Dynamic Equations

At this moment, we would like to introduce the equations for fluid motions.

Lagrange (material) derivative versus Eulerian (local) derivative

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} + w\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial t} + \vec{v} \bullet \nabla \psi \tag{1.1}$$

For a Eulerian perspective, see Marshall and Plumb, Chapter 6 or the Dynamic equation notes under reference books on the course webpage, which is taken from Holton's Dynamic Meteorology book. It is somewhat easier mathematically to look at this from the Lagrangian perspective by using the Reynolds' transport theorem. Consider a finite material volume V(t), i.e. one that contains a fixed collection of matter. Consider how the integral property changes with time

$$\begin{aligned} &\frac{D}{Dt}\int_{V(t)} \psi(x,y,z,t)dV' \\ &= \int_{V(t)} \frac{\partial \psi}{\partial t} dV' + \int_{S(t)} \psi \vec{v} \bullet \vec{n} dS' \end{aligned}$$

The equality is a generalization of the Leibniz's rule for differentiating an integral with variable limits (the volume is changing). Now apply the Gauss' theorem, we have

$$\begin{aligned} &\frac{D}{Dt}\int_{V(t)} \psi(x,y,z,t)dV' \\ &= \int_{V(t)} \left\{ \frac{\partial \psi}{\partial t} + \nabla \bullet (\psi \vec{v}) \right\} dV' \qquad (1.2) \\ &= \int_{V(t)} \left\{ \frac{D\psi}{Dt} + \psi \nabla \bullet \vec{v} \right\} dV' \end{aligned}$$

This is the Reynolds' transport theorem. Now we can derive the equations of fluid motion more readily.

Conservation of mass:

For a fixed collection of matter, the mass does not change. Since Eq. (1.2) holds for arbitrary material volume, we have:

$$\frac{D\rho}{Dt} + \rho \nabla \bullet \vec{v} = 0$$

For incompressible flow, we have $\nabla \bullet \vec{v} = 0$.

Momentum equation:

Now take $\psi = \rho \vec{v}$, we have

$$\frac{D}{Dt}\int_{V(t)}\rho\vec{v}dV'$$
$$=\int_{V(t)}\left\{\frac{D\rho\vec{v}}{Dt}+\rho\vec{v}\nabla\bullet\vec{v}\right\}dV'$$
$$=\int_{V(t)}\left\{\rho\frac{D\vec{v}}{Dt}+\vec{v}\frac{D\rho}{Dt}+\rho\vec{v}\nabla\bullet\vec{v}\right\}dV'$$
$$=\int_{V(t)}\left\{\rho\frac{D\vec{v}}{Dt}+\vec{v}\left[\frac{D\rho}{Dt}+\rho\nabla\bullet\vec{v}\right]\right\}dV'$$
$$=\int_{V(t)}\left\{\rho\frac{D\vec{v}}{Dt}\right\}dV'$$

We have used the mass conservation in the last step.
Now consider the Newton's second law on this fixed collection of matter:

$$\frac{D}{Dt}\int_{V(t)}\rho\vec{v}dV'=\int_{V(t)}\rho\vec{f}dV'+\int_{S(t)}\vec{\tau}\bullet\vec{n}dS'$$

where f is the (specific) body force, and tau is the stress tensor acting on its surface.
Again use the Gauss' theorem, and recognize this works for arbitrary volume, we have

$$\rho\frac{D\vec{v}}{Dt}=\rho\vec{f}+\nabla\bullet\tau$$

For motions in the atmosphere and ocean, the body force is gravity, and the second term on the right hand side can be split into a pressure gradient term and a drag term due to viscosity.

$$\frac{D\vec{v}}{Dt}=\vec{g}-\frac{1}{\rho}\nabla p-\vec{D}$$

The same can be done to the first law of thermodynamics.