Lecture 3

First: In univ prop of $U \otimes V$, the fxn $i: U \times V \to U \otimes V$ is required to be bilinear. I thought I could be fancy, but I was just being wrong.

But w/ i bilinear, $U \otimes V$ satisfies the usual property you've probably seen before:

$U \times V \xrightarrow{i} U \otimes V$ is the bilinear map through which any other bilinear map from $U \times V$ must factor; and uniquely!

Last time:

- Prop: $\{e_i\}$, $\{f_j\}$ basis for U, V respectively $\Rightarrow \{e_i \otimes f_j\}$ basis for $U \otimes V$

and

- examples of G representations.

Today: Develop the analogies

Vec Spaces		G-reps
linear map, isomorphisms		
subspace	$\leftrightsquigarrow$	
$\oplus$		
$\otimes$		
one-dimensional		

Among the examples I gave, one gets a special name! Recall that G acts on itself by left multiplication. As a result, G acts on the vector space

$$\Bbbk\langle G\rangle := \left\{ \sum_{g\in G} a_g e_g \;\middle|\; a_g \in \Bbbk,\ \text{finitely many } a_g \text{ non-zero} \right\}.$$

by

$$\begin{aligned} G \times \Bbbk\langle G\rangle &\longrightarrow \Bbbk\langle G\rangle \\ (g, \textstyle\sum a_h e_h) &\longmapsto \textstyle\sum a_h e_{gh}. \end{aligned}$$

<u>Notation</u>

When the set $X = G$, we write
$$\Bbbk\langle X\rangle =: \Bbbk G.$$
Also, instead of writing $\sum a_g e_g$ w/ e_g basis vectors, we will write $\sum a_g g$.

<u>Rmk</u> $\Bbbk G$ is also a ring; extend group operation linearly:

$$\left(\sum a_g g\right)\left(\sum b_h h\right) := \sum a_g b_h (gh).$$

This is called the <u>group ring</u>.

Defn The above rep. of G on $\mathbb{k}G$ is called the regular representation of G.

Rmk If G finite, it is a $|G|$-dimensional representation.

Finally, whenever you have a new mathematical concept (like gp rep), you should ask what kinds of maps relate them:

Defn Let U, V be two G-representations. A linear map $\phi: U \longrightarrow V$ is called G-equivariant if

$$\phi(gu) = g\,\phi(u) \qquad \forall\, u \in U,\ g \in G.$$

ϕ is also called

- a map of G-representations, or
- a G-map.

Defn Let U, V be two G-representations. We let

$$\hom_G(U, V)$$

be the collection of all G-maps $U \to V$.

<u>Rmk</u> $\hom_G(U,V)$ is a $\mathbb{k}$-vec. space:

$$\phi(gu) = g\phi(u) \Longrightarrow a\phi(gu) = g \cdot a\phi(u)$$

and ϕ_1, ϕ_2 G-maps $\Longrightarrow (\phi_1 + \phi_2)(gu) = g(\phi_1 + \phi_2)(u)$

<u>Defn</u> A G-map ϕ is an <u>isomorphism</u> of G-representations if it is a bijection

<u>Rmk</u> $\Longrightarrow \phi^{-1}$ is a G-map, as

$$\begin{aligned} g\phi^{-1}(v) &= g\phi^{-1}(\phi(u)) \\ &= gu \\ &= \phi^{-1}(\phi(gu)) \\ &= \phi^{-1}(g\phi(u)) \\ &= \phi^{-1}(gv). \end{aligned}$$

<u>Defn</u> Let V be a G-rep. A vector subspace $W \subset V$ is called a <u>subrepresentation</u> if $G \cdot W = W$. That is, $\forall g \in G, w \in W$,

$$gw \in W.$$

<u>Ex</u> If G acts trivially on V, any subspace is a subrepresentation.

We'll see plenty of examples.

For now, let's keep building vocabulary.

<u>Defn</u> Let $\mathcal{U}, V$ be G-reps.

Then $\mathcal{U} \oplus V$ is the G-rep.

$$g(u,v) = (gu, gv).$$

Direct sum of reps

And $\mathcal{U} \otimes V$ is the G-rep.

$$g\left(\sum u_i \otimes v_i\right) = \sum gu_i \otimes gv_i .$$

Tensor product of reps.

<u>Exer</u> Let $G = C_n = \mathbb{Z}/n\mathbb{Z}$, and let $\mathcal{U}$ be the 1-dimensional vec space $\mathbb{C}$ w/ G-action

$$a \cdot z = \zeta^a z$$

where ζ is, say, $e^{\frac{2\pi i}{n}}$, and $a \in C_n$. Note $\mathcal{U} \otimes \mathcal{U}$ is again 1-dimensional. What is the G-rep. given by $\mathcal{U} \otimes \mathcal{U}$? By $\underbrace{\mathcal{U} \otimes \cdots \otimes \mathcal{U}}_{n \text{ times}}$?

<u>Sol'n</u> In general, $\mathcal{U}^{\otimes k} = \underbrace{\mathcal{U} \otimes \cdots \otimes \mathcal{U}}_{k \text{ times}}$

is the G-rep of $\mathbb{C}$ w/ action

$$a \cdot z = \zeta^{ak} z.$$

So $\otimes$ can produce (interesting?) new reps.

How do we classify or understand G-reps?
How do we understand algebraic things generally?
If this is only your third algebra class, you might not have a feel for this. So let's take the example of understanding finite groups (which is far harder than classifying G-reps):

Given a gp H, ask for normal subgps.

If $H' \trianglelefteq H$, can write injection and surjection
$$H' \hookrightarrow H \twoheadrightarrow H/H'.$$

Both H' and H/H' are smaller groups, so maybe you can understand H from these smaller pieces. Some scenarios, from best to worst:

- $H \cong H' \times H/H'$
- $H \cong H' \rtimes H/H'$
- H/H' does NOT lift to a subgp of H.
- H has only trivial normal subgps to begin with. (ie, H is simple.)

This last means we need to study simple groups one-by-one; they're also the basic building blocks.

The analogue of simplicity for G-represcentations:

Defn A represcentation V is called <u>irreducible</u> if the only subrepresentations of V are 0 and V.

Ex Any 1-dim rep. is irreducible.

Ex Let C_n, $n \geq 3$, act on $\mathbb{R}^2$ by rotation by $2\pi/n$. This is an irreducible C_n-rep.

Notation "irrep" is shorthand for "irreducible represcentation."

The following fact makes G-reps (for G finite) far easier to classify than, say, groups:

Prop'n Let G be a finite group, and suppose $\mathbb{k} = \mathbb{C}$. If V is a represcentation of G, and if $W \subset V$ is a subrepresentation, $\exists$ another subrep W' s.t.

$$V \cong W \oplus W'.$$

<u>Cor</u> (Complete reducibility) Let G be finite and V a fin-dim rep.
Then $\exists$ irreps V_α st

$$V \cong \bigoplus_\alpha V_\alpha.$$

The uniqueness of this decomposition will come soon.

(Imagine if $\forall$ $H' < H$, $\exists$ H'' s.t.

$$H \cong H' \times H''$$

for groups. Things would be much easier.)

The proposition uses finiteness of G by <u>averaging</u>.

<u>Pf</u> (of Prop.) Assume $\mathbb{k} = \mathbb{C}$, and fix any Hermitian inner product on V. Define a new inner product by

$$\langle u, v \rangle := \frac{1}{|G|} \sum_{g \in G} \langle gu, gv \rangle_{\text{old}}.$$

inner products aren't so nice in positive charac., so we'll have another proof soon. (See next page.) As an example, try putting inner product on $\mathbb{F}_2 \oplus \mathbb{F}_2$. Then this "average" will fail to be non-deg. for the non-trivial C_3 action.

This new inner product is G-invariant, in that

$$\langle gu, gv \rangle = \langle u, v \rangle.$$

<u>Pf</u>

$$\langle gu, gv \rangle = \frac{1}{|G|} \sum_{h \in G} \langle hgu, hgv \rangle$$

$$= \frac{1}{|G|} \sum_{h \in G} \langle hu, hv \rangle$$

$$= \langle u, v \rangle.$$

(The map $\{hg\}_{h \in G} \xrightarrow{(\,)g^{-1}} \{h\}_{h \in G}$ is obviously a bijection.) //

Then $W^\perp := \{u \mid \langle u, w \rangle = 0 \ \forall w \in W\}$ is a G-subspace, and

$$W \oplus W^\perp \longrightarrow V$$

is an isomorphism. //

Here's another proof:

Pf. Choose *any* complement V to W' (not nec. a subrep), so

$$W \cong W' \oplus V$$

as $\mathbb{k}$-vector spaces. This induces a projection

$$p: W \to W'.$$

Consider the map

$$p^{av} := \frac{1}{|G|} \sum_{g \in G} g \circ p \circ g^{-1}.$$

Note this is still a map $W \to W'$ because W' is a subrepresentation.

Moreover, if $w \in W'$,

$$\sum_{g \in G} g \circ p \circ g^{-1}(w)$$

$$= \sum g g^{-1}(w)$$

$$= |G| w$$

$w \in W' \Rightarrow g^{-1}w \in W'$

$\Rightarrow p(g^{-1}w) = g^{-1}w.$

$\neq 0$ unless $w = 0$. ($\operatorname{char} \mathbb{k} \nmid |G|$).

Hence $\operatorname{Ker}(p^{av}) \cap W' = \{0\}$, meaning

$$\operatorname{Ker}(p^{av}) \oplus W' \cong W.$$

OTOH, $p^{av}(w) = 0$ implies

$$\sum_g gpg^{-1}(h^{-1}w) = \sum_g h^{-1}(hg)p(hg)^{-1}w$$

$$= h^{-1} \sum_s sps^{-1} w$$

$$= h^{-1} 0 = 0. //$$

so $G\operatorname{Ker}(p^{av}) = \operatorname{Ker}(p^{av})$.

Now that we see any G-rep is a $\oplus$ of irreps, let's state a lemma that's both obvious and remarkably useful:

<u>Lemma</u> (Schur's Lemma) Fix $\mathbb{k}$ alg closed. Let U, V be irreps. Then

$$\hom_G(U,V) \cong \begin{cases} \mathbb{k} & U \cong V \\ 0 & \text{otherwise} \end{cases}$$

Moreover, if $\phi: U \to U$ is a G-map (w/ U an irrep), $\exists$ $\lambda \in \mathbb{C}$ s.t. $\phi(v) = \lambda v$ $\forall v$.

Pf. First part: If $\phi: U \to V$ G-map, $\ker(\phi) \subset U$ and $\operatorname{image}(\phi) \subset V$ are sub-reps. If both U, V are irreps,

- $\ker = U$, image $= V \implies V \cong 0$
- $\ker = U$, image $= \{0\} \implies \phi = 0$
- $\ker = 0$, image $= V \implies \phi$ an $\cong$.
- $\ker = 0$, image $= 0 \implies U = \{0\}$.

Next, if $\phi: U \to U$ is a G-map, then so is

$$\lambda I - \phi : U \to U. \quad \text{for any } \lambda \in \mathbb{k},$$

and either $\ker(\lambda I - \phi) = U$ or $= 0$. Since $\mathbb{k}$ is alg closed, $\exists$ root to char. polynomial, so for some λ, $\ker(\lambda I - \phi) \supsetneq \{0\}$, hence $\ker(\lambda I - \phi)$ is U by irreducibility, hence $\phi = \lambda I$.

Finally, if U is isomorphic to V by some map $\psi: U \to V$, ψ and ψ^{-1} induce an isomorphism

$$\hom_G(U,U) \longrightarrow \hom_G(U,V)$$
$$\phi \longmapsto \psi \circ \phi$$

hence $\hom(U,U) \cong \mathbb{k}$

$\implies \hom(U,V) \cong \mathbb{k}$. //

So now the name of the game is to find irreps. Miraculously, it turns out that to every representation V, we can assign a function

$$\chi_V : G \longrightarrow \mathbb{k},$$

and χ_V characterizes V up to isomorphism:

<u>**Thm**</u> Let $\mathbb{k} = \mathbb{C}$. Then two G-reps $V \cong W$ iff $\chi_V = \chi_W$.

For this reason, χ_V will be called the <u>character</u> of V. Even better, we'll see

<u>**Thm**</u> Let $\frac{G}{G}$ be the set of conjugacy classes of G. Then, up to $\cong$, $\exists$ exactly $\left|\frac{G}{G}\right|$ irreps over $\mathbb{k} = \mathbb{C}$

<u>**Rmk**</u> Sometimes in life, you'll encounter two related sets, be able to prove they're the same size, but not be able to give a nice or natural bijection between them. This is one of those situations. In general, there's no natural correspondence between conjugacy classes and irreps.

Let's start constructing these characters. The story is surprisingly straightforward:

<u>Defn</u> Let V be fin-dim G-rep. The <u>character</u> of V is the fxn

$$\chi_V : G \longrightarrow \Bbbk$$
$$g \longmapsto \text{trace}(\rho(g)) =: \text{tr}(\rho(g)).$$

We're lazy, so we write "tr" instead of "trace."

Here, $\rho : G \longrightarrow GL(V)$ is the homomorphism given by the representation.

<u>Rmk</u> (Definitions of trace). Let $f : V \to V$ be any endomorphism of any fin-dim vec space V. Then

- If you fix a basis $\{e_i\}$ for V, you can represent f as a matrix. $\text{tr}(f)$ can be defined as the sum of the diagonal entries.
- Let $\det(tI - f)$ be the characteristic polynomial of f. Then $\text{tr}(f)$ is the coefficient of $t^{\dim V - 1}$.
- $\text{tr}(f)$ is the composition

$$\Bbbk \to V^\vee \otimes V \xrightarrow{\text{id}_{V^\vee} \otimes f} V^\vee \otimes V \xrightarrow{ev} \Bbbk$$

in a basis, $\Bbbk \to V^\vee \otimes V$ is $1 \mapsto \sum e_i^\vee \otimes e_i$.

Recall:

Prop'n For two endomorphisms

$f, g : V \longrightarrow V$

we have

$$\mathrm{tr}(fg) = \mathrm{tr}(gf)$$

Pf • Choose basis, so f, g have matrix form. Then

$$\mathrm{tr}(fg) = \sum_i (fg)_{ii} = \sum_i \sum_j f_{ij} g_{ji}$$
$$= \sum_j \sum_i g_{ji} f_{ij}$$
$$= \sum_j (gf)_{jj}$$
$$= \mathrm{tr}(gf) \; //$$

Cor If f invertible,

$$\mathrm{tr}(fgf^{-1}) = \mathrm{tr}(g).$$

Cor If χ_V is a character, χ_V factors

$$\begin{tikzcd} G \arrow[r, "\chi_V"] \arrow[d] & \mathbb{k} \\ \frac{G}{G} \arrow[ur] & \end{tikzcd}$$

ie, $\chi_V(g)$ is sensitive only to the conjugacy class of $g \in G$.

Defn A fxn

$$G \longrightarrow \mathbb{k}$$

which takes conjugate elements to the same value is called a class function.

The collection of class fxns is a vector space of dimension $\# \frac{G}{G}$. Moreover,

Propn

$$\chi_{U \oplus V} = \chi_U + \chi_V.$$

Pf Choose a basis for U, V so the action of $g \in G$ is represented by a matrix

$$\begin{pmatrix} \rho_U(g) & 0 \\ 0 & \rho_V(g) \end{pmatrix}$$

where $\rho_U : G \longrightarrow GL(V) \cong GL(\mathbb{k}^{\dim V})$

$\rho_V : G \longrightarrow GL(U) \cong GL(\mathbb{k}^{\dim U})$.

Then obviously the sum of the diagonal entries gives

$$\chi_{U \oplus V}(g) = \chi_U(g) + \chi_V(g). //$$

The way we'll prove the theorems, in fact is:

<u>Thm</u> $\exists$ a Hermitian inner product on

$$\{\text{class fxns } G \to \mathbb{C}\}$$

s.t. the collection

$$\{\chi_V\}_{V \text{ irrep}}$$

forms an orthonormal basis.

Moreover, for U, V any two G-reps,

$$\langle \chi_U, \chi_V \rangle = \dim\left(\hom_G(U,V)\right).$$

<u>Cor</u> For any rep. V, if

$$V \cong \bigoplus_\alpha W_\alpha \cong \bigoplus_\beta W_\beta$$

where W_α, W_β are all irreps, $\exists$ bijection

$$\{\alpha\} \xrightarrow{i} \{\beta\}$$

s.t. $W_{i(\alpha)} \cong W_\alpha$.

In particular, given an irrep W, the # of times its isom class appears in $\{W_\alpha\}$, $\{W_\beta\}$ is the same.