Lecture 2

Last time, constructed $U \otimes V$.
If this is your first encounter, $U \otimes V$ should feel foreign.
The construction

$$\Bbbk\langle U \times V \rangle / \text{Bilinear rel'ns}$$

is opaque.

$\Bbbk\langle U \times V\rangle :=$ vec space generated by set $U \times V$. Has basis in bijection w/ $U \times V$, and we let $e_{(u,v)}$ denote a basis element last time. $B \subset \Bbbk\langle U \times V\rangle$ was subspace generated by vectors:

- $a e_{(u,v)} - e_{(au,v)}$
- $e_{(u_1+u_2,v)} - e_{(u_1,v)} - e_{(u_2,v)}$.
- etc.

For example, if $\dim U_i = d_i$, $i = 1, 2$, then what is

$$\dim U_1 \otimes U_2 \,?$$

First, some notation.
Recall that, regardless of the construction, $U \otimes V$ receives a bilinear map from the set $U \times V$:

$$i : U \times V \longrightarrow U \otimes V.$$

$$\begin{tikzcd}
U \times V \arrow[r, "f", "\text{given bilinear}"'] \arrow[d, "i"', "\text{bilinear}"] & W \\
U \otimes V \arrow[ur, "\exists!\, j"'] &
\end{tikzcd}$$

universal property of $U \otimes V$

For brevity (?) we will denote the image of $(u, v) \in U \times V$ by $u \otimes v$.
That is,

$$u \otimes v := i(u, v) \in U \otimes V.$$

Why does construction work?

(An elaboration on last time)

By definition, f is <u>bilinear</u> iff

- $f(au, v) = f(u, av) = af(u,v)$
- $f(u_1+u_2, v) = f(u_1, v) + f(u_2, v)$
- $f(u, v_1+v_2) = f(u, v_1) + f(u, v_2)$

and, hence, by defn of $\tilde{f}$ last time,

$$a\tilde{f}(e_{(u,v)}) - \tilde{f}(e_{(au,v)}) = af(u,v) - f(au,v) = 0.$$

That is, bilinearity of f guarantees $\tilde{f}$ vanishes on B. Likewise, if $\tilde{f}$ vanishes on B, and if $\tilde{f}$ is extended linearly from some $f: U \times V \to W$, then f must be bilinear.

Okay, so let's get a feel for $U \otimes V$.

Q: Is $i : U \times V \longrightarrow U \otimes V$ injective?

A: No. For example:

$$i((au, v)) = [e_{(au,v)}] \longleftarrow \text{ie, equiv class of } e_{(au,v)} \text{ mod } B.$$

$$= [a e_{(u,v)}] \longleftarrow \text{since } e_{(au,v)} - a e_{(u,v)} \in B.$$

$$= [e_{(u,av)}] \longleftarrow \text{since } a e_{(u,v)} - e_{(u,av)} \in B.$$

$$= i((u, av)).$$

More succinctly,

$$(au) \otimes v = a(u \otimes v) = u \otimes (av)$$

in $U \otimes V$. However,

Lemma i is bilinear

Pf
$$i(au_1 + u_2, v) = [e_{(au_1+u_2, v)}]$$
$$= [e_{(au_1, v)} + e_{(u_2,v)}]$$
$$= [a e_{(u_1,v)} + e_{(u_2,v)}]$$
$$= a[e_{(u_1,v)}] + [e_{(u_2,v)}]$$
$$= a\, i(u_1, v) + i(u_2, v).$$

Likewise in the v variable. //

But the following makes $U \otimes V$ seem not so bad.

Lemma Let $\{e_i\}_{i \in I}$ be a spanning set in U, and $\{f_j\}_{j \in J}$ a spanning set for V. Then

$$\{e_i \otimes f_j\}_{(i,j) \in I \times J}$$

spans $U \otimes V$.

Pf For this proof, we'll use our construction from last time. Since any quotient map is a surjection, and $\Bbbk\langle U \times V \rangle$ is spanned by $\{e_{(u,v)}\}_{(u,v) \in U \times V}$, any element of $U \otimes V = \Bbbk\langle U \times V \rangle / B$ is a finite linear combination

$$\sum a_{(u,v)} [e_{(u,v)}]$$

$\longleftarrow$ summation runs over some finite cllxn of $(u,v) \in U \times V$

$$\| $$

$$\sum a_{(u,v)}\, u \otimes v$$

$$\|$$

$$\sum a_{(u,v)} \left(\sum u_i e_i\right) \otimes \left(\sum v_j f_j\right)$$

since $u = \sum u_i e_i$, $u_i \in \Bbbk$
$v = \sum v_j e_j$, $v_j \in \Bbbk$

$$\|$$

$$\sum_{(u,v),\, i,j} a_{(u,v)} u_i v_j \; e_i \otimes f_j .$$

//

Using that $\left(\sum_i u_i e_i\right) \otimes v = \sum_i u_i (e_i \otimes v)$ and likewise for $v_j f_j$.

And in fact,

Prop'n If $\{e_i\}_{i\in I}$ is a basis for U, and $\{f_j\}_{j\in J}$ is a basis for V, then

$$\{e_i \otimes f_j\}_{(i,j)\in I\times J}$$

is a basis for $U\otimes V$.

Cor If $\dim U$, $\dim V$ are finite, then

$$\dim(U\otimes V) = \dim U \times \dim V.$$

Pf: $|I\times J| = |I| \times |J|$ when I and J are finite. //

Pf (of Prop): Since $\{e_i\}$, $\{f_j\}$ span, so does $\{e_i\otimes f_j\}$. So we need only prove linear independence of the collection $\{e_i\otimes f_j\}$. Assume $\exists$ some cllxn $a_{ij}\in k$ such that

$$\sum a_{ij}\, e_i\otimes f_j = 0 \quad \in U\otimes V.$$

Then any linear fxn on $U\otimes V$ sends $\sum a_{ij}\, e_i\otimes f_j$ to 0. OTOH, define a bilinear fxn on $U\times V$ as follows:

$$\phi_{i_0 j_0}\left(\sum_i u_i e_i, \sum_j v_j f_j\right) := a_{i_0 j_0}\, u_{i_0} v_{j_0}$$

Here, we use that $\{e_i\}, \{f_j\}$ form a basis to make sure ϕ_{i_0,j_0} is well-defined.

The linear fxn induced on $U\otimes V$ sends $\sum a_{ij}\, e_i\otimes f_j$ to $a_{i_0 j_0}$; hence $a_{i_0 j_0} = 0$ $\forall\, i_0, j_0$. //

Last time, we said

- direct product $\times$
- direct sum $\oplus$
- tensor product $\otimes$, and
- subspaces

were ways to make new vec. spaces out of old. Here's another:

<u>Defn</u> Let V be a vec space/$\Bbbk$. We let

$$V^\vee := \hom_{\Bbbk}(V, \Bbbk)$$

$$:= \{\phi : V \to \Bbbk \text{ s.t. } \phi \text{ is } \Bbbk\text{-linear}\}$$

denote the collection of linear functionals on V. We call $V^\vee$ the <u>dual</u> vector space of V.

The superscript is a "check" mark. $V^\vee$ is vocalized as "Vee check," or "Vee dual."

<u>Ex</u> If V is the vector space of $n \times 1$ column vectors, $V^\vee$ can be identified w/ the space of $1 \times n$ row vectors:

$$[b_1 \cdots b_n] : \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mapsto \sum_{i=1}^{n} b_i a_i$$

__Rmk__ When V is finite-dim, $V^\vee$ is isomorphic to V, but not naturally. Such an isomorphism requires extra data — for example, a choice of basis. (If $e_1, \cdots, e_n$ is a basis for V, let $e_1^\vee, \cdots, e_n^\vee : V \to \mathbb{k}$ be the linear functions s.t.

$$e_i^\vee(e_j) = \begin{cases} 1 & i=j \\ 0 & \text{otherwise.} \end{cases}$$

The map $e_i \mapsto e_i^\vee$ induces a linear isom $V \xrightarrow{\cong} V^\vee$.)

Now let's combine $\otimes$ and $^\vee$.

__Prop__ If U is finite-dim., and V is any vec space, $\exists$ natural isomorphism

$$U^\vee \otimes V \xrightarrow{\cong} \hom_{\mathbb{k}}(U, V)$$

$\mathbb{k}$-vec space of all $\mathbb{k}$-linear maps from U to V.

__Pf__: Your homework.

Rmk When U, V are finite-dim, you've seen this before in some form. For example, let $U = \mathbb{R}^n$, $V = \mathbb{R}^m$. Then

$$\hom_{\mathbb{R}}(U,V) \cong M_{m \times n}(\mathbb{R})$$

$$:= \{m \times n \text{ matrices w/ entries in } \mathbb{R}\}.$$

What does the matrix

$$\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \begin{matrix} 0 & 0 & 0 \\ 0 & \underline{1} & 0 \\ 0 & 0 & 0 \end{matrix} & & \\ & & & \ddots & \end{pmatrix}$$

(1 in $(i,j)^{th}$ spot, 0 elsewhere) represent? It takes $e_j \in \mathbb{R}^n$ and returns the i^{th} basis vector $e_i \in \mathbb{R}^m$. This can be thought of as

$$e_j^\vee \otimes e_i \in U^\vee \otimes V.$$

In general, the matrix entries a_{ij} of a matrix A are the coefficients in

$$\sum a_{ij}\, e_i^\vee \otimes f_j.$$

Under the isomorphism $U^\vee \otimes V \to \hom(U,V)$,

$$\sum a_{ij}\, e_i^\vee \otimes f_j \longmapsto A.$$

Tensor products will return very soon! Let's begin talking about group representations.

Defn Fix a group G. A representation of G is the data of:

- a vector space V
- a linear G-action $G \times V \longrightarrow V$.

Rmk Let $GL(V)$ denote the group of invertible linear maps $V \to V$. Then a G-representation of V is the same thing as a homomorphism

$$\rho : G \longrightarrow GL(V).$$

Ex The *trivial* rep. is the action

$$\begin{aligned} G \times \Bbbk &\longrightarrow \Bbbk \\ (g, t) &\longmapsto t. \end{aligned}$$

This the same data as the constant map

$$\begin{aligned} G &\longrightarrow \Bbbk^{\times} = GL(\Bbbk) = \Bbbk \setminus \{0\} \\ g &\longmapsto 1 \end{aligned}$$

Ex Let $G = C_n := \mathbb{Z}/n\mathbb{Z}$,
the cyclic group of order n.
Set $\mathbb{k} = \mathbb{C}$, and let ζ
be an n^{th} root of unity. Then

$$\begin{aligned} G \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (a, z) &\longmapsto \zeta^a z \end{aligned}$$

is a representation — it just rotates complex numbers. It's the same data as the group homomorphism

$$\begin{aligned} C_n &\longrightarrow \mathbb{C}^\times \\ a &\longmapsto \zeta^a \end{aligned}.$$

Ex Let G act on a set X.
Then G acts on $\mathbb{k}\langle X \rangle$, the free vector space generated by X.
Specifically, $\mathbb{k}\langle X \rangle$ has basis

$$\{e_x\}_{x \in X}$$

and we declare $g e_x := e_{gx}$.

using G-action on X.

Extending by linearity,

$$g\left(\sum_x a_x e_x\right) = \sum_x a_x e_{gx}.$$

Here are two sub-examples:

<u>Ex</u> Any group G acts on itself by left multiplication:

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned}$$

So we have a representation of G on $\Bbbk\langle G\rangle$. This is called the <u>regular representation</u> of G.

<u>Ex</u> Let $G = S_n$ be the symmetric group on n letters. By definition, S_n acts on the set $\{1, \dots, n\}$. Hence S_n acts on $\Bbbk\langle\{1, \dots, n\}\rangle = \Bbbk^n$.

<u>Some notation/conventions:</u>

- When G acts on V or X, we will write $G \circlearrowright V$ or $G \circlearrowright V$.
- "Let V be a G-rep." implies the action.
- $\Bbbk\langle G\rangle$ will be written $\Bbbk G$ when G is a gp. (The group ring.)