September 11

Exer. Let $\phi : G \to K$ be a group hom. Then, $G \times K \to K$, $(g, k) \mapsto \phi(g)\cdot k$ defines a left group ac.

Exer. Putting an equiv. rel on X is the same as writing X as a disjoint union ~~of disjoint~~ of subsets.

① a) $(e,k) \mapsto \phi(e)\cdot k = e_K k = k$

b) $\mu(gh,k) = \phi(gh)\cdot k = \phi(g)\phi(h)\cdot k = \phi(g)(\phi(h)k) = \mu(g, \mu(h,k))$

② (1) x is in the same subset as itself $\Leftrightarrow (x,x) \in R$

(2) x and y are in the same subset $\Leftrightarrow (x,y) \in R \Rightarrow (y,x) \in R$

(3) x in the same subset as y and y in the same subset as z $\Leftrightarrow (x,y), (y,z) \in R$.

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

x in the same subset as z $\qquad$ $(x,z) \in R$

Soln. Recall eq. rels define eq. classes

$[x] = \{ y \in X \text{ s.t. } y \sim x, \text{ i.e. } (y,x) \in R \}$

we know from last time $x \in [x]$

$y \sim x \Rightarrow [y] = [x]$

every el. belongs in some eq. class

$\Downarrow$

$\coprod [x] = X$

disjoint union: no overlap.

Rmk. Fix a gp. action $G \times X \to X$, $(g, x) \mapsto gx$ & consider the set of orbits of this action $G \backslash X$

Ex. $G = S^1$, $X = \mathbb{C}$

There's an orbit for every $r \in \mathbb{R}_{\geq 0}$, i.e. there is a bijection

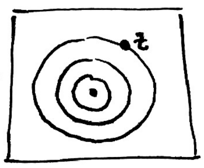
$S^1 \backslash \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$

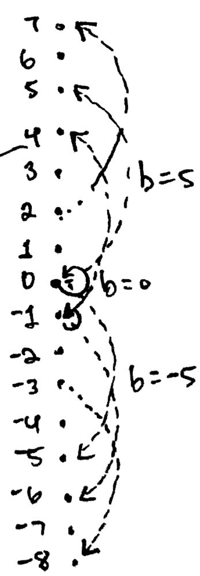
$\cup$

$[z] = \mathcal{O}_z \longmapsto \|z\|$

$\|$

$\{ z' \mid \|z'\| = \|z\| \}$





Ex. Let $G = \mathbb{Z}$, $H = \{0, \pm N, \pm 2N, \dots\}$ for a fixed N

Then the function $G \times H \to G$, $(a, b) \mapsto a+b$

then $\mathbb{Z}/N\mathbb{Z} \cong \{0, 1, \dots, N-1\}$

Explicitly, $\mathbb{Z}/N\mathbb{Z} = \{ \mathcal{O}_0 = \{0, \pm N, \pm 2N, \dots\}, \mathcal{O}_1 = \{1, 1 \pm N, \dots\}, \dots, \mathcal{O}_{N-1} = \{N-1, N-1 \pm N, \dots\} \}$

So If something is a left action $G \times X \to X$, you mod out on the left: $\frac{X}{G}$.

Defn. Let $X \subset G$ be a subgroup. For any $g \in G$ let

$$gH = \{g' \in G \mid g' = gh \text{ for some } h \in H\}$$
$$= \{\text{set of all } gh\}$$
$$= \mathcal{O}_g \text{ under the right group action of } H \text{ on } G.$$

Exer. Show $g_1 H = g_2 H$ iff $g_2^{-1} g_1 \in H$.

If $g_1 H = g_2 H$ then for $g' \in g_1 H$, $g' = g_1 h_1 = g_2 h_2$ for some $h_1, h_2 \in H$.
Then $g_2^{-1} g_1 h_1 = g_2^{-1} g_2 h_2 = h_2 \Rightarrow g_2^{-1} g_1 = g_2^{-1} g_1 h_1 h_1^{-1} = h_2 h_1^{-1} \in H$.
If $g_2^{-1} g_1 \in H$ then let $g' \in g_1 H$
$\Rightarrow g' = g_1 h$ for $h \in H$
$\Rightarrow g_2^{-1} g' = g_2^{-1} g_1 h \in H$
$\Rightarrow g_2^{-1} g' \in H$
$\Rightarrow g_2 (g_2^{-1} g') = g' \in g_2 H$

Similarly, $g' \in g_2 H \Rightarrow g' \in g_1 H$.

Defn. gH is called the <u>left coset</u> of H in G respect to g.

orbit of subgroup acting on parent group.

Quotient Groups

Motivation: In class last time we defined a group structure on $\mathbb{Z}/n\mathbb{Z}$ by
$m(a,b) = a+b \mod n$.

On the other hand we wrote out a set of orbits

$$G/H = \mathbb{Z}/n\mathbb{Z} = \begin{Bmatrix} \mathcal{O}_0 \\ \mathcal{O}_1 \\ \vdots \\ \mathcal{O}_{n-1} \end{Bmatrix}$$

whenever we have a subgroup $H \subset G$ that satisfies some property, this set of orbits G/H admits/inherits a group structure

<u>Q</u>: when does this happen?

<u>Question</u>: when does the fn

$$G/H \times G/H \to G/H$$
$$(g_1 H, g_2 H) \mapsto g_1 g_2 H$$

make G/H a group.