on the rotating parabolic surface during one rotation period of the table $\frac{2\pi}{\Omega}$in (a) the inertial frame and (b) the rotating frame of reference. The parabola is rotating in an anticlockwise (cyclonic) sense.

$$\frac{du_{rot}}{dt} - 2\Omega v_{rot} = 0; \frac{dv_{rot}}{dt} + 2\Omega u_{rot} = 0 \tag{6.34}$$

$$u_{rot} = \frac{dx}{dt}; v_{rot} = \frac{dy}{dt}.$$

If we launch the puck from the origin of our coordinate system $x(0) = 0$; $y(0) = 0$ (chosen to be the center of the rotating dish) with speed $u_{rot}(0) = 0$; $v_{rot}(0) = v_o$, the solution to Eq.(6.34) satisfying these boundary conditions is:

$$u_{rot}(t) = v_o \sin 2\Omega t;\ \ v_{rot}(t) = v_o \cos 2\Omega t$$

$$x(t) = \frac{v_o}{2\Omega} - \frac{v_o}{2\Omega} \cos 2\Omega t;\ \ y(t) = \frac{v_o}{2\Omega} \sin 2\Omega t \tag{6.35}$$

The puck's trajectory in the rotating frame is a circle (see Fig.6.15) with a radius of $\frac{v_o}{2\Omega}$. It moves around the circle in a clockwise direction (anticyclonically) with a period $\frac{\pi}{\Omega}$, known as the 'inertial period'. Note from Fig.6.14

that in the rotating frame the puck is observed to complete two oscillation periods in the time it takes to complete just one in the inertial frame.

Inertial frame Now let us consider the same problem but in the non-rotating frame. The acceleration in a frame rotating at angular velocity $\mathbf{\Omega}$ is related to the acceleration in an inertial frame of reference by Eq.(6.27). And so, if the balance of forces is $\frac{D\mathbf{u}_{rot}}{Dt} = -2\mathbf{\Omega} \times \mathbf{u}_{rot}$ these two terms cancel out in Eq.(6.27), and it reduces to[11]:

$$\frac{d\mathbf{u}_{in}}{dt} = \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r} \tag{6.36}$$

If the origin of our inertial coordinate system lies at the center of our dish, then the above can be written out in component form thus:

$$\frac{du_{in}}{dt} + \Omega^2 x = 0; \frac{dv_{in}}{dt} + \Omega^2 y = 0 \tag{6.37}$$

where the subscript $_{in}$ means inertial. This should be compared to the equation of motion in the rotating frame — see Eq.(6.34).

The solution of Eq.(6.37) satisfying our boundary conditions is:

$$u_{in}(t) = 0;\ v_{in}(t) = v_o \cos \Omega t$$

$$x_{in}(t) = 0;\ y_{in}(t) = \frac{v_o}{\Omega} \sin \Omega t \tag{6.38}$$

The trajectory in the inertial frame is a straight line shown in Fig.6.15. Comparing Eqs.(6.35) and (6.38), we see that the length of the line marked out in the inertial frame is **twice** the diameter of the inertial circle in the rotating frame and the frequency of the oscillation is **one-half** that observed in the rotating frame, just as observed in Fig.6.14.

The above solutions go a long way to explaining what is observed in GFD Lab V and expose many of the curiosities of rotating versus non-rotating

[11] Note that if there are no frictional forces between the puck and the parabolic surface, then the *rotation* of the surface is of no consequence to the trajectory of the puck. The puck just oscillates back and forth according to:

$$\frac{d^2 r}{dt^2} = -g\frac{dh}{dr} = -\Omega^2 r$$

where we have used the result that h, the shape of the parabolic surface, is given by Eq.(6.33). This is another (perhaps more physical) way of arriving at Eq.(6.37).

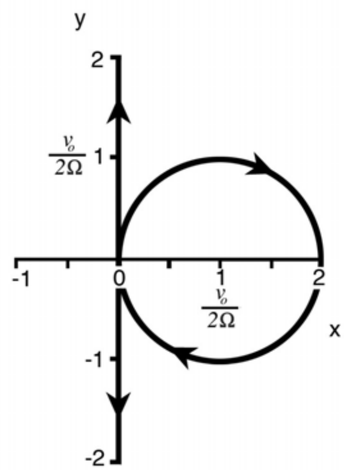

Figure 6.15: Theoretical trajectory of the puck during one complete rotation period of the table, $\frac{2\pi}{\Omega}$, in GFD Lab V in the inertial frame (straight line) and in the rotating frame (circle). We launch the puck from the origin of our coordinate system $x(0) = 0$; $y(0) = 0$ (chosen to be the center of the rotating parabola) with speed $u(0) = 0$; $v(0) = v_o$. The horizontal axes are in units of $\frac{v_o}{2\Omega}$. Observed trajectories are shown in Fig.6.14.

frames of reference. The deflection 'to the right' by the Coriolis force is indeed a consequence of the rotation of the frame of reference: the trajectory in the inertial frame is a straight line!

Before going on we note in passing that the theory of inertial circles discussed here is the same as that of the 'Foucault pendulum', named after the French experimentalist who in 1851 demonstrated the rotation of the earth by observing the deflection of a giant pendulum swinging inside the Pantheon in Paris.

Observations of inertial circles Inertial circles are not just a quirk of this idealized laboratory experiment. They are in fact a common feature of oceanic flows. For example Fig.6.16 shows inertial motions observed by a current meter deployed in the main thermocline of the ocean at a depth of 500 m. The period of the oscillations is:

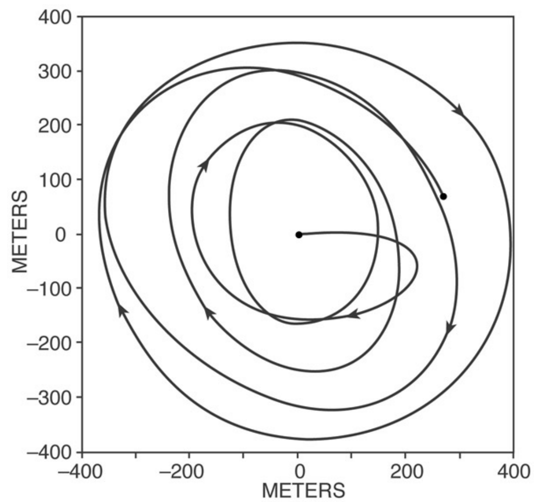

Figure 6.16: Inertial circles observed by a current meter in the main thermocline of the Atlantic Ocean at a depth of 500 m; 28°N, 54°W. Five inertial periods are shown. The inertial period at this latitude is 25.6 h and 5 inertial periods are shown. Courtesy of Carl Wunsch, MIT.

$$\text{Inertial Period} = \frac{\pi}{\Omega \sin \varphi} \tag{6.39}$$

where φ is latitude; the $\sin \varphi$ factor (not present in the theory developed in Section 6.6.4) is a geometrical effect due to the sphericity of the Earth, as we now go on to discuss. At the latitude of the mooring, 28°N, the period of the inertial circles is 25.6 hrs.

6.6.5 Putting things on the sphere

Hitherto our discussion of rotating dynamics has made use of a laboratory turntable in which $\mathbf{\Omega}$ and $\mathbf{g}$ are parallel or antiparallel to one another. But on the spherical Earth $\mathbf{\Omega}$ and $\mathbf{g}$ are not aligned and we must take into account these geometrical complications, illustrated in Fig.6.17. We will now show that, by exploiting the thinness of the fluid shell (Fig.1.1) and the overwhelming importance of gravity, the equations of motion that govern the fluid on the rotating spherical Earth are essentially the same as those that govern the motion of the fluid of our rotating table if $2\mathbf{\Omega}$ is replaced by (what

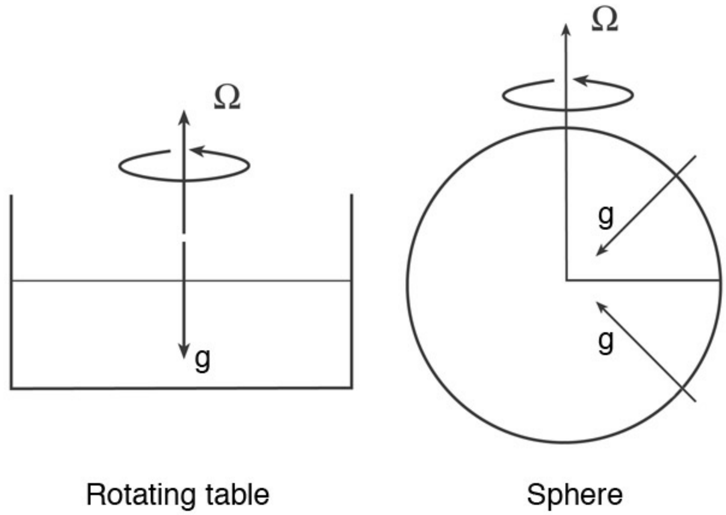

Figure 6.17: In the rotating table used in the laboratory Ω and g are always parallel or (as sketched here) anti-parallel to one another. This should be contrasted with the sphere.

is known as) the 'Coriolis parameter' $f = 2\Omega \sin\varphi$. This is because it turns out that a fluid parcel on the rotating earth "feels" a rotation rate of only $2\boldsymbol{\Omega}\sin\varphi$ — $2\boldsymbol{\Omega}$ resolved in the direction of gravity, rather than the full $2\boldsymbol{\Omega}$.

The centrifugal force, modified gravity and geopotential surfaces on the sphere

Just as on our rotating table, so on the sphere the centrifugal term on the right of Eq.(6.28) modifies gravity and hence hydrostatic balance. For an inviscid fluid at rest *in the rotating frame*, we have

$$\frac{1}{\rho}\nabla p = -g\widehat{\mathbf{z}} - \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r} \ .$$

Now, consider Fig. 6.18.

The centrifugal vector $\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}$ has magnitude $\Omega^2 r$, directed outward normal to the rotation axis, where $r = (a+z)\cos\varphi \simeq a\cos\varphi$, a is the mean Earth radius, z is the altitude above the spherical surface with radius a, and

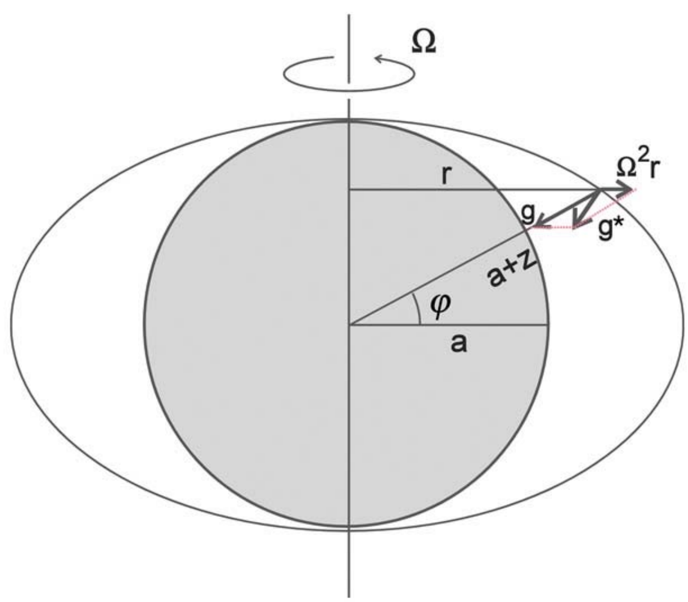

Figure 6.18: The centrifugal vector $\Omega \times \Omega \times r$ has magnitude $\Omega^2 r$, directed outward normal to the rotation axis. Gravity, g, points radially inwards to the center of the Earth. Over geological time the surface of the Earth adjusts to make itself an equipotential surface — close to a reference ellipsoid — which is always perpendicular the the vector sum of $\mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}$ and $\mathbf{g}$. This vector sum is 'measured' gravity: $\mathbf{g}^* = -g\widehat{\mathbf{z}} - \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}$.

φ is latitude, and where the 'shallow atmosphere' approximation has allowed us to write $a + z \simeq a$. Hence on the sphere Eq.(6.30) becomes:

$$\phi = gz - \frac{\Omega^2 a^2 \cos^2 \varphi}{2}$$

defining the modified gravitational potential on the Earth. At the axis of rotation the height of a geopotential surface is geometric height, z (because $\varphi = \frac{\pi}{2}$). Elsewhere geopotential surfaces are defined by:

$$z^* = z + \frac{\Omega^2 a^2 \cos^2 \varphi}{2g} \,. \tag{6.40}$$

We can see that Eq.(6.40) is exactly analogous to the form we derived in Eq.(6.33) for the free surface of a fluid in solid body rotation in our rotating table when we realize that $r = a \cos \varphi$ is the distance normal to the axis of rotation. A plumb line is always perpendicular to z^* surfaces, and modified gravity is given by $\mathbf{g}^* = -\nabla z^*$.

Since (with $\Omega = 7.27 \times 10^{-5}\,\mathrm{s}^{-1}$ and $a = 6.37 \times 10^6\,\mathrm{m}$) $\Omega^2 a^2/2g \approx 11\,\mathrm{km}$, geopotential surfaces depart only very slightly from a sphere, being 11 km higher at the equator than at the pole. Indeed, the figure of the Earth's surface — the geoid — adopts something like this shape, actually bulging more than this at the equator (by 21 km, relative to the poles)[12]. So, if we adopt these (very slightly) aspherical surfaces as our basic coordinate system, then relative to these coordinates the centrifugal force disappears (being subsumed into the coordinate system) and hydrostatic balance again is described (to a very good approximation) by Eq.(6.8). This is directly analogous, of course, to adopting the surface of our parabolic turntable as a coordinate reference system in GFD Lab V.

Components of the Coriolis force on the sphere: the Coriolis parameter

We noted in Chapter 1 that the thinness of the atmosphere allows us (for most purposes) to use a local Cartesian coordinate system, neglecting the Earth's curvature. First, however, we must figure out how to express the Coriolis force in such a system. Consider Fig.6.19.

[12] The discrepancy between the actual shape of the Earth and the prediction Eq.(6.40) is due to the mass distribution of the equatorial bulge which is not taken in the calculation presented here.

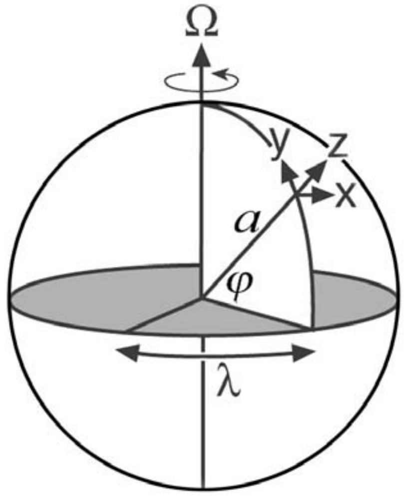

Figure 6.19: At latitude φ, longitude, λ, we define a local coordinate system such that the three coordinates in the (x, y, z) directions point (eastward, northward, upward): $dx = a\cos\varphi d\lambda$; $dy = ad\varphi$; $dz = dz$ where a is the radius of the earth. The velocity is $\mathbf{u} = (u, v, w)$ in the directions (x, y, z). See also Section 13.2.3.

At latitude φ, we define a local coordinate system such that the three coordinates in the (x, y, z) directions point (eastward, northward, upward), as shown. The components of $\mathbf{\Omega}$ in these coordinates are $(0, \Omega\cos\varphi, \Omega\sin\varphi)$. Therefore, expressed in these coordinates,

$$\begin{aligned}\mathbf{\Omega}\times\mathbf{u} &= (0, \Omega\cos\varphi, \Omega\sin\varphi)\times(u, v, w) \\ &= (\Omega\cos\varphi\, w - \Omega\sin\varphi\, v, \Omega\sin\varphi\, u, -\Omega\cos\varphi\, u)\ .\end{aligned}$$

We can now make two (good) approximations. First, we note that the vertical component competes with gravity and so is negligible if $\Omega u \ll g$. Typically, in the atmosphere $|\mathbf{u}| \sim 10\,\mathrm{m\,s^{-1}}$, so $\Omega u \sim 7\times10^{-4}\,\mathrm{m\,s^{-2}}$, which is utterly negligible compared with gravity (we will see in Chapter 9 that ocean currents are even weaker and so $\Omega u \ll g$ there too). Second, because of the thinness of the atmosphere and ocean, vertical velocities (typically $\leq 1\,\mathrm{cm\ s^{-1}}$) are very much less than horizontal velocities; so we may neglect the term involving w in the x-component of $\mathbf{\Omega}\times\mathbf{u}$. Hence we may write the Coriolis term as

$$\begin{aligned}2\mathbf{\Omega}\times\mathbf{u} &\simeq (-2\Omega\sin\varphi\, v, 2\Omega\sin\varphi\, u, 0) \qquad (6.41)\\ &= f\widehat{\mathbf{z}}\times\mathbf{u}\ ,\end{aligned}$$

latitude	f ($\times 10^{-4}\,\mathrm{s}^{-1}$)	β ($\times 10^{-11}\,\mathrm{s}^{-1}\,\mathrm{m}^{-1}$)
90°	1.46	0
60°	1.26	1.14
45°	1.03	1.61
30°	0.73	1.98
10°	0.25	2.25
0°	0	2.28

Table 6.1: Values of the Coriolis parameter, $f = 2\Omega \sin\varphi$ — Eq.(6.42) — and its meridional gradient, $\beta = \frac{df}{dy} = \frac{2\Omega}{a}\cos\varphi$ — Eq.(10.10) — tabulated as a function of latitude. Here Ω is the rotation rate of the Earth and a is the radius of the Earth.

where:

$$f = 2\Omega \sin\varphi \tag{6.42}$$

is known as the *Coriolis parameter*. Note that $\Omega \sin\varphi$ is the *vertical component* of the Earth's rotation rate — this is the only component that matters (a consequence of the thinness of the atmosphere and ocean). For one thing this means that since $f \to 0$ at the equator, rotational effects are negligible there. Furthermore, $f < 0$ in the southern hemisphere — see Fig.6.17(right). Values of f at selected latitudes are set out in Table 6.1.

We can now write Eq.(6.29) as (rearranging slightly),

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \boldsymbol{\nabla}\phi + f\widehat{\mathbf{z}} \times \mathbf{u} = \mathcal{F}\,. \tag{6.43}$$

where $2\boldsymbol{\Omega}$ has been replaced by $f\widehat{\mathbf{z}}$. Writing this in component form for our *local* Cartesian system (see Fig.6.19), and making the hydrostatic approximation for the vertical component, we have

$$\begin{aligned}
\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv &= \mathcal{F}_x\,, \\
\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu &= \mathcal{F}_y\,, \\
\frac{1}{\rho}\frac{\partial p}{\partial z} + g &= 0\,,
\end{aligned} \tag{6.44}$$

where $(\mathcal{F}_x, \mathcal{F}_y)$ are the (x, y) components of friction (and we have assumed the vertical component of $\mathcal{F}$ to be negligible compared with gravity).

The set, Eq.(6.43) — in component form, Eq.(6.44) — is the starting point for discussions of the dynamics of a fluid in a thin spherical shell on a rotating sphere, such as the atmosphere and ocean.

6.6.6 GFD Lab VI: An experiment on the Earth's rotation

A classic experiment on the Earth's rotation was carried out by Perrot in 1859[13]. It is directly analogous to the radial inflow experiment, GFD Lab III, except that the Earth's spin is the source of rotation rather than a rotating table. Perrot filled a large cylinder with water (the cylinder had a hole in the middle of its base plugged with a cork, as sketched in Fig.6.20) and left it standing for two days. He returned and released the plug. As fluid flowed in toward the drain-hole it conserved angular momentum, thus 'concentrating' the rotation of the Earth, and acquired a 'spin' that was cyclonic (in the same sense of rotation as the Earth)

According to theory below, we expect to see the fluid spiral in the same sense of rotation as the Earth. The close analogue with the radial inflow experiment is clear when one realizes that the container sketched in Fig 6.20 is on the rotating Earth and experiences a rotation rate of $\Omega \times \sin lat$!

Theory We suppose that a particle of water initially on the outer rim of the cylinder at radius r_1 moves inwards conserving angular momentum until it reaches the drain hole, at radius r_o — see Fig. 6.21. The earth's rotation Ω_{earth} resolved in the direction of the local vertical is $\Omega_{earth} \sin\varphi$ where φ is the latitude. Therefore a particle initially at rest relative to the cylinder at radius r_1, has a speed of $v_1 = r_1 \Omega_{earth} \sin\varphi$ in the inertial frame. Its angular momentum is $A_1 = v_1 r_1$. At r_o what is the rate of rotation of the particle?

If angular momentum is conserved, then $A_o = \Omega_o r_o^2 = A_1$, and so the rate of rotation of the ball at the radius r_o is:

$$\Omega_o = \left(\frac{r_1}{r_o}\right)^2 \Omega_{earth} \sin\varphi$$

[13] Perrot's experiment can be regarded as the fluid-mechanical analogue of Foucault's 1851 experiment on the Earth's rotation using a pendulum.

Figure 6.20: A leveled cylinder is filled with water, covered by a lid and left standing for several days. Attached to the small hole at the center of the cylinder is a hose (also filled with water and stopped by a rubber bung) which hangs down in to a pail of water. On releasing the bung the water flows out and, according to theory, should acquire a spin which has the same sense as that of the Earth.

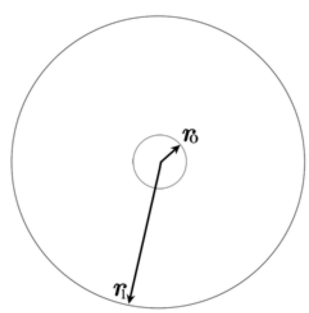

Figure 6.21: The Earth's rotation is magnified by the ratio $\left(\frac{r_1}{r_o}\right)^2 >> 1$, if the drain hole has a radius r_o, very much less than the tank itself, r_1.

Thus if $\frac{r_1}{r_o} >> 1$ the earth's rotation can be 'amplified' by a large amount. For example, at a latitude of 42°N, appropriate for Cambridge, Massachusetts, $\sin\varphi = 0.67$, $\Omega_{earth} = 7.3 \times 10^{-5}\,\mathrm{s}^{-1}$, and if the cylinder has a radius of $r_1 = 30\,\mathrm{cm}$ and the inner hole has radius $r_o = 0.15\,\mathrm{cm}$, we find that $\Omega_o = 1.96$ rad s^{-1}, or a complete rotation in only 3 seconds!

Perrot's experiment, although based on sound physical ideas, is rather tricky to carry out: the initial (background) velocity has to be very tiny ($v << fr$) for the experiment to work, thus demanding great care in set-up. Apparatus such as that shown in Fig.6.20 can be used but the experiment must be repeated many times. More often than not the fluid does indeed acquire the spin of the Earth, swirling cyclonically as it exits the reservoir.

6.7 Further reading

Discussions of the equations of motion in a rotating frame can be found in most texts on atmospheric and ocean dynamics, such as Holton (2004).

6.8 Problems

1. Consider the zonal-average zonal flow, u, shown in Fig.5.20. Concentrate on the vicinity of the subtropical jet near 30°N in winter (DJF). If the $x-$component of the frictional force per unit mass in Eq.(6.44) is
$$\mathcal{F}_x = \nu\nabla^2 u \ ,$$
where the kinematic viscosity coefficient for air is $\nu = 1.34\times10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ and $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, compare the magnitude of this eastward force with the northward or southward Coriolis force and thus convince yourself that the frictional force is negligible. [10° of latitude $\simeq$ 1100 km; the jet is at an altitude of about 10 km. An order-of-magnitude calculation will suffice to make the point unambiguously.]

2. Using only the equation of hydrostatic balance and the rotating equation of motion, show that a fluid cannot be motionless unless its density is horizontally uniform. (Do *not* assume geostrophic balance, but you should assume that a motionless fluid is subjected to no frictional forces.)

3. a. What is the value of the centrifugal acceleration of a particle fixed to the earth at the equator and how does it compare to $\mathbf{g}$? What is the deviation of a plumb line from the true direction to the centre of the earth at $45°N$?

   b. By considering the centrifugal acceleration on a particle fixed to the surface of the earth, obtain an order-of-magnitude estimate for the earth's ellipticity $\left[\frac{r_1 - r_0}{r_0}\right]$, where r_1 is the equatorial radius and r_0 is the polar radius. As a simplifying assumption the gravitational contribution to $\mathbf{g}$ may be taken as constant and directed toward the centre of the earth. Discuss your estimate given that the ellipticity is observed to be $\frac{1}{297}$. You may assume that the mean radius of the earth is 6000 km.

4. A punter kicks a football a distance of 60 m on a field at latitude 45°N. Assuming the ball, until being caught, moves with a constant forward velocity (horizontal component) of $15\,\mathrm{m\,s^{-1}}$, determine the lateral deflection of the ball from a straight line due to the Coriolis effect. [Neglect friction and any wind or other aerodynamic effects.]

5. Imagine that Concord is (was) flying at speed u from New York to London along a latitude circle. The deflecting force due to Coriolis is toward the south. By lowering the left wing ever so slightly the pilot (or perhaps more conveniently the computer on board) can balance this deflection. Draw a diagram of the forces — gravity, uplift normal to the wings and Coriolis — and use it to deduce that the angle of tilt, γ, of the aircraft from the horizontal required to balance the Coriolis force is

$$\tan\gamma = \frac{2\Omega\sin\varphi \times u}{g},$$

   where Ω is the Earth's rotation, the latitude is φ and gravity is g. If $u = 600\,\mathrm{m\,s^{-1}}$, insert typical numbers to compute the angle. What analogies can you draw with atmospheric circulation? [Hint: cf Eq.(7.8).]

6. Consider horizontal flow in circular geometry in a system rotating about a vertical axis with a steady angular velocity Ω. Starting from Eq.(6.29), show that the equation of motion for the azimuthal flow in this geometry is, in the rotating frame (neglecting friction and assuming

2-dimensional flow)

$$\frac{Dv_\theta}{Dt} + 2\Omega v_r \equiv \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + 2\Omega v_r = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \quad (6.45)$$

where (v_r, v_θ) are the components of velocity in the (r, θ) = (radial, azimuthal) directions (see Fig.6.8). [Hint: write out Eq.(6.29) in cylindrical coordinates (see Appendix 13.2.3) noting that $v_r = \frac{Dr}{Dt}$; $v_\theta = r\frac{D\theta}{Dt}$ and that the gradient operator is $\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta})$.]

(a) Assume that the flow is axisymmetric (*i.e.*, all variables are independent of θ). For such flow, angular momentum (relative to an inertial frame) is conserved. This means, since the angular momentum per unit mass is

$$A = \Omega r^2 + v_\theta r \ , \quad (6.46)$$

that

$$\frac{DA}{Dt} \equiv \frac{\partial A}{\partial t} + v_r \frac{\partial A}{\partial r} = 0 \ . \quad (6.47)$$

Show that Eqs.(6.45) and (6.47) are mutually consistent for axisymmetric flow.

(b) When water flows down the drain from a basin or a bath tub, it usually forms a vortex. It is often said that this vortex is anticlockwise in the northern hemisphere, and clockwise in the southern hemisphere. Test this saying by carrying out the following experiment.

Fill a basin or a bath tub (preferably the latter — the bigger the better) to a depth of at least 10 cm, let it stand for a minute or two, and then let it drain. When a vortex forms[14], estimate, as well as you can, its angular velocity, direction, and radius (use small floats, such as pencil shavings, to help you see the flow). Hence calculate the angular momentum per unit mass of the vortex.

[14] A clear vortex (with a "hollow" center, as in Fig.6.7) may not form. As long as there is an identifiable swirling motion, however, you will be able to proceed; if not, try repeating the experiment.

Now, suppose that, at the instant you opened the drain, there was no motion (relative to the rotating Earth). Now if only the vertical component of the Earth's rotation matters, calculate the angular momentum density due to the Earth's rotation at the perimeter of the bath tub or basin. [Your tub or basin will almost certainly not be circular, but assume it is, with an effective radius R such that the area of your tub or basin is πR^2 in order to determine m.]

(c) Since angular momentum should be conserved, then if there was indeed no motion at the instant you pulled the plug, the maximum possible angular momentum per unit mass in the drain vortex should be the same as that at the perimeter at the initial instant (since that is where the angular momentum was greatest). Compare your answers and comment on the importance of the Earth's rotation for the drain vortex. Hence comment on the validity of the saying.

(d) In view of your answer to (c), what are your thoughts on Perrot's experiment, GFD Lab VI?

7. We specialize Eq.(6.44) to two-dimensional, inviscid ($\mathcal{F} = 0$) flow of a homogeneous fluid of density ρ_{ref} thus:

$$\begin{aligned} \frac{Du}{Dt} + \frac{1}{\rho_{ref}}\frac{\partial p}{\partial x} - fv &= 0 \\ \frac{Dv}{Dt} + \frac{1}{\rho_{ref}}\frac{\partial p}{\partial y} + fu &= 0 \end{aligned}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ and the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

(a) By eliminating the pressure gradient term between the two momentum equations and making use of the continuity equation, show that the quantity $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)$ is conserved following the motion: i.e.

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) = 0.$$

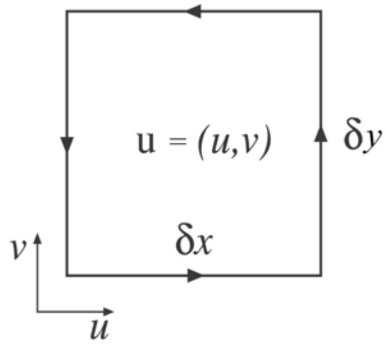

Figure 6.22: Circulation integral schematic.

(b) Convince yourself that

$$\widehat{\mathbf{z}}.\nabla \times \mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{6.48}$$

(see Appendix 13.2.2), i.e. that $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the vertical component of a vector quantity known as the vorticity, $\nabla \times \mathbf{u}$, the curl of the velocity field.

The quantity $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f$ is known as the 'absolute' vorticity and is made up of 'relative' vorticity (due to motion relative to the rotating planet) and 'planetary' vorticity, f, due to the rotation of the planet itself.

(c) By computing the 'circulation' — the line integral of $\mathbf{u}$ about the rectangular element in the (x, y) plane shown in Fig.6.22 — show that:

$$\frac{\text{circulation}}{\text{area enclosed}} = \begin{array}{c}\text{average normal component} \\ \text{of vorticity}\end{array} .$$

Hence deduce that if the fluid element is in solid body rotation then the average vorticity is equal to twice the angular velocity of its rotation.

(d) If the tangential velocity in a hurricane varies like $v = \frac{10^6}{r}\,\mathrm{m\,s^{-1}}$ where r is the radius, calculate the average vorticity between an inner circle of radius 300 km and an outer circle of radius 500 km.

Express your answer in units of planetary vorticity f evaluated at $20^o N$. What is the average vorticity within the inner circle?