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Fix a group action of G on X $(G \times X \to X)$

<u>Def.</u> A subset $I \subset X$ is called an <u>orbit</u> of this action if the following hold

(1) $I \neq \emptyset$

(2) $\forall y_1, y_2 \in I, \exists g \in G$ s.t. $gy_1 = y_2$ (you can get between any two elements w/ the group action)

(3) $\forall y \in I, \forall g \in G, gy \in I$

<u>Prop</u>. Let I be an orbit. Then $\forall x \in I$, $I = \mathcal{O}_x$ $(:= \{y \in X \mid y = gx\})$

<u>Proof.</u> $I \subset \mathcal{O}_x$ by (2): If $y \in I$, $y = gx$

$\mathcal{O}_x \subset I$ by (3): Set $x = y$ then $gy \in I$ $\forall g \in G$ $\square$

<u>Corollary</u>. (i) Every element $x \in X$ is contained in some orbit $(\mathcal{O}_x)$

(ii) If I, J are orbits, either they have no intersection $I \cap J = \emptyset$ or they are equal $I = J$

(iii) $\mathcal{O}_x = \mathcal{O}_y \iff \exists g \in G$ s.t. $gx = y$

<u>(Trick) Question</u>: For what H does the fn

$$G/H \times G/H \to G/H$$
$$g_1 H, g_2 H \mapsto (g_1 g_2) H$$

make G/H a group?

My thoughts:

$g_1^{-1} H = (g_1 H)^{-1}$

$g_1^{-1} h$ $\forall h$

$g h$ $\forall h$

$(g_1 h)^{-1} = h^{-1} g_1^{-1} = g_1^{-1} h'$ $\in g_1 H$

$\boxed{g_1^{-1} = h g_1^{-1} h'}$

$g_1 h g_1^{-1} h'$

$H \overset{?}{=} \{(g_1 g_2 g_1^{-1} g_2^{-1})\}$

$hg \in gH$

$hg = gh'$

$\Downarrow$

$hgh'^{-1}g^{-1} = e$

<u>Problem</u>: Suppose $g_1' H = g_1 H$ and $g_2' H = g_2 H$. Does $g_1 g_2 H = g_1' g_2' H$?

My thoughts:

$g_1 g_2 h = g_1' g_2'$ $\Rightarrow h = g_2^{-1} g_1^{-1} g_1' g_2'$ $\qquad g_1' = h_1 g_1 \qquad g_2' = h_2 g_2$

$\Rightarrow$ $\cancel{h = g_2^{-1} g_1^{-1} h_1 g_1 h_2 g_2}$

$\cancel{g_2 g_1 h g_2^{-1} = h_1 g_1 h_2}$ ← ugly

Prop. Suppose H satisfies $\forall g \in G,\ gHg^{-1} = H$
Then the problem ~~as~~ has an affirmative answer.

Ex. • All abelian groups G, you can make any subgroup H work
• For any group G, $H = \{e_G\}$ works and G/H isomorphic to G.

$\boxed{\text{Proof of proposition}}$ (the idea) ("good midterm question")

To simplify, assume $g_2' = g_2$

Note: $g_1'H = g_1 \Rightarrow g_1' = g_1 h_1$ for some ~~h~~ $h \in H$

To show $g_1' g_2 H = g_1 g_2 H$, want to show that for $h \in H$

$$g_1' g_2 h \in g_1 g_2 H$$

Since $g_1' \in g_1 H$

$$g_1' g_2 h = g_1 h_1 g_2 h$$
$$= g_1 (g_2 h_2 g_2^{-1}) g_2 h$$
$$= g_1 g_2 h_2 h \in g_1 g_2 H$$

conjugation by g_2

Def. A subgroup $H \subset G$, is called "normal" (in G) iff. $\forall g \in G,\ gHg^{-1} = H$.

not a property of H itself but of how H is inside G.

So we answered the question!!

Thm. If H is a normal subgroup of G then the fn

$$G/H \times G/H \to G/H$$
$$(g_1 H, g_2 H) \mapsto g_1 g_2 H$$

~~makes~~ makes G/H into a group.

Note: $\exists$ fn $G \to G/H$. This is a group homomorphism because
$g \mapsto gH$

Pf. Call the fn $q : G \to G/H$.

$$q(g_1 g_2) = g_1 g_2 H = g_1 H g_2 H = q(g_1) q(g_2)$$

We are "collapsing H":

~~Prop~~. $q^{-1}(e_{G/H}) = \{g \in G \text{ s.t. } q(g) = e_{G/H}\}$
$= \{g \in G \text{ s.t. } q(g) = H\}$
$= H.$

In general, when we have homomorphisms between groups, we wanna know what gets collapsed.

<u>Defn</u>. Let $\phi: G \to K$ be a group homomorphism. The kernel of ϕ, denoted $\ker(\phi)$, is the set $\phi^{-1}(e_K) = \{g \in G \mid \phi(g) = e_K\}$

<u>Thm</u>. A subgroup $H \subset G$ is normal (in G) <u>iff</u> $H = \ker(\phi)$ for some group homomorphism

$$\phi: G \to K.$$