Sep. 14

Last time on Math 122

- G a group $H \subseteq G$ subgroup

  Note: $G \xrightarrow{\pi} G/H$ is always defined, but H normal makes G/H into a group

  $g \longmapsto gH$

- **Thm.** Assume H is normal in G then the function

  $$G/H \times G/H \longrightarrow G/H$$

  $$(g_1H, g_2H) \longmapsto g_1g_2H$$

  is well-defined and makes G/H a group.

- **Def.** A subgroup $H \subseteq G$ is called normal in G when $\forall g \in G$

  $$gHg^{-1} = H$$

  $$gHg^{-1} = \{x \in G \mid x = ghg^{-1}, h \in H\}$$

  (so $\forall g \in G, h \in G, ghg^{-1} \in H$)

\lhd on LaTeX

Defn. When H is normal we write $H \lhd G$. G/H (when $H \lhd G$) is called the quotient gp of G by H.

Pf. that this multiplication fn is well defined

Want to prove: if $g_1H = g_1'H \iff \exists h_1$ s.t. $g_1 = g_1'h_1$

$g_2H = g_2'H \iff \exists h_2$ s.t. $g_2 = g_2'h_2$

then $g_1g_2H = g_1'g_2'H$ **Show:** $g_1g_2 = g_1'g_2'h$ for some $h \in H$

We know $g_1 g_2 = (g_1' h_1)(g_2' h_2)$
$= g_1'(g_2' h_1' (g_2')^{-1}) g_2' h_2 \longleftarrow$ Since $H \trianglelefteq G$ i.e. $gHg^{-1} = H$ $\forall h_1 \ \exists h_1'$ s.t. $h_1 = g_2' h_1' (g_2')^{-1}$ $\Big\}$ $H \subset g_2' H (g_2')^{-1}$
$= g_1' g_2' \underbrace{h_1' h_2}_{h \in H}$ $\square$

Defn. Fix gp. hom. $\phi : G \to L$
The kernel of ϕ is the set
$\ker \phi = \{ g \in G \mid \phi(g) = e_L \}$

Excs. Fix $\phi : G \to L$
Prove: ϕ injection
$\Updownarrow$
$\ker \phi = \{e_G\}$

ϕ injection $\Rightarrow$ If $g \neq e_G$ then $f(g) \neq e_L$
$\Rightarrow \ker \phi = \{e_G\}$

$\ker \phi = \{e_G\} \Rightarrow$ Let $\phi(g) = \phi(h)$
Then $e_L = \phi(gg^{-1})$
$= \phi(h)\phi(g^{-1})$
$= \phi(hg^{-1})$
$\Rightarrow hg^{-1} = e_G$
$\Rightarrow h = g$

Pf ($\Downarrow$) ϕ injective $\Rightarrow \left(\begin{array}{l} \phi(g_1) = \phi(g_2) \\ \Rightarrow g_1 = g_2 \end{array} \right)$
$\Rightarrow$ we know $\phi(e_G) = e_L$
so $\phi(g_1) = e_L \Rightarrow g_1 = e_G$

($\Uparrow$) Suppose $\phi(g_1) = \phi(g_2)$
$\Rightarrow \phi(g_2)^{-1}\phi(g_1) = e_L$
$\Rightarrow \phi(g_2^{-1})\phi(g_1) = e_L$
$\Rightarrow \phi(g_2^{-1} g_1) = e_L$
$\Rightarrow g_2^{-1} g_1 \in \ker \phi$
$\Rightarrow g_2^{-1} g_1 = e_G$
$\Rightarrow g_2 = g_1$ $\square$

Defn. Fix $\phi : G \to L$ gp hom. Then the image of ϕ is
$\mathrm{Im}(\phi) := \{ \ell \in L \mid \ell = \phi(g) \text{ for some } g \in G \}$

Excs (1) $\operatorname{Im}\phi \subset L$ is a subgroup
(2) $\ker \phi \subset L$ is " "
(3) $\ker \phi \trianglelefteq G$

Proof. (1) Need to show $\operatorname{Im}\phi \ni e_L$
closed under inv.
closed under mult.

$\ell = \phi(g) \Rightarrow \ell^{-1} = \phi(g)^{-1} = \phi(g^{-1})$

$\ell_i = \phi(g_i) \Rightarrow \ell_1 \ell_2 = \phi(g_1)\phi(g_2) = \phi(g_1 g_2)$

(2) similar

(3) Fix $g \in G$, $k \in \ker\phi$
Show $gkg^{-1} \in \ker\phi$

$\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1})$
$= \phi(g)\, e_L\, \phi(g)^{-1}$
$= e_L$

$\square$

subgroup: closed under mult & inverses, has Id

(a) $\operatorname{Im}\phi \ni e_L$ because $\phi(e_G) = e_L$
If $\ell \in \operatorname{Im}\phi$ then $\ell = \phi(g) \Rightarrow$
$\phi(g^{-1}) = \phi(g)^{-1} = \ell^{-1}$
If $\ell_1, \ell_2 \in \operatorname{Im}\phi$ $\ell_1 = \phi(g_1)$ $\ell_2 = \phi(g_2)$
$\ell_1 \ell_2 = \phi(g_1 g_2)$

(b) $\ker\phi \ni e_G$ b/c $\phi(e_G) = e_L$
If $g \in \ker\phi$ then $\phi(g^{-1}) = \phi(g)^{-1} = e_L^{-1} = e_L$
$\Rightarrow g^{-1} \in \ker\phi$
$g_1 g_2 \in \ker \Rightarrow \phi(g_1 g_2) = \phi(g_1)\phi(g_2) = e_L$
$\Rightarrow g_1 g_2 \in \ker$

(c) let $h \in \ker\phi$ ~~then~~ and pick any $g \in G$
$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1})$
$= \phi(g)\, e_L\, \phi(g^{-1})$
$= \phi(g)\phi(g)^{-1}$
$= e_L$
$\Rightarrow ghg^{-1} \in \ker\phi$

Rmk. Every subgroup $L' \subset L$ is the image of some group homomorphism.

$L' \to L$
$\ell \mapsto \ell$ ← the inclusion

Thm. (1^{st} isomorphism thm)
Fix a homom $\phi: G \to L$
There exists a natural homomorphism ← we'll prove this later!
$G/\ker\phi \to \operatorname{Im}\phi$ and this is an isomorphism.

Thm Fix $\phi: G \to L$ a gp. hom. Assume $K \subset G$ is a subgroup s.t. $K \subset \ker\phi$
Then

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & L \\ G/K \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

a fn.

$\exists!$ fn $\psi: G/K \to L$ s.t. $\psi \circ \pi = \phi$
(the diagram commutes)
unique

Moreover if $K \trianglelefteq G$ then ψ is a group homomorphism.

Ex. $G = \mathbb{Z}$
L = whatever ($\mathbb{C}$?)

$$\begin{tikzcd} \phi: \mathbb{Z} \arrow[r, "\phi"] \arrow[d] & L \\ \mathbb{Z}/\ker\phi \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

$\mathbb{Z}/\ker\phi$
$=$
$\mathbb{Z}/n\mathbb{Z}$ for some n

so the fn ϕ actually didn't care about the integer, it just cared about the integer mod n.

Pf of "$\exists ! \varphi$" thm

Note that if φ exists, it has to be unique b/c π is a surjection

Define φ to be: $gK \mapsto \phi(g)$

Need to show: If $g_1K = g_2K$ $\phi(g_1) = \phi(g_2)$

$g_1K = g_2K \Rightarrow g_1 = g_2k$ for $k \in K$

$$\begin{aligned} \Rightarrow \phi(g_1) &= \phi(g_2k) \\ &= \phi(g_2)\phi(k) \\ &= \phi(g_2)e_L \\ &= \phi(g_2) \end{aligned}$$

$\square$

When $K \trianglelefteq G$ need to show $\varphi(g_1Kg_2K) = \varphi(g_1K)\varphi(g_2K)$

$$\begin{aligned} \varphi(g_1Kg_2K) &= \varphi(g_1g_2K) = \phi(g_1g_2) \\ &= \phi(g_1)\phi(g_2) = \varphi(g_1K)\varphi(g_2K) \end{aligned}$$

"Universal property of quotient groups"